Grobner Bases for polytopal affinoid algebras

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Joint work with M. Barkatou and T. Vaccon

FELIM 2024

Outline: Grobner theory

$K[X_1, \ldots, X_n]$

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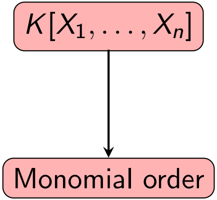

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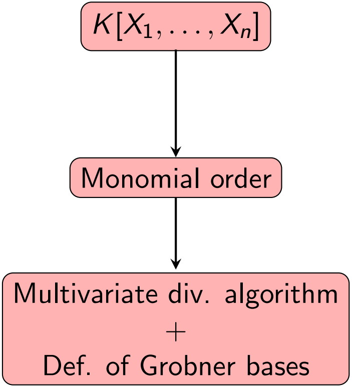

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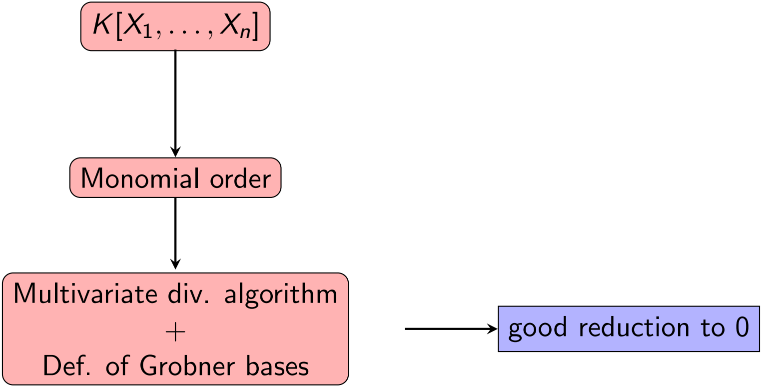

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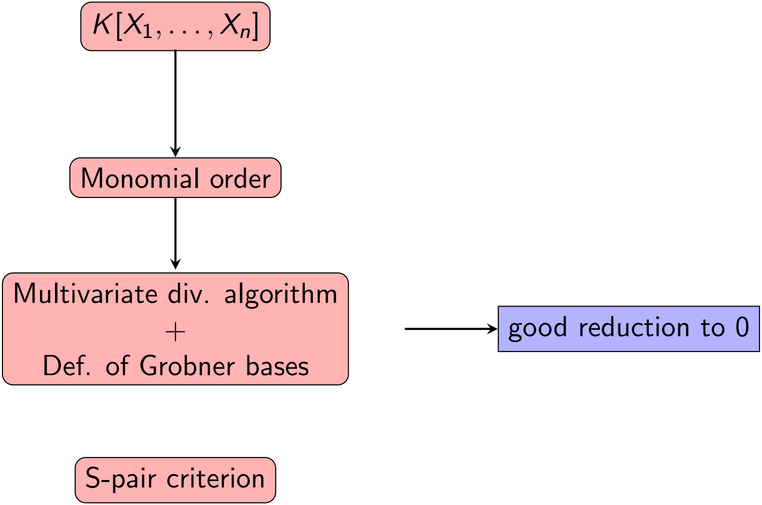

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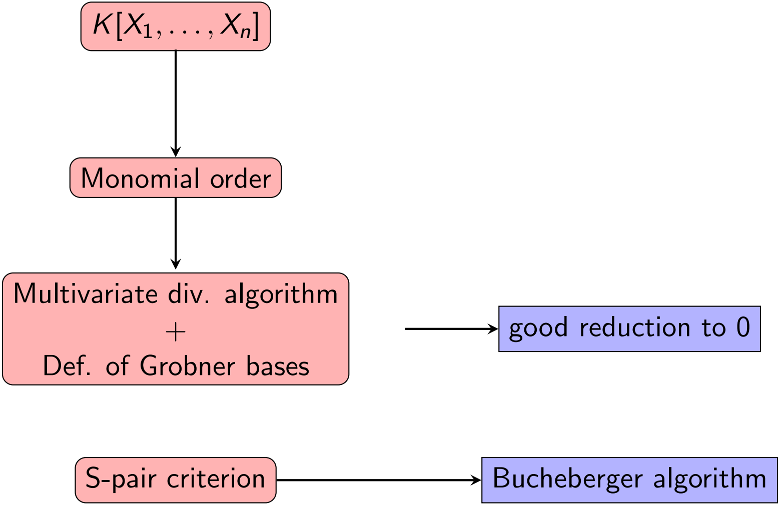

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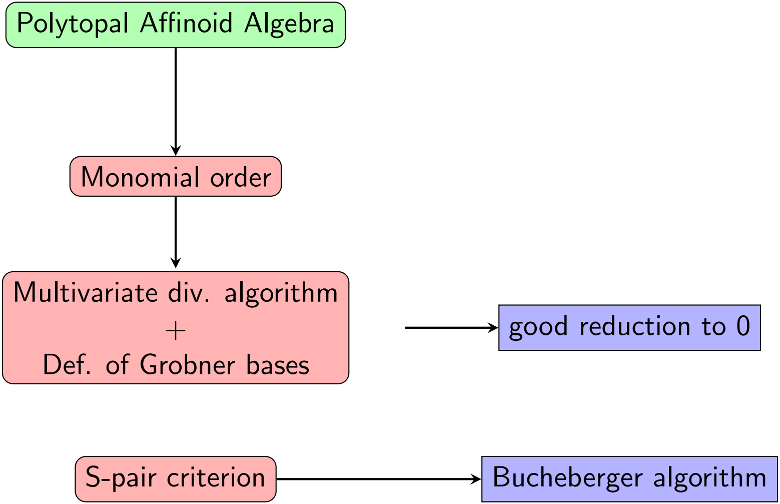

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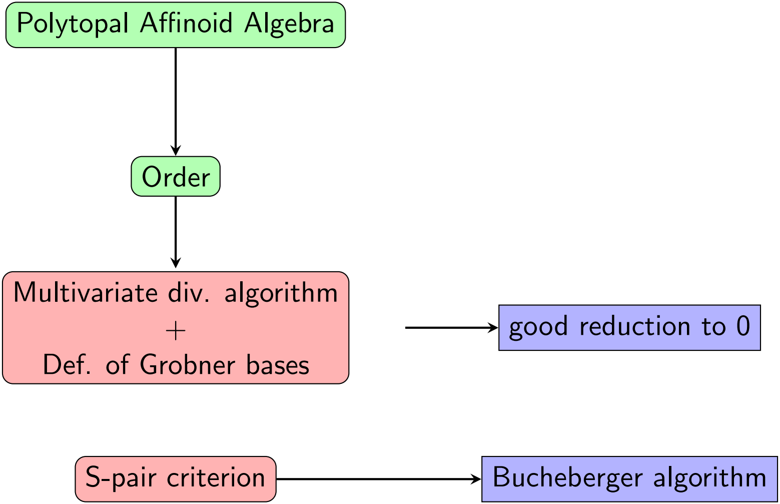

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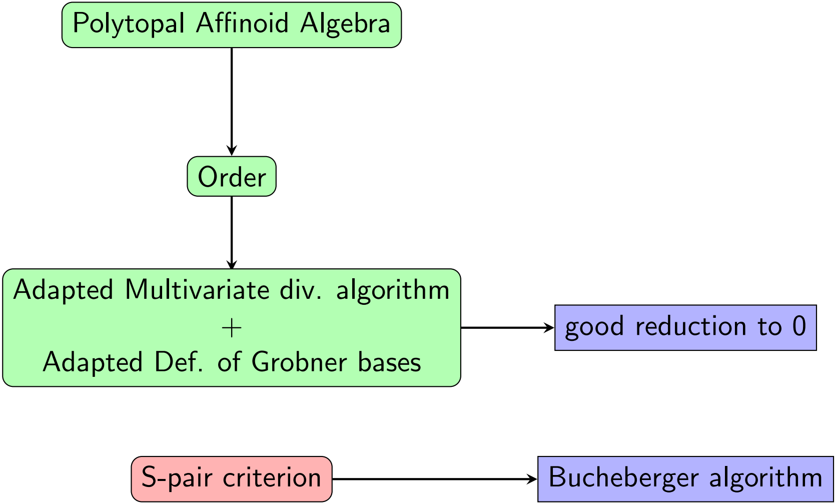

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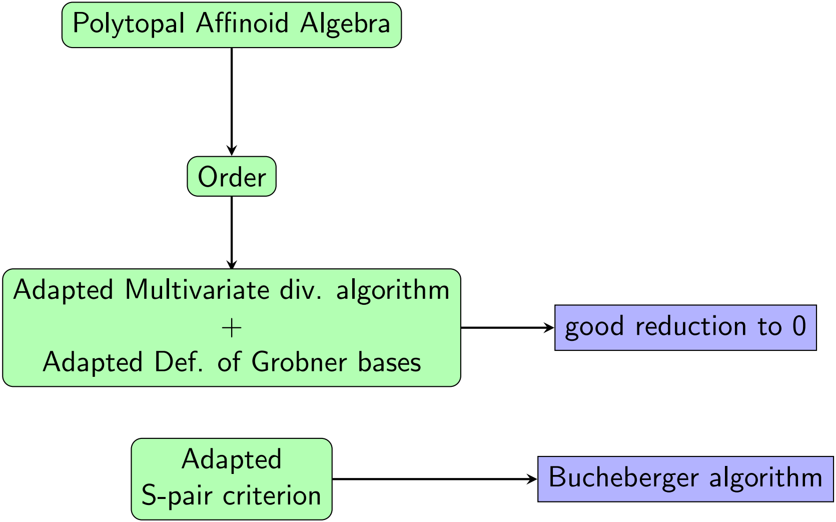

Non-archimedean setting

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Notations :

- $n \in \mathbb{N}^*$
- $\mathbf{X} := (X_1, \ldots, X_n)$
- $u \in \mathbb{Z}^n, \mathbf{X}^u := X_1^{u_1} \cdots X_n^{u_n}$

Non-archimedean setting

A series $\sum_{i=0}^{+\infty} a_i$ is convergent if and only if $val(a_i) \to +\infty$.

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$\text{val}_P(f)$ is the minimum valuation reached by f on $\text{val}^{-1}(P)$.

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- Can we use Grobner bases for PAA to demonstrate the existence of "tropical bases" for ideals in $K\{\mathbf{X}; P\}$?

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But we can use "generalized monomial order" as defined by Pauer-Unterkircher in:
PU:1999

Generalized monomial order

Put $T := \{\mathbf{X}^u,\ u \in \mathbb{Z}^n\}$.

Definition

A conic decomposition of T is a finite family $(T_i)_{i \in I}$ of finitely generated submonoids of T such that

1. for each i, the only invertible element in the monoid T_i is 1 and the group generated by T_i is T.
2. the union of all the T_i's equal T

Figure: Conic decompositions for $n = 2$

Generalized monomial order

Definition

Let $(T_i)_{i\in I}$ be a conic decomposition of T. A generalized monomial order (or g.m.o) on T for the decomposition $(T_i)_{i\in I}$ is a total order $<$ on T such that

1. $\forall t \in T, 1 \leq t$
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For $f \in K[\mathbf{X}^{\pm 1}], K\{\mathbf{X}; P\}\dots$ and $t \in T$, we generally have :

$$\mathrm{lt}(tf) \neq t\,\mathrm{lt}(f)$$

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Lemma

Write $\mathrm{lm}(uf) = ut_u \in T_i$ *and* $\mathrm{lm}(vf) = vt_v \in T_i$ *for some monomials* t_u, t_v *of* f. *Then* $t_u = t_v$.

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For $i \in I$ define $T_i(f) := \{t \in T,\ \operatorname{Im}(tf) \in T_i\}$.

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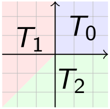
Definition

For $t \in T_i(f)$, define $\operatorname{Im}_i(f) := \operatorname{Im}(tf)t^{-1}$.

An example

Take the conic decomposition:



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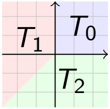

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$$(-\min(0, x_1, y_1) \leq -\min(0, x_2, y_2)) \text{ or}$$
$$(-\min(x_1, y_1) = -\min(x_2, y_2) \text{ and } (x_1, y_1) \leq_{lex} (x_2, y_2))$$

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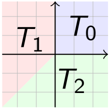

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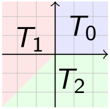

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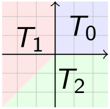

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- $xy^{-2} > x^{-1}y^{-2} > x^{-2}y^{-2} > y^2$,
- $\text{lm}_0(f) = \text{lm}_2(f) = \text{lm}(f) = xy^{-2}$
- $\text{lm}_1(f) = x^{-2}y^{-2}$

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$$2 = 2 = 2 < 6 = 6 < 8 \dots$$

Grobner basis

Fix a g.m.o $\leq$ for a conic decomposition $(T_i)_{i \in I}$.
Let J be an ideal in $K\{\mathbf{X}; P\}$ and G be a finite subset of $J \setminus \{0\}$.
We always have the containment:

$$\{\mathrm{lm}(f), f \in J\} \supset \bigcup_{g \in G} \{\mathrm{lm}(tg), t \in T\} = \bigcup_{g \in G, i \in I} T_i(g)\mathrm{lm}_i(g)$$

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Definition

We say that G is a Gröbner basis of J when:

$$\{\mathrm{lm}(f), f\in J\} = \bigcup_{g\in G}\{\mathrm{lm}(tg), t\in T\} = \bigcup_{g\in G, i\in I} T_i(g)\mathrm{lm}_i(g)$$

Multivariate division algorithm in $K\{\mathbf{X};P\}$

input : $f, g_1, \dots, g_m \in K\{\mathbf{X};P\}$
output: $q_1, \dots, q_m, r$

$q_1, \dots, q_m, r \leftarrow 0$;
while $f \neq 0$ **do**
  while $\exists (i,j) \in I \times [1,m]$ *such that* $\mathrm{lm}\left(\frac{\mathrm{lm}(f)}{\mathrm{lm}_i(g_j)} g_j\right) = \mathrm{lm}(f)$ **do**
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return $q_1, \dots, q_m, r \in K\{\mathbf{X};P\}$

Multivariate division algorithm in $K\{\mathbf{X}; P\}$

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output: $q_1, \ldots, q_m, r$

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Take $i \in I, f, g \in K\{\mathbf{X}; P\}$.

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The T_i-module

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Let $U(i, f, g)$ be a finite system of generator and $v \in U(i, f, g)$.

Definition (S-pair)

$$S(i, f, g, v) := \mathrm{lc}_i(g)\frac{v}{\mathrm{Im}_i(f)}f - \mathrm{lc}_i(f)\frac{v}{\mathrm{Im}_i(g)}g.$$

S-pairs criterion

Let $h_1, \ldots, h_m \in K\{\mathbf{X}; P\}$ and $i \in I$. For $1 \leq j \leq m-1$, let $U(i, h_j, h_{j+1})$ be a finite system of generators. Suppose that there are terms $\{t_1, \ldots, t_m\}$, $u \in T_i$ and $c \in \mathrm{val}(K^\times)$ such that

- for all $j \in \{1, \ldots, m\}$, $\mathrm{lt}(t_j h_j) = c_j u$ with $\mathrm{val}(c_j) = c$
- $\mathrm{lt}(\sum_{i=1}^m t_j h_j) < c_1 u$.

Then there are elements $d_j \in K$, $v_j \in U(i, h_j, h_{j+1})$ for $1 \leq j \leq m-1$ and $t'_m \in K\{\mathbf{X}; P\}$ such that:

1. $\sum_{j=1}^m t_j h_j = \sum_{j=1}^{m-1} d_j \frac{u}{v_j} S(i, h_j, h_{j+1}, v_j) + t'_m h_m$.
2. $\mathrm{val}_P(t'_m h_m) > \mathrm{val}_P(u c_1)$.
3. $\frac{u}{v_j} \in T_i$ for all $j < m$.
4. For all $j < m$, $\mathrm{val}\left(d_j \mathrm{lc}_i(h_j) \mathrm{lc}_i(h_{j+1})\right) \geq c$.

Buchberger algorithm in $K\{\mathbf{X}; P\}$

input : $J = (h_1, \dots, h_m)$ an ideal of $K\{\mathbf{X}; P\}$
output: a Gröbner basis of J

$H \leftarrow \{h_1, \dots, h_m\}$; $B \leftarrow \{(h_i, h_j),\ 1 \le i < j \le m\}$
while $B \neq \emptyset$ **do**
  $(f, g) \leftarrow$ element of B; $B \leftarrow B \setminus \{(f, g)\}$;
  for $i \in I$ **do**
    $U(i, f, g) \leftarrow$ finite set of generators of $\mathrm{lm}_i(f)T_i(f) \cap \mathrm{lm}_i(g)T_i(g)$;
    for $v \in U(i, f, g)$ **do**
      $\_, r \leftarrow \mathrm{division}(S(i, f, g, v), H)$;
      if $r \neq 0$ **then**
        $B \leftarrow B \cup \{(h, r),\ h \in H\}$;
        $H \leftarrow H \cup \{r\}$
return H

Example

- $T_0 := \{X^k,\ k \in \mathbb{N}^n\}$
- for $1 \leq j \leq n$ let T_j be the monoid generated by $\{X_1^{-1} \dots X_n^{-1}\} \cup \{X_1, \dots, \hat{X}_j, \dots, X_n\}$

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Theorem

For all $f, g \in K\{\mathbf{X}; P\}$ and $i \in I$, $U(i, f, g)$ is monogenous as a T_i-module.