Algebraicity of Hypergeometric Functions with Arbitrary Parameters

joint work with S. Yurkevich (arXiv:2308.12855)

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Functional Equations in Limoges (FELIM 2024)

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Overview

Definitions

Hypergeometric differential equation:

$$x(\theta + a_1)\cdots(\theta + a_p)F(x) = \theta(\theta + b_1 - 1)\cdots(\theta + b_q - 1)F(x) \qquad (\theta = x\frac{\mathrm{d}}{\mathrm{d}x})$$

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Solutions: **Hypergeometric function:**

$$F(x) = {}_pF_q\left[\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}; x\right] := \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \cdot \frac{x^n}{n!},$$

where $(a)_n := a(a+1)\cdots(a+n-1)$ denotes the **rising factorial.**

Examples

- Logarithm:

$$ {}_2F_1\left[\begin{matrix}1,1\\2\end{matrix};x\right] = -\frac{\log(1-x)}{x} = 1 + \frac{1}{2}x + \frac{1}{3}x^2 + \frac{1}{4}x^3 + \ldots \in \mathbb{Q}[\![x]\!] $$

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- **Catalan numbers:**

$$ C_n = \binom{2n}{n}\frac{1}{n+1} \in \mathbb{Z}, \quad \sum_{n\geq 0} C_n x^n = {}_2F_1\left[\begin{matrix}\frac{1}{2},1\\2\end{matrix};4x\right] $$

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- **Chebychev numbers:**

$$ a_n = \frac{(30n)!n!}{(15n)!(10n)!(6n)!} \in \mathbb{Z}, \quad \sum_{n\geq 0} a_n x^n = {}_8F_7\left[\begin{matrix}\frac{1}{30},\frac{7}{30},\frac{11}{30},\frac{13}{30},\frac{17}{30},\frac{19}{30},\frac{23}{30},\frac{29}{30}\\ \frac{1}{2},\frac{1}{3},\frac{2}{3},\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5}\end{matrix};\frac{30^{30}}{6^6 10^{10} 15^{15}}x\right] $$

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- Some other algebraic series, such as

$$ {}_3F_2\left[\begin{matrix}1/2,\sqrt{2}+1,-\sqrt{2}+1\\ \sqrt{2},-\sqrt{2}\end{matrix};4x\right] = \frac{(7x-1)(2x-1)}{(1-4x)^{5/2}} = 1+x-6x^2+\cdots \in \mathbb{Z}[\![x]\!] $$

Definitions

A power series $f(x) \in \mathbb{Q}[\![x]\!]$ is called **algebraic** (over $\mathbb{Q}(x)$) if there is $P(x, y) \in \mathbb{Q}[x, y]$, $P(x, y) \neq 0$, such that $P(x, f(x)) = 0$.

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A power series $f(x) \in \mathbb{Q}[\![x]\!]$ is called **globally bounded** if there are $\alpha, \beta \in \mathbb{Z} \setminus \{0\}$, such that $\beta f(\alpha x) \in \mathbb{Z}[\![x]\!]$ and its convergence radius is nonzero and finite.
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Theorem (Eisenstein 1852, Heine 1854)

Any algebraic $f(x) \in \mathbb{Q}[\![x]\!]$ is a polynomial or globally bounded.

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Theorem (Folklore, Abel 1827)

Any algebraic $f(x) \in \mathbb{Q}[\![x]\!]$ is D-finite.

Any hypergeometric function $F(x) \in \mathbb{Q}[\![x]\!]$ is D-finite as it satisfies the hypergeometric differential equation.

Classical Question (Fuchs, Liouville, ...)

Which D-finite functions are algebraic? Which differential equations have algebraic solutions?

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The hypergeometric function

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clearly is **not algebraic.** It is not even globally bounded.

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Gaussian Hypergeometric Functions

Schwarz 1873: Classification of all algebraic **Gaussian hypergeometric functions**, i.e., all $F(x) = {}_2F_1([a_1, a_2], [b_1]; x)$, with rational parameters $a_1, a_2, b_1 \in \mathbb{Q}$ by essentially providing a finite list.

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Landau 1904, 1911 and Errera 1913 exploited Eisenstein's Theorem, leading to an arithmetic criterion for algebraicity of Gaussian hypergeometric functions with rational parameters:

Theorem (Landau, Errera)

Let $F(x) = {}_2F_1([a_1, a_2], [b_1]; x)$ with $a_1, a_2, b_1, a_1 - b_1, a_2 - b_1 \notin \mathbb{Z}$. Then $F(x)$ is globally bounded iff it is algebraic and iff for all $1 \leq \lambda \leq N$ coprime to the common denominator N of a_1, a_2, b_1 we have

$$\langle \lambda a_1 \rangle < \langle \lambda b_1 \rangle < \langle \lambda a_2 \rangle \quad \text{or} \quad \langle \lambda a_2 \rangle < \langle \lambda b_1 \rangle < \langle \lambda a_1 \rangle,$$

where $\langle \cdot \rangle$ denotes the fractional part.

Christol's Interlacing Criterion

Define $\langle \cdot \rangle : \mathbb{R} \to (0, 1]$ as the fractional part, where integers are assigned 1 instead of 0.
Define $\preceq$ on $\mathbb{R}^2$ via $a \preceq b$ if $\langle a \rangle < \langle b \rangle$ or $\langle a \rangle = \langle b \rangle$ and $a \geq b$.

Theorem (Christol, 1986)

Let

$$F(x) = {}_pF_{p-1}\left[\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_{p-1} \end{matrix} ; x\right],$$

with rational parameters, $a_j, b_k \notin -\mathbb{N}$, denote by N the least common denominator of all parameters, and set $b_p = 1$. Then $F(x)$ is globally bounded if and only if for all $1 \leq \lambda \leq N$ with $\gcd(\lambda, N) = 1$ we have for all $1 \leq k \leq p$ that

$$|\{\lambda a_j \preceq \lambda b_k : 1 \leq j \leq p\}| - |\{\lambda b_j \preceq \lambda b_k : 1 \leq j \leq p\}| \geq 0.$$

Christol's Interlacing Criterion

For $a_j - b_k \notin \mathbb{Z}$ the criterion can be interpreted graphically:

Draw the sets $\{\exp(2\pi i\lambda a_j)\}$ in red and $\{\exp(2\pi i\lambda b_k)\}$ in blue on the unit circle for all $1 \leq \lambda \leq N$ with $\gcd(\lambda, N) = 1$. Then F is globally bounded iff there are always at least as many red as blue points going counter-clockwise starting after 1 (count with multiplicity).

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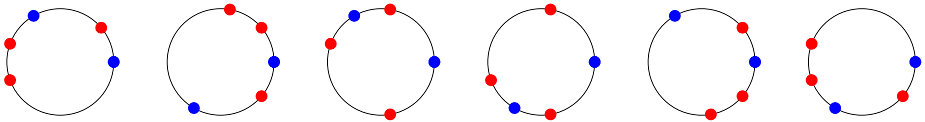
Example

${}_3F_2([1/9, 4/9, 5/9], [1/3, 1]; x)$ is globally bounded, as one can deduce from the pictures below. They correspond to $\lambda = 1, 2, 4, 5, 7, 8$ respectively.



Beukers–Heckman Interlacing Criterion

Theorem (Christol 1986, Beukers–Heckman 1989, Katz 1990)

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$$F(x) = {}_pF_{p-1}\left[\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_{p-1} \end{matrix}; x\right],$$

with rational parameters $a_j, b_k \notin -\mathbb{N}$ such that $a_j - b_k, a_j \notin \mathbb{Z}$, denote by N the least common denominator of all parameters, and set $b_p = 1$. Then $F(x)$ is algebraic if and only if for all $1 \leq \lambda \leq N$ with $\gcd(\lambda, N) = 1$ we have for all $1 \leq k \leq p$ that

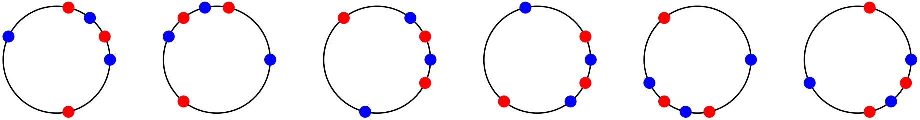
$$|\{\langle \lambda a_j \rangle \leq \langle \lambda b_k \rangle : 1 \leq j \leq p\}| - |\{\langle \lambda b_j \rangle \leq \langle \lambda b_k \rangle : 1 \leq j \leq p\}| = 0. \qquad \text{(IC)}$$

In other words, $F(x)$ is algebraic, if and only if the sets $\{\exp(2\pi i \lambda a_j)\}$ and $\{\exp(2\pi i \lambda b_k)\}$ **interlace** *on the unit circle for all λ.*

Beukers–Heckman Interlacing Criterion

Example

$F(x) = {}_3F_2([1/14, 3/14, 11/14], [1/7, 3/7]; x)$ is algebraic:



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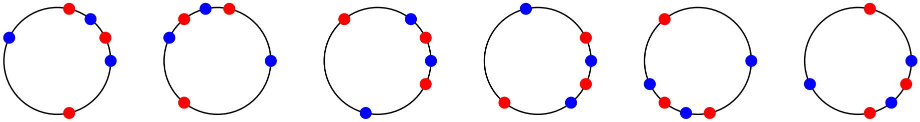

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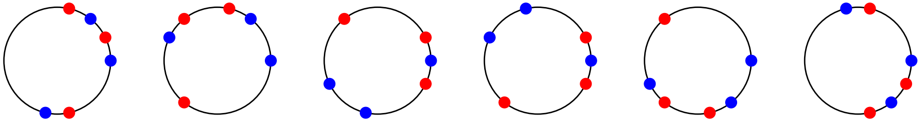
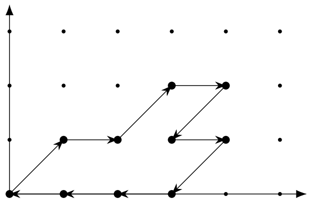
$F(x) = {}_3F_2([1/14, 3/14, 11/14], [1/7, 5/7]; x)$ is not algebraic:



Example from Combinatorics: Gessel Excursions

Lattice walks in the quaterplane with step set $\{\rightarrow, \leftarrow, \nearrow, \swarrow\}$: **Gessel walks**



Consider the generating function

$$G(x) = \sum_{n \geq 0} g_n x^n$$

of **excursions** of length n, i.e., walks with n steps that start and end at $(0, 0)$.

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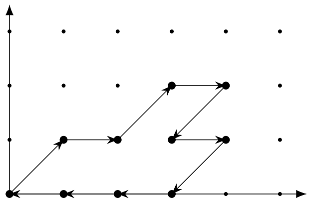

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Theorem (conjectured by Gessel 2001, Kauers–Koutschan–Zeilberger 2009, Bousquet-Mélou 2016, Bostan–Kurkova–Raschel 2017)

$$G(x) = \sum_{n\geq 0} \frac{(5/6)_n(1/2)_n}{(2)_n(5/3)_n} 16^n x^{2n} = {}_3F_2\left[\begin{matrix} \frac{5}{6}, \frac{1}{2}, 1 \\ 2, \frac{5}{3} \end{matrix}; 16x^2\right].$$

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Direct application of the interlacing criterion is not possible, as $a_3 = 1 \in \mathbb{Z}$.
Trick: use identities for hypergeometric functions:

$$G(x) = \frac{1}{2x^2}\left({}_2F_1\left[\begin{matrix}-1/2, -1/6\\ 2/3\end{matrix}; 16x^2\right] - 1\right),$$

which is algebraic by Schwarz' classification.

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Algebraicity of $G(x)$ was overlooked until Bostan and Kauers proved the algebraicity of the trivariate generating function $Q(x, y, t)$ of Gessel walks ending at $(i, j) \in \mathbb{N}^2$ in 2010.

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Minimal polynomial of $G(x)$:

$$\begin{gathered}27x^{14}y^8 + 108x^{12}y^7 + 189x^{10}y^6 + 189x^8y^5 - 9x^6(32x^4 + 28x^2 - 13)y^4\\ -9x^4(64x^4 + 56x^2 - 5)y^3 - 2x^2(256x^6 - 312x^4 + 156x^2 - 5)y^2\\ -(32x^2 - 1)(4x^2 - 6x + 1)(4x^2 + 6x + 1)y - 256x^6 - 576x^4 + 48x^2 - 1\end{gathered}$$

Irrational Parameters

The function

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Recall: The interlacing criterion of Beukers and Heckman treats the case of $a_j, b_k \in \mathbb{Q} \setminus -\mathbb{N}$ with $a_j - b_k, a_j \notin \mathbb{Z}$.

Aim

An easy to use criterion to account for irrational parameters and integer differences.

Change of Setting

Define

$$\mathcal{F}\left[\begin{matrix} c_1, \ldots, c_r \\ d_1, \ldots, d_s \end{matrix}; x\right] := \sum_{n\geq 0} \frac{(c_1)_n \cdots (c_r)_n}{(d_1)_n \cdots (d_s)_n} x^n.$$

Note:

$$\begin{aligned} {}_pF_q\left[\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}; x\right] &= \mathcal{F}\left[\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q, 1 \end{matrix}; x\right] \\ \mathcal{F}\left[\begin{matrix} c_1, \ldots, c_r \\ d_1, \ldots, d_s \end{matrix}; x\right] &= {}_{r+1}F_s\left[\begin{matrix} c_1, \ldots, c_r, 1 \\ d_1, \ldots, d_s \end{matrix}; x\right]. \end{aligned}$$

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$F(x)$ is **contracted** if $c_j - d_k \notin \mathbb{N}$. $F(x)$ is **reduced** if $c_j - d_k \notin \mathbb{Z}$.

The **contraction** $F^c(x)$ of $F(x)$ is obtained from $F(x)$ by removing pairs of parameters (c_j, d_k) with minimal difference $c_j - d_k \in \mathbb{N}$. It is contracted by definition.

If $F(x)$ is given as ${}_pF_q$, convert to $\mathcal{F}$ first.

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Example

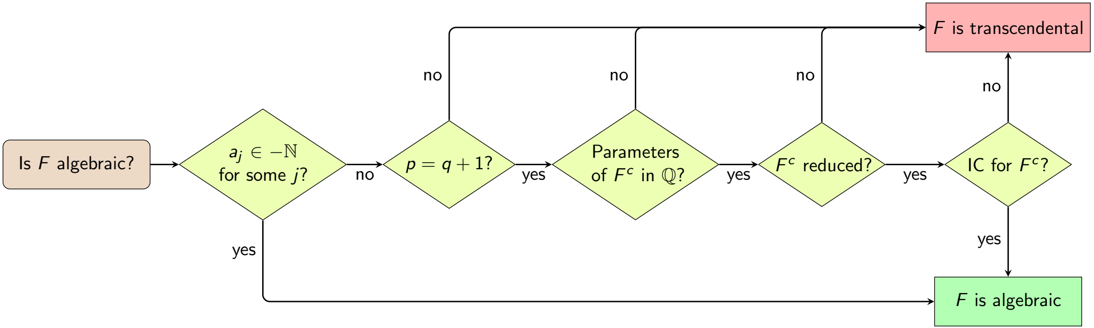
$${}_4F_3\left[\begin{matrix} \frac{1}{3}, \frac{1}{2}, 2, 4 \\ \frac{3}{2}, 3, 1 \end{matrix}; x\right]^c = \mathcal{F}\left[\begin{matrix} \frac{1}{3}, \frac{1}{2}, 2, 4 \\ \frac{3}{2}, 3, 1, 1 \end{matrix}; x\right]^c = \mathcal{F}\left[\begin{matrix} \frac{1}{3}, \frac{1}{2} \\ \frac{3}{2}, 1 \end{matrix}; x\right].$$

This contraction is not reduced, as $1/2 - 3/2 \in \mathbb{Z}$.

The Criterion

Theorem (F.–Yurkevich 2023)

For any hypergeometric function $F(x) = {}_pF_q([a_1, \ldots, a_p], [b_1, \ldots, b_q]; x) \in \mathbb{Q}[\![x]\!]$ *the following decision tree answers the question whether it is algebraic over* $\mathbb{Q}(x)$.



Ideas of the Proof

$(\theta + a_1)F(x)$ is a hypergeometric function with the same parameters as $F(x)$, except for a_1, which is increased by 1. With this one can show that $F(x)$ is algebraic if and only if $F^c(x)$ is algebraic.

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If $F(x)$ is contracted, its minimal differential equation is the hypergeometric one. If $F(x)$ has irrational parameters, the equation has an irrational local exponent and $F(x)$ cannot be algebraic.

If $F(x)$ is not reduced, define $G(x)$ by removing all pairs of parameters with integer differences. Then the interlacing criterion for global boundedness for $F(x)$ and for algebraicity for $G(x)$ cannot be fulfilled at the same time, contradicting the algebraicity of $F(x)$.

Example 1

$$f(x) = \mathcal{F}\left[\begin{matrix} \frac{1}{14}, \frac{3}{14}, \frac{11}{14}, 1+i\sqrt{3}, 1-i\sqrt{3} \\ \frac{1}{7}, \frac{3}{7}, i\sqrt{3}, -i\sqrt{3}, 3 \end{matrix}; x\right] = {}_6F_5\left[\begin{matrix} \frac{1}{14}, \frac{3}{14}, \frac{11}{14}, 1+i\sqrt{3}, 1-i\sqrt{3}, 1 \\ \frac{1}{7}, \frac{3}{7}, i\sqrt{3}, -i\sqrt{3}, 3 \end{matrix}; x\right].$$

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$$f(x) = \mathcal{F}\left[\begin{matrix} \frac{1}{14}, \frac{3}{14}, \frac{11}{14}, 1+i\sqrt{3}, 1-i\sqrt{3} \\ \frac{1}{7}, \frac{3}{7}, i\sqrt{3}, -i\sqrt{3}, 3 \end{matrix} ; x\right] = {}_6F_5\left[\begin{matrix} \frac{1}{14}, \frac{3}{14}, \frac{11}{14}, 1+i\sqrt{3}, 1-i\sqrt{3}, 1 \\ \frac{1}{7}, \frac{3}{7}, i\sqrt{3}, -i\sqrt{3}, 3 \end{matrix} ; x\right].$$

Contraction has rational parameters and is reduced:

$$f^c(x) = \mathcal{F}\left[\begin{matrix} \frac{1}{14}, \frac{3}{14}, \frac{11}{14} \\ \frac{1}{7}, \frac{3}{7}, 3 \end{matrix} ; x\right] = {}_4F_3\left[\begin{matrix} \frac{1}{14}, \frac{3}{14}, \frac{11}{14}, 1 \\ \frac{1}{7}, \frac{3}{7}, 3 \end{matrix} ; x\right].$$

Example 1

$$f(x) = \mathcal{F}\left[\begin{matrix} \frac{1}{14}, \frac{3}{14}, \frac{11}{14}, 1+i\sqrt{3}, 1-i\sqrt{3} \\ \frac{1}{7}, \frac{3}{7}, i\sqrt{3}, -i\sqrt{3}, 3 \end{matrix}; x\right] = {}_6F_5\left[\begin{matrix} \frac{1}{14}, \frac{3}{14}, \frac{11}{14}, 1+i\sqrt{3}, 1-i\sqrt{3}, 1 \\ \frac{1}{7}, \frac{3}{7}, i\sqrt{3}, -i\sqrt{3}, 3 \end{matrix}; x\right].$$

Contraction has rational parameters and is reduced:

$$f^c(x) = \mathcal{F}\left[\begin{matrix} \frac{1}{14}, \frac{3}{14}, \frac{11}{14} \\ \frac{1}{7}, \frac{3}{7}, 3 \end{matrix}; x\right] = {}_4F_3\left[\begin{matrix} \frac{1}{14}, \frac{3}{14}, \frac{11}{14}, 1 \\ \frac{1}{7}, \frac{3}{7}, 3 \end{matrix}; x\right].$$

We have already seen that $f^c(x)$ is **algebraic** by the interlacing criterion, thus so is $f(x)$.

Example 2

$$u_n = \frac{3}{2}\binom{4n}{n}\frac{n+2}{(n+1)(n+3)}.$$

Generating function:

$$f(x) = {}_6F_5\left[\begin{matrix}\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 3, 3, 1\\ \frac{1}{3}, \frac{2}{3}, 4, 2, 2\end{matrix}; \frac{256}{27}x\right]$$

Contraction:

$$f^c(x) = {}_4F_3\left[\begin{matrix}\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\\ \frac{1}{3}, \frac{2}{3}, 4\end{matrix}; \frac{256}{27}x\right]$$

Interlacing criterion: $f(x)$ **algebraic**.

Example 2

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Generating function:

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Contraction:

$$f^c(x) = {}_4F_3\left[\begin{matrix}\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\\ \frac{1}{3}, \frac{2}{3}, 4\end{matrix}; \frac{256}{27}x\right]$$

Interlacing criterion: $f(x)$ **algebraic**.

$$v_n = \frac{3}{2}\binom{4n}{n}\frac{n+2}{(n+1)^2}.$$

Generating function:

$$g(x) = {}_6F_5\left[\begin{matrix}\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 3, 1, 1\\ \frac{1}{3}, \frac{2}{3}, 2, 2, 2\end{matrix}; \frac{256}{27}x\right]$$

Contraction:

$$g^c(x) = {}_5F_4\left[\begin{matrix}\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, 1\\ \frac{1}{3}, \frac{2}{3}, 2, 2\end{matrix}; \frac{256}{27}x\right]$$

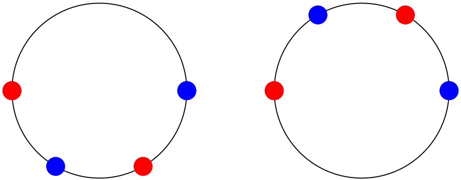
Not reduced: $g(x)$ **not algebraic**.

Example 3 – Gessel Revisited

Recall the generating function of Gessel excursions

$$G(x) = {}_3F_2\left[\begin{matrix}\frac{5}{6}, \frac{1}{2}, 1\\ 2, \frac{5}{3}\end{matrix}; 16x^2\right] = \mathcal{F}\left[\begin{matrix}\frac{5}{6}, \frac{1}{2}\\ 2, \frac{5}{3}\end{matrix}; x\right].$$

$G(x)$ is contracted, reduced, has only rational parameters and satisfies the interlacing criterion:



The End

Thank you for your attention!