G functions and hypergeometric series

Thomas Dreyfus [1], Tanguy Rivoal [2]

[1]Université Bourgogne, France
[2]Université Grenoble, France

E-functions

6

A definition

Definition

A power series $F(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n \in \overline{\mathbb{Q}}[[x]]$*, is an E-function if*

(i) *F is solution of a linear differential equation with coefficients in* $\overline{\mathbb{Q}}(x)$*.*

(ii) $\exists\ C > 0$ *such that* $\forall \sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), n \geq 0, |\sigma(a_n)| \leq C^{n+1}$*.*

(iii) $\exists D > 0, d_n \in \mathbb{N}^{\mathbb{N}}$*, with* $1 \leq d_n \leq D^{n+1}$*, such that* $d_n a_m$ *are algebraic integers for all* $m \leq n$*.*

Example

$\exp(x)$, $\cos(x)$...

Basic properties

Proposition

- *E-functions form a ring.*
- *Derivative of an E-function is an E-function.*

(weak version of) Siegel-Shidlovsky theorem

Theorem

Let F be a E-function and assume that F is transcendental over $\overline{\mathbb{Q}}(x)$. Then, for any $0 \neq \alpha \in \overline{\mathbb{Q}}$ that is not a singularity of the differential equation, $F(\alpha) \notin \overline{\mathbb{Q}}$.

Example

For all $0 \neq \alpha \in \overline{\mathbb{Q}}$, $\exp(\alpha) \notin \overline{\mathbb{Q}}$.

6

Hypergeometric series

Definition

$$ {}_pF_q\left[\begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix}; x\right] := \sum_{n=0}^{\infty} \frac{(a_1)_n\cdots(a_p)_n}{(1)_n(b_1)_n\cdots(b_q)_n} x^n $$

where $(a)_n := a(a+1)\cdots(a+n-1)$ *for* $n \geq 1$, $(a)_0 := 1$, *and* $a_j \in \mathbb{C}$, $b_j \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$.

Siegel's question

Is it possible to write any E-function as a polynomial with coefficients in $\overline{\mathbb{Q}}$ of E-functions of the form

$$ {}_pF_q[a_1, \ldots, a_p; b_1, \ldots, b_q; \gamma x^{q-p+1}], $$

with

- $q \geq p \geq 0$,
- $a_j \in \mathbb{Q}$, $b_j \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$
- $\gamma \in \overline{\mathbb{Q}}$?

Positive answer would contradicts a generalization to exponential periods of Grothendieck's Period Conjecture

6

Siegel's question

Is it possible to write any E-function as a polynomial with coefficients in $\overline{\mathbb{Q}}$ of E-functions of the form

$${}_pF_q[a_1, \ldots, a_p; b_1, \ldots, b_q; \gamma x^{q-p+1}],$$

with

- $q \geq p \geq 0$,
- $a_j \in \mathbb{Q}$, $b_j \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$
- $\gamma \in \overline{\mathbb{Q}}$?

Negative answer by Fresan-Jossen.

G-functions

A definition

Definition

A power series $F(x) = \sum_{n=0}^{\infty} a_n x^n \in \overline{\mathbb{Q}}[[x]]$*, is an G-function if*

(*i*) *F is solution of a linear differential equation with coefficients in* $\overline{\mathbb{Q}}(x)$*.*

(*ii*) $\exists\ C > 0$ *such that* $\forall \sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), n \geq 0, |\sigma(a_n)| \leq C^{n+1}$*.*

(*iii*) $\exists D > 0, d_n \in \mathbb{N}^{\mathbb{N}}$*, with* $1 \leq d_n \leq D^{n+1}$*, such that* $d_n a_m$ *are algebraic integers for all* $m \leq n$*.*

Example

$${}_pF_{p-1}\left[\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_{p-1} \end{matrix}; x\right].$$

Basic properties

Proposition

- *G-functions form a ring*
- *Derivative of an G-function is an G-function.*
- *algebraic function analytic at* 0 *are G-functions.*
- *G-functions have a positive radius of convergence.*

Fichler-Rivoal's question

Is it possible to write any G-function as a polynomial with coefficients in $\overline{\mathbb{Q}}$ of functions of the form

$$\mu(x) \cdot {}_pF_{p-1}[a_1, \ldots, a_p; b_1, \ldots, b_{p-1}; \lambda(x)],$$

with

- $p \geq 1$,
- $a_j \in \mathbb{Q}$, $b_j \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$,
- $\lambda, \mu \in \overline{\mathbb{Q}}[[x]]$ algebraic over $\overline{\mathbb{Q}}(x)$, and $\lambda(0) = 0$?

Slight extension of Fischler-Rivoal's question

Is it possible to write any G-function as a polynomial with coefficients in $\overline{\mathbb{C}(x)}$ of solutions of functions of the form

$$ {}_pF_{p-1}[a_1, \dots, a_p; b_1, \dots, b_{p-1}; \lambda(x)], $$

with

- $p \geq 1$,
- $a_j \in \mathbb{C}$, $b_j \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$
- $\lambda \in \overline{\mathbb{C}(x)}$?

Main result (toward a negative answer)

Theorem (D-Rivoal)

Let $M \in \mathbb{N}^*$. There exists a G-function which is not an element of the field of rational functions with coefficients in $\overline{\mathbb{C}(x)}$ of functions of the form

$$ {}_pF_{p-1}[a_1, \dots, a_p; b_1, \dots, b_{p-1}; \lambda(x)], $$

with

- $p \geq 1$,
- $a_j \in \mathbb{C}$, $b_j \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$
- $\lambda \in \mathbb{C}(x)$ *with coprime numerators and denominators of degree less than M.*

Differential Galois theory

Picard-Vessiot extension

Let $\partial_x Y = AY$, with $A \in \mathrm{Mat}_n(\mathbb{C}(x))$.

Definition

A Picard-Vessiot extension is a field extension $K|\mathbb{C}(x)$ such that

(i) $\exists U \in \mathrm{GL}_n(K)$, *s.t.* $\partial_x U = AU$.

(ii) $K = \mathbb{C}(x)(U)$.

(iii) $K^{\partial_x} = \{\alpha \in K | \partial_x \alpha = 0\} = \mathbb{C}(x)^{\partial_x} = \mathbb{C}$.

Proposition

Existence and uniqueness of the Picard-Vessiot extension.

Differential Galois group

Let $\partial_x Y = AY$, with $A \in \mathrm{Mat}_n(\mathbb{C}(x))$ be a differential system.

Definition

The differential Galois group is

$$\mathrm{Gal}(K|\mathbb{C}(x)) = \{\sigma \in \mathrm{Aut}(K|\mathbb{C}(x)) | \sigma \circ \partial_x = \partial_x \circ \sigma\}.$$

Algebraic group structure

Theorem

$$\begin{array}{rcl} \mathrm{Gal}(K|\mathbb{C}(x)) & \to & \mathrm{GL}_n(\mathbb{C}) \\ \sigma & \mapsto & U^{-1}\sigma(U). \end{array}$$

The latter representation identifies $\mathrm{Gal}(K|\mathbb{C}(x))$ *with a linear algebraic subgroup* $G \subset \mathrm{GL}_n(\mathbb{C})$.

Galois correspondence

Let $G = \mathrm{Gal}(K|\mathbb{C}(x)) \subset \mathrm{GL}_n(\mathbb{C})$.

Theorem

Let $\mathcal{G}$ be the set of algebraic subgroups of G and let $\mathcal{F}$ be the set of differential subfields of K containing $\mathbb{C}(x)$. Then, the following holds.

1. *The map $H \mapsto K^H$ defines a bijection between $\mathcal{G}$ and $\mathcal{F}$. Its inverse is given by $F \mapsto \mathrm{Gal}(K|F)$.*
2. *Let $H \in \mathcal{G}$. Then, H is a normal subgroup of G if and only if $F := K^H$ is stable under the action of G.*

From several Picard-Vessiot extensions to only one

Proposition

Let $f, f_1, \ldots, f_k$ be solutions of a linear differential equations with coefficients in $\mathbb{C}(x)$ whose differential Galois group we denote by G_f, G_{f_i} and with Picard-Vessiot extension K_f, K_{f_i} containing f, f_i. Assume that $f \in \mathbb{C}(x)(f_1, \ldots, f_k) \setminus \mathbb{C}(x)$.
If G_f is non commutative and has no normal algebraic subgroups other than itself and the trivial group, then $\exists i$ such that $K_f \subset K_{f_i}$.

Similar ideas in Fresan-Jossen proof

Consequence in the problem

Theorem (D-Rivoal)

Let $M \in \mathbb{N}^*$. There exists a G-function which is not an element of the field of rational functions with coefficients in $\overline{\mathbb{C}(x)}$ of functions of the form ${}_pF_{p-1}[a_1, \dots, a_p; b_1, \dots, b_{p-1}; \lambda(x)]$, with $p \geq 1$, $a_j \in \mathbb{C}$, $b_j \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$, and $\lambda \in \mathbb{C}(x)$ with coprime numerators and denominators of degree less than M.

- Assume that f belongs to that field and G_f is non commutative and has no normal algebraic subgroups other than itself and the trivial group.
- Then $K_f \subset K_{f_i}$ for $f_i = {}_pF_{p-1}[a_1, \dots, a_p; b_1, \dots, b_{p-1}; \lambda(x)]$.
- Then the singularities of f are inside the singularities of f_i.

Sketch of proof

What we are looking for?

Let us find a G-function f such that

- G_f is non commutative and has no normal algebraic subgroups other than itself and the trivial group.
- f has sufficiently many singularities.

Definition of the G-function (1/3)

We start with the generating series of the sequence of Apéry's numbers:

$$\alpha(x) := \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{n}^2 \right) x^n \in \mathbb{Z}[[x]].$$

It is a solution of the differential equation

$$x^2(1-34x+x^2)y'''(x) + x(3-153x+6x^2)y''(x) + (1-112x+7x^2)y'(x) + (x-5)y(x) = 0. \qquad (1)$$

The Galois group is not connected, we need to modify α.

Definition of the G-function (2/3)

Proposition

The G-function $\xi(x) := x(x^2 - 34x + 1)^{1/2}\alpha(x)$ *has a Galois group that is* $\mathrm{PSL}_2(\mathbb{C})$. *Moreover, the points* $(\sqrt{2} - 1)^4$ *and* $(\sqrt{2} + 1)^4$ *are non-polar singularities of* ξ.

Definition of the G-function (3/3)

Proposition

Let $\varphi \in \mathbb{C}(x) \setminus \mathbb{C}$. The G-function $\xi \circ \varphi(x)$ has a Galois group that is $\mathrm{PSL}_2(\mathbb{C})$.

Let $M \in \mathbb{N}^*$. Choose a convenient φ to have $\xi \circ \varphi(x)$ with at least $3M + 1$ singularities.

Theorem (D-Rivoal)

The G-function $\xi \circ \varphi(x)$ is not an element of the field of rational functions with coefficients in $\overline{\mathbb{C}(x)}$ of functions of the form

$$ {}_pF_{p-1}[a_1, \dots, a_p; b_1, \dots, b_{p-1}; \lambda(x)], $$

with $p \geq 1$, $a_j \in \mathbb{C}$, $b_j \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$, and $\lambda \in \mathbb{C}(x)$ with coprime numerators and denominators of degree less than M.

Sketch of proof

- Let $M \in \mathbb{N}^*$. Choose a convenient φ to have $\xi \circ \varphi(x)$ with at least $3M + 1$ singularities.
- To the contrary assume that $\xi \circ \varphi(x)$ is rational functions with coefficients in $\overline{\mathbb{C}(x)}$ of functions of the form ${}_pF_{p-1}[a_1, \ldots, a_p; b_1, \ldots, b_{p-1}; \lambda(x)]$, with $p \geq 1$, $a_j \in \mathbb{C}$, $b_j \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$, and $\lambda \in \mathbb{C}(x)$ with coprime numerators and denominators of degree less than M.
- The differential Galois group is $\mathrm{PSL}_2(\mathbb{C})$.
- Then, $\xi \circ \varphi \in K_{f_i}$ for $f_i = {}_pF_{p-1}[a_1, \ldots, a_p; b_1, \ldots, b_{p-1}; \lambda(x)]$.
- Then, $\xi \circ \varphi$ has at most $3M$ singularities. A contradiction.