Symbolic integration on planar differential foliations

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Consider a solution $y(x)$ of a differential equation

$$\frac{d}{dx}y(x) = F(x, y(x)) \qquad F \in \mathbb{K}(x, y) \tag{1}$$

The field $\mathbb{K}$ will be a finite extension of $\mathbb{Q}$.

We are interested in the symbolic integration of expressions of the form

$$I(x) = \int G(x, y(x))dx, \qquad G \in \mathbb{K}(x, y)$$

What do we mean by symbolic?

Definition

We say that I is elementary if it there exists a tower of field $K_n \supset K_{n-1} \supset \cdots \supset K_0 = \mathbb{C}(x, y(x))$ with $I \in K_n$, and where $K_{i+1} = K_i(f_i)$ and f_i is either

- *algebraic over K_i*
- *the exponential of an element of K_i*
- *the log of an element of K_i*

Remark that $I(x)$ is not always elementary, as

$$\int e^{x^2} dx = \operatorname{erf}(x)$$

We want to consider a larger class of functions than elementary when $y(x)$ is transcendental.

Our framework allows to consider many kinds of integrals

$$\int \frac{dx}{\ln x}, \int \frac{xdx}{e^x+1}, \int \frac{e^{x^2}}{\operatorname{erf}(x)}dx, \int x\sqrt{\ln x}dx, \int \frac{x\sqrt{x^3+1}dx}{\int \sqrt{x^3+1}dx}$$

When $y(x)$ is not algebraic, the action Galois group sends $y(x)$ to any (non algebraic) solution of (1).

Thus we can replace $y(x)$ by $y(x,h)$, where h parametrizes a family of solutions.

The integral now has a parameter h

$$I(x,h)=\int G(x,y(x,h))dx$$

How it behaves as a function of h?

Definition

A m variables function f is called Liouvillian if it there exists a tower of field $K_n \supset K_{n-1} \supset \cdots \supset K_0 = \mathbb{C}(x_1,\dots,x_m)$ with $f \in K_n$, and where $K_{i+1} = K_i(f_i)$ and f_i is either

- *algebraic over K_i*
- *the exponential of an element of K_i*
- *the integral of a closed 1-form with coefficients in K_i*

The integral

$$I(x,h)=\int \frac{dx}{\ln x + h}$$

is obviously Liouvillian as a single variable function of x.

However, as a function of h, this expression is not enough to conclude, we need to rewrite it

$$I(x,h)=e^{-h}\int \frac{e^h dx + e^h x dh}{\ln x + h}$$

And what about

$$I(x,h)=\int \frac{dx}{x+\ln x + h} \quad ?$$

Theorem

If $y(x,h)$ satisfies a differential equation $P(y)=0$ where $P \in \mathcal{O}(h)[x,y,\partial_h y,\partial_h^2 y,\dots]$, then equation (1) admits a symbolic first integral in one of the 4 classes

- *A rational first integral $\mathcal{F} \in \mathbb{C}(x,y)$*
- *A k-Darbouxian first integral, $(\partial_y \mathcal{F})^k = R \in \mathbb{C}(x,y), k \in \mathbb{N}^*$*
- *A Liouvillian first integral, $\partial_{yy}\mathcal{F}/\partial_y \mathcal{F} = R \in \mathbb{C}(x,y)$*
- *A Ricatti first integral, $\mathcal{F} = \mathcal{F}_1/\mathcal{F}_2$ where $\mathcal{F}_1, \mathcal{F}_2$ are a $\mathbb{C}(x)$ basis of solutions of a differential equation of the form $\partial_{yy}\mathcal{F}/\mathcal{F} = R \in \mathbb{C}(x,y)$.*

Theorem

If $I(x,h)$ satisfies a non constant differential equation in $\mathcal{O}(h)[x,y,\partial_h y,\partial_h^2 y,\dots,I,\partial_h I,\partial_h^2 I,\dots]$ and $y(x,h)$ is not algebraic in x, then up to reparametrization in h, it satisfies a differential equation of the form $LI=(\partial_h y)^{ord(L)}H$ where $H\in\mathbb{C}(x,y)$ and

- *$L\in\mathbb{C}[\partial_h^k]\partial_h^j,\ j\in\{0,\dots,k-1\}$ when equation (1) has a k-Darbouxian first integral.*
- *$L=\partial_h^j, j\in\mathbb{N}$ when equation (1) has a Liouvillian first integral.*
- *$L=\partial_h^j,\ j\in\{0,1\}$ when equation (1) has a Ricatti first integral or y differentially transcendental.*

We call such differential relation a telescoper.

Idea of proof: Assume we have a relation

$$P\left(x, y(x,h), \int G(x,y(x,h))dx, \int \partial_h G(x,y(x,h))dx, \dots\right) = 0$$

The Galois group of the integrals over the differential field

$$\mathbb{L} = \mathcal{O}(h)(x, y(x,h), \partial_h y(x,h), \dots).$$

of these integrals is either identity or additive.

It acts as translations

$$\left(\int G(x,y(x,h))dx, \int \partial_h G(x,y(x,h))dx, \int \partial_h^2 G(x,y(x,h))dx, \dots\right) \rightarrow$$

$$\left(\int G(x,y(x,h))dx, \int \partial_h G(x,y(x,h))dx, \int \partial_h^2 G(x,y(x,h))dx, \dots\right) + v$$

If all the possible translations vectors v satisfy a linear relation, then $I(x,h)$ satisfies a linear differential equation.

Else P is constant in the integrals, the relation is trivial!

If there exists a symbolic first integral, we can write up to reparametrization

$$\mathcal{F}(x, y(x, h)) = h$$

The differential Galois group acts as

- $\mathcal{F} \to \xi\mathcal{F} + \beta,\ \xi^k = 1$ in the k-Darbouxian case
- $\mathcal{F} \to \alpha\mathcal{F} + \beta$ in the Liouvillian case
- $\mathcal{F} \to \frac{\alpha\mathcal{F}+\beta}{\gamma\mathcal{F}+\delta}$ in the Ricatti case
- $\mathcal{F} \to \phi(\mathcal{F})$ in the differentially transcendental case

$\Rightarrow$ This restricts the coefficients to be constants and of a specific form!

If L exists, then I satisfies the PDE system

$$LI = (\partial_h y)^{\operatorname{ord}(L)} H, \quad \partial_x I = G$$

which has finite dimensional space of solutions.

Noting $I(x,h) = J(x, y(x,h))$, we have

- For $L = 1$, J is rational
- For $L = \partial_h$, we can write

$$J(x,y) = \int (G(x,y) - H(x,y)F(x,y))dx + H(x,y)dy$$

  and thus J is elementary
- In other cases, a k-Darbouxian first integral $\mathcal{F}$ should exist, and we note $U = \partial_y \mathcal{F}$, called the integrating factor.

Remarking that $\partial_h y = U^{-1}$, J is then solution of the holonomic system

$$D_x J = G, \ \sum_{i=0}^{\text{ord}} a_i D_h^i J = U^{-\text{ord}} H \text{ where } D_x = \partial_x + F\partial_y, \ D_h = U^{-1}\partial_y$$

Solutions of the homogeneous part are of the form

$$\mathcal{F}^k e^{\alpha\mathcal{F}}, \ \alpha \in \mathbb{C}, \ k \in \mathbb{N}$$

$\Rightarrow$ By variation of constants

$$J(x,y) = \sum_{\lambda\in S} \sum_{r=0}^{v_\lambda} e^{\lambda\mathcal{F}} \mathcal{F}^r \int e^{-\lambda\mathcal{F}} \sum_{i=1}^{m_\lambda} \sum_{j\in\mathbb{Z}} \mathcal{F}^i U^j \omega_{\lambda,i,j,r}$$

This is not always elementary, but always Liouvillian as a two variables function.

Proposition

If $\int G(x,y(x))dx$ is elementary, then $I(x,h)$ admits a telescoper.

If $\int G(x,y(x))dx$ is elementary, then so is $I(x,h)$ for any h

$$I(x,h) = F_0(x,y(x,h)) + \sum_{i=1}^{\ell} \lambda_i(x,y(x,h)) \ln F_i(x,y(x,h)).$$

Differentiating, we see that λ_i should be functions of h only. Thus applying a suitable operator $L \in \mathcal{O}(h)[\partial_h]$, we can ensure that

$$LI \in \mathcal{O}(h)[x, y(x,h), \partial_h y(x,h), \dots].$$

By the previous theorem, up to reparametrization, $I(x,h)$ admits a telescoper.

Example

$$\int \frac{x^3\ln x + x^3 + x^2\ln x + x^2 + x\ln x + x + \ln x}{x\ln x(1+\ln x)}dx$$

Noting $y = \ln x$, we have

$$y' = \frac{1}{x} = F(x,y)$$

$$\partial_x I = \frac{x^3y + x^3 + x^2y + x^2 + xy + x + y}{xy(1+y)}$$

A 1-Darbouxian first integral exists,

$$\mathcal{F}(x,y) = y - \ln x$$

A telescoper is found

$$L = \partial_h^4 + 6\partial_h^3 + 11\partial_h^2 + 6\partial_h$$

with certificate

$$-8y^{-4}(1+y)^{-4}(16x^3y^6 + 16x^3y^5 + 24x^2y^6 - 29y^8 - 48x^3y^4 + 32x^2y^5 + 48xy^6 - 164y^7 - 32x^3y^3 - 64x^2y^4 + 112xy^5 - 230y^6 + 112x^3y^2 - 96x^2y^3 + 16xy^4 - 180y^5 + 144x^3y + 56x^2y^2 - 96xy^3 - 37y^4 + 48x^3 + 128x^2y + 16xy^2 + 48x^2 + 112xy + 48x)$$

Integration of the ∂-finite PDE system gives

$$-x^3e^{-3y}Ei(-3y) - x^2e^{-2y}Ei(-2y) - xe^{-y}Ei(-y) + \ln(1+y)$$

The result is defined up to a linear combination of

$$1, e^{\ln x - y}, e^{2(\ln x - y)}, e^{3(\ln x - y)}$$

These are first integrals of (1), and thus functions of h.

For any closed loop γ_h on the complex Riemann surface $\ln x - y = h$, we have

$$L \int_{\gamma_h} \frac{x^3y + x^3 + x^2y + x^2 + xy + x + y}{xy(1+y)} = 0$$

The integrand is defined on an infinite genus Riemann surface, but the monodromy maps the infinite dimensional homotopy group to a finite dimensional vector space.

In the case of Liouvillian, Riccati or no first integrals, we look for

$$Q(x,y(x))\partial_h^{\ell} I(x,h) = U^{-\ell} P(x,y(x))$$

In the k-Darbouxian case, we look for

$$\sum_{i=0}^{\lfloor \ell/k \rfloor} Q(x,y(x,h)) a_i \partial_h^{ki+(l \bmod k)} I(x,h) = U^{-(\ell \bmod k)} P(x,y(x,h))$$

However, the unknowns P, Q, a_i appear non linearly!

Definition

A k-pseudo telescoper is of the form with $Q_i, P \in \mathbb{C}[x,y]$,

$$\sum_{i=0}^{\lfloor l/k \rfloor} Q_i(x,y(x,h)) \partial_h^{ki+(l \bmod k)} I(x,h) = U^{-(l \bmod k)} P(x,y(x,h))$$

Proposition

Assume equation (1) admits a k-Darbouxian first integral but not rational first integral. If a non trivial k-pseudo telescoper exists, then a true telescoper exists. The algorithm <u>ReduceTelescoper</u> *always terminate and compute such telescoper. It runs in* $\tilde{O}(Nord^{\omega+3})$.

<u>ReduceTelescoper</u>

1. Note L_1 the initial telescoper. Assign $i=1$. While $\text{rank}_{\mathbb{K}(x,y)}((L_j)_{j=1\dots i}) = i$ do
$$L_{i+1} := \partial_x L_i,\ i := i+1$$
2. Build a row echelon form of the matrix L, and note $(Q_r,\dots,Q_0,P)$ its shortest non zero line. Return
$$\sum_{i=0}^{r} \frac{Q_i}{Q_r} \partial_h^{ki+(l \bmod k)} I(x,h) = U^{-(l \bmod k)} \frac{P}{Q_r}$$

How to check if a telescoper is correct? We differentiate it in x

$$\sum_{i=0}^{\lfloor l/k \rfloor} a_i D_h^{ki+(l \bmod k)} G = U^{-(l \bmod k)}(D_x H + HU^{-1} D_x U)$$

This a an equality of rational functions which can thus be checked.

Integrating it in x, we recover the telescoper up to a function of h

$$\sum_{i=0}^{\lfloor l/k \rfloor} a_i \partial_h^{ki+(l \bmod k)} I(x,h) = U^{-(l \bmod k)} H(x, y(x,h)) + f(h)$$

The integrating constant $f(h)$ can be removed by subtracting to $I(x,h)$ a function $g(h)$ solution of the equation

$$\sum_{i=0}^{\lfloor l/k \rfloor} a_i \partial_h^{ki+(l \bmod k)} g(h) = f(h).$$

FindTelescoper

1. Note $M = \frac{1}{2}(N+1)(N+2)(\lceil \text{ord}/k \rceil + 2)$.
2. Compute at order M the list LG of $(\partial_h^j G(x, y(x,h))_{j=0\dots\text{ord}}$.
3. Compute J the list of list of $y(x)^i \int_0^x (LG_j(x, y(x)))_{|\partial_h y = U^{-1}} dx, j = 0 \dots \text{ord}, i = 0 \dots N$
4. For ℓ from 0 to ord do
    1. If U does not exist and $\ell \leq 1$, look for a telescoper of the form $Q(x, y(x,h))\partial_h^\ell I(x,h) = \partial_h y P(x, y(x,h))$.
    2. If U is not algebraic, look for a telescoper of the form $Q(x, y(x,h))\partial_h^\ell I(x,h) = U^{-\ell} P(x, y(x,h))$
    3. Else look for a k pseudo telescoper with valuation ℓ in ∂_h vanishing on the series at order M.
    4. If a non trivial solution found, note it T.
    5. $T =$ReduceTelescoper(T, G, F, U). If T is correct return T else return FAIL.
5. Return "None".

Proposition

If $\int G(x, y(x))dx$ admits a telescoper of order ord and degree $\leq N$, then <u>FindTelescoper</u> *returns either a correct telescoper, or FAIL.*
If <u>FindTelescoper</u> *returns "None", then no telescoper of order $\leq$ ord and certificate degree $\leq N$ and with structure according to given U exists.*
If <u>FindTelescoper</u> *returns FAIL, then (x_0, y_0) belongs to a codimension 1 algebraic set.*
The complexity is $\tilde{O}(N^{\omega+1} ord^{\omega-1} + N ord^{\omega+3})$.

Example 1: $y=\ln x,\ \partial_x y=\frac{1}{x},\ U=1$

$$I_1=\int \frac{x^2}{(\ln x)^2}dx, \quad D_h I_1+3I_1=\frac{4x^3-y^2}{4y^2}, \quad I_1=-\frac{x^3}{\ln x}+3Ei(3\ln x)$$

$$I_2=\int \frac{x^3+(\ln x)^3+x^2}{(x+1)\ln x}dx, \quad D_h^4 I_2+3D_h^3 I_2=\frac{16y^4-162x^3}{27y^4}$$

$$I_2=2\ln x Li_2(-x)-2Li_3(-x)+Ei(3\ln x)+(\ln x)^2\ln(x+1)$$

$$I_3=\int \frac{2x(\ln x)^2+(\ln x)^3+(\ln x)^2-x-\ln x}{x(x+\ln x)(\ln x)^2},$$

$$D_h I_3=-\frac{\frac{4}{45}xy^2+\frac{4}{45}y^3-y^2+x+y}{y^2(y+x)},$$

$$I_3=\ln x+(\ln x)^{-1}+\ln(x+\ln x)$$

$$I_4 = \int \frac{x^2+2x\ln x+\ln x}{x(x+\ln x)\ln x}dx,$$

$$D_h^2 I_4 + D_h I_4 = \frac{14x^2y^2+28xy^3+14y^4-225x^3-450x^2y+225y^3-225y^2}{225y^2(y+x)^2},$$

$$I_4 = Ei(\ln x) + \ln(x+\ln x)$$

$$\int \sum_{m=1}^{n} \frac{x^m}{\ln x + h} dx = \left(\sum_{m=1}^{n} e^{m(y-\ln x)} Ei(my) \right)_{|y-\ln x = h}$$

Telescoper order m, degree $2m$

n	1	2	3	4	5	6
time	0.s	0.13s	0.8s	12s	96s	583s

$$\int -\frac{e^x x^2}{2(e^x-1)}dx$$

Considering the differential equation and first integral

$$y' = y, \quad \mathcal{F}(x,y) = x - \ln y$$

Integral rewrites

$$I(x,h) = \int_{x-\ln y=h} -\frac{yx^2}{2(y-1)}dx$$

Telescoper found

$$\partial_h^3 I = \frac{x^2y^2 + x^2y + 2xy^2 - 12y^3 - 2xy + 38y^2 - 40y + 14}{2(y-1)^3}$$

Integration of the connection gives

$$Li_3(y) - xLi_2(y) + \frac{1}{2}x^2 Li_1(y)$$

Proposition

An integral of $G \in \mathbb{C}(x, \ln x)$ admitting a telescoper can be written

$$\int G(x)dx = \sum_{p\in\mathbb{Z}^*,\lambda\in\mathbb{C}} a_{p,\lambda} Ei(p\ln x+\lambda) + \sum_{p,r\in\mathbb{N}^*,\lambda\in\mathbb{C}^*} b_{p,r,\lambda}(\ln x)^r Li_p(\lambda x) +$$

$$\sum_{\lambda\in\mathbb{C}} \lambda \ln(K_\lambda(x,\ln x)) + H(x,\ln x)$$

where K_λ, H are rational functions.

IntegrateLn

1. Note $G = P/Q$. Look for $R \in Q^{-1}\mathbb{C}[x, y]$ such that $G - (\partial_x R + \frac{1}{x}\partial_y R)$ has only simple poles outside $x = 0$. If possible, note $\tilde{G}$ the resulting fraction.
2. Compute the residue in y along poles of $\tilde{G}$ of the form $y = \lambda$. If in $\mathbb{C}[x, 1/x]$, remove them from $\tilde{G}$.
3. Compute the residue in x along poles of $\tilde{G}$ of the form $x = \lambda$, $\lambda \neq 0$. If in $\mathbb{C}[y]$, remove them from $\tilde{G}$.
4. Look for an integral of $\tilde{G}$ of the form

$$S(x, y) + \sum \lambda_i \ln Q_i(x, y)$$

where $Q_i \mid Q$,and $S \in \mathbb{C}[x, \frac{1}{x}, y]$. If all previous steps succeeded, return the expression, else return "None".

Proposition

An integral of $G \in \mathbb{C}(x, x^\alpha)$, $\alpha \notin \mathbb{Q}$ admitting a telescoper can be written

$$\int G(x)dx = \sum_{(p,q,r)\in(\mathbb{Z}*)^3,\lambda\in\mathbb{C}} a_{p,q,r,\lambda} x^r \Phi\left(\lambda x^{p\alpha+q}, 1, \frac{r}{p\alpha+q}\right) +$$

$$\sum_{(p,q)\in(\mathbb{Z}^2)*,\lambda\in\mathbb{C}} b_{p,q,\lambda} x^{p\alpha+q} \Phi\left(\lambda x, 1, p\alpha+q\right) +$$

$$\sum_{\lambda\in\mathbb{C}} \lambda \ln(K_\lambda(x, x^\alpha)) + H(x, x^\alpha)$$

where K_λ, H are rational functions.

The extension x^α can be replaced by $h(x)^\alpha$ with h homography.

Is it possible to generalize these results?

Consider the Darbouxian first integral

$$\mathcal{F}(x,y)=\ln(1+x^2y)+\sqrt{2}\ln\left(\frac{x^2+y\sqrt{2}}{x^2-y\sqrt{2}}\right)$$

x is a Darboux polynomial, and $\mathcal{F}$ is smooth along $x=0$.

$$I_9=\int_{\mathcal{F}(x,y)=h}\frac{y^2x(x^2y^6-4y^7-4x^2y^4+19y^5+x^2y^2+2y^3+2x^2-8y)}{(x^4+4x^2y-2y^2+4)(y^2+2)^5}dx=$$

$$\frac{10x^4y^8+242x^4y^6-81x^2y^7+402x^4y^4-162x^2y^5+590x^4y^2-81y^6+376x^4-324y^4-324y^2}{648(y^2+2)^4x^4}$$

The integrand has no poles along $x=0$, but the integral has a pole of order 4 along $x=0$!

This simplification occurs because the integral is a series expansion of $1/\mathcal{F}^4$ which is meromorphic along $x=0$.

Proposition

If equation (1) has no rational first integral, and for any algebraic solution Γ, there exists an order $k \in \mathbb{N}^*$ such that the normal variational equation of order k near Γ has an infinite Galois group, then there exists an algorithm to decide the existence of a telescoper.

New poles appearing in the telescoper can only be Darboux polynomials.

Their order increase is bounded by k.

If a non rational symbolic first integral exists which is meromorphic near a particular algebraic orbit, the order of the pole can increase arbitrary.

$\Rightarrow$ We are not able to define a Hermite reduction near such poles!

Example 2: $y=\left(\frac{x-\sqrt{2}}{x+\sqrt{2}}\right)^{\sqrt{2}}, \ \partial_x y=\frac{4y}{x^2-2}, \ U=\frac{1}{y}$

$$I_6=\int \frac{(x^2+2)\left(\frac{x-\sqrt{2}}{x+\sqrt{2}}\right)^{\sqrt{2}}}{(x^2-2)^2\left(\left(\frac{x-\sqrt{2}}{x+\sqrt{2}}\right)^{\sqrt{2}}+1\right)}dx,$$

$$D_h^2 I_6-\frac{1}{2}I_6=\frac{2x^2y^2+5x^2y+2xy^2+2x^2+2xy-4y^2-6y-4}{4(y+1)^2(x^2-2)}$$

$$I_6=-\frac{(6x^5+40x^3+24x)\sqrt{2}+x^6+30x^4+60x^2+8}{(32x^5-128x)\sqrt{2}+8x^6+80x^4-160x^2-64}\Phi\left(-\left(\frac{x-\sqrt{2}}{x+\sqrt{2}}\right)^{\sqrt{2}},1,-\frac{\sqrt{2}}{2}\right)$$

$$-\frac{(2x\sqrt{2}+x^2+2)(x^2-2)}{(32x^3+64x)\sqrt{2}+8x^4+96x^2+32}\Phi\left(-\left(\frac{x-\sqrt{2}}{x+\sqrt{2}}\right)^{\sqrt{2}},1,\frac{\sqrt{2}}{2}\right)-\frac{(x+2)(x-1)}{x^2-2}$$

Example 3:

$$\int \frac{-xAi(x)^4 - 4Ai(x)^3 xAi'(x) + (4x^3+4x+1)Ai'(x)^2 Ai(x)^2 + 4Ai(x)Ai'(x)^3 - (4x^2-4x+6)Ai'(x)^4}{Ai'(x)(-x^2Ai'(x) + Ai(x) - Ai'(x))(Ai(x)^2 - 2Ai'(x)^2)} dx$$

The function $y(x) = Ai'(x)/Ai(x)$ satisfies the equation

$$y' = xy^2 - 1 = F(x,y)$$

Integral rewrites $I(x,h) =$

$$\int_{\frac{Ai(x)+yAi'(x)}{Bi(x)+yBi'(x)}=h} -\frac{4x^3y^2 - xy^4 + 4xy^3 + 4xy^2 - 4x^2 + y^2 + 4x - 4y - 6}{(x^2+y+1)(y^2-2)}$$

Telescoper found

$$\partial_h I = (\partial_h y)\frac{-4x^2 + y^2 - 4y - 6}{(x^2+y+1)(y^2-2)}$$

Integration gives

$$\ln(x^2+y+1) + \sqrt{2}\ln\left(\frac{y+\sqrt{2}}{y-\sqrt{2}}\right)$$

Example 4:

$$\int 2\cos(2\Pi(x,-1,2)) + \frac{2\sqrt{4x^4-5x^2+1}}{(4x^6-x^4-4x^2+1)\sin(2\Pi(x,-1,2))}dx$$

This expression is rational in x and

$$y(x) = \frac{\tan(\Pi(x,-1,2))}{\sqrt{4x^4-5x^2+1}}$$

$$y' = \frac{1+(4x^4-5x^2+1)y^2+(-8x^5-3x^3+5x)y}{(4x^2-1)(x^4-1)}$$

This equation has a 2-Darbouxian first integral

$$\mathcal{F}(x,y) = \Pi(x,-1,2)) - \arctan(y\sqrt{4x^4-5x^2+1})$$

The integrating factor U is

$$U = \frac{\sqrt{4x^4-5x^2+1}}{(4x^4-5x^2+1)y^2+1}$$

The integral rewrites $I(x,h) =$

$$\int_{\Pi(x,-1,2))-\arctan(y\sqrt{4x^4-5x^2+1})=h} \frac{(16x^8-40x^6+33x^4-10x^2+2)y^4+\dots+(8x^6-2x^4-8x^2+2)y+1}{(x^2+1)(4x^4y^2-5y^2x^2+y^2+1)y(4x^4-5x^2+1)}$$

The telescoper is

$$\partial_h^3 I + 4\partial_h I = \dots$$

Integration gives

$$e^{2i(\Pi(x,-1,2))-\arctan(y\sqrt{4x^4-5x^2+1}))}\int e^{-2i\Pi(x,-1,2))}dx+$$

$$e^{-2i(\Pi(x,-1,2))-\arctan(y\sqrt{4x^4-5x^2+1}))}\int e^{2i\Pi(x,-1,2))}dx+$$

$$\frac{1}{2}\ln(y^2(4x^4-5x^2+1))$$

Example 5: application to integration of a planar vector field

$$\dot{x} = -\frac{xy(y^2-2)}{xy^2-2x+4y}, \quad \dot{y} = -\frac{y(y^2-2)^2}{4xy^2-8x+16y}$$

We wish to find (if possible) a Liouvillian expression of the solutions.

$$\frac{\partial y}{\partial x} = \frac{(xy^2-2x+4y)(y^2-2)}{(4xy^2-8x+16y)x}$$

This equation admits a Darbouxian first integral

$$\mathcal{F}(x,y) = \frac{1}{4}\ln(x) - \frac{\sqrt{2}}{2}\operatorname{arctanh}\left(y\frac{\sqrt{2}}{2}\right)$$

with integrating factor

$$U = \frac{1}{2(y^2-1)}$$

To then perform the time integration, it is necessary to compute the integral

$$\int_{\mathcal{F}(x,y)=h} \frac{xy^2-2x+4y}{xy(y^2-2)}dx$$

A telescoper is found, giving an implicit expression for the solutions

$$\tfrac{1}{4}\ln(x) - \tfrac{\sqrt{2}}{2}\operatorname{arctanh}\left(y\tfrac{\sqrt{2}}{2}\right) = h$$

$$xe^{-2\sqrt{2}\operatorname{arctanh}\left(\frac{\sqrt{2}}{2}y\right)}\int e^{2\sqrt{2}\operatorname{arctanh}\left(\frac{\sqrt{2}}{2}y\right)}y^{-2}dy - \tfrac{25}{4}\ln x +$$

$$\tfrac{21\sqrt{2}}{2}\operatorname{arctanh}\left(\tfrac{\sqrt{2}}{2}y\right) + \frac{4xy^2-25y^3+16y^2-8x+50y}{4y(y^2-2)} = t+c$$