Functional Equations in LIMoges (FELIM) 2024

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Functional Equations in LIMoges (FELIM) 2024 is the fourteenth in a series of annual international gatherings for researchers in functional equations. This conference, held annually at the University of Limoges since 2008, aims to present recent advances in symbolic or symbolic-numeric algorithms which treat systems of linear or nonlinear, ordinary or partial, differential equations, (q-)difference equations,... Additionally, FELIM emphasizes on the development state of related software implementations and the publicity of such codes. Topics include, but are not limited to, ordinary differential equations, difference equations, partial differential equations; integration of dynamical systems; methods for local solving (formal, symbolic-numeric, modular); methods for global solving or simplification (e.g., decomposition, factorisation); applications and software applications.

The conference website with links to the previous editions can be found at:

`https://indico.math.cnrs.fr/event/11249/page/768-overview`

1 Invited talks abstracts

Loïc Teyssier (Université de Strasbourg, France)
Numeric-symbolic computation of the separatrix graph of a holomorphic polynomial differential equation.

`Monday, March 25, 11:00-12:00`

We consider real-time differential equations $\dot{z} = P(z)$ where P is a complex polynomial and $z \in \mathbb{C}$. The qualitative behavior of the solutions (the topological class of the corresponding phase portrait) is encoded by Smale's separatrix graph. This combinatorial data can be computed exactly, and we have a tractable bound on the complexity in terms of the degree of P and its closeness to a bifurcation. I'll show examples of usage of the actual Python library. The inverse problem (to find a polynomial having a specified class) is harder. The algebraic formulation can be solved using Gröbner bases, provided the universe last long enough. Neural networks can be trained to solve it, and it is expected that the method will show generalizing properties (i.e. the ability to answer correctly in cases of degrees higher than those in the training dataset). Applications range from dynamical systems (the main topic), computer graphics (representation of planar trees) to algebraic geometry (Grothendick's dessins d'enfants). I'll discuss at the end of the talk generalizations when P is a rational or an algebraic function. This is joint work with R. Schilling (Strasbourg).

Carlos Arreche (The University of Texas at Dallas, USA)
Galois groups of functional equations: theory, algorithms, and applications.

`Tuesday, March 26, 9:30-10:30`

I discuss recent theoretical and algorithmic developments concerning Galois groups of differential equations and difference equations, and a sample of applications to diverse areas such as combinatorics, number theory, and mathematical physics.

Thierry Combot (Université de Bourgogne, France)
Symbolic integration on differential foliations.

`Tuesday, March 26, 15:00-16:00`

Consider a non algebraic solution $y(x)$ of a differential equation $y' = F(x, y)$, and the integral $\int G(x, y(x))dx$, where F, G are rational. As y is transcendental, there is a Galois action on y which naturally defines a parametrized integral $I(x, h) = \int G(x, y(x, h))dx$. We will prove that I is either differentially transcendental, or satisfies a linear differential equation in h we call a telescoper. We will present an algorithm to look for a telescoper up to some bound. If a telescoper exists, the integral I is Liouvillian in both variables x, h, and can be written using single variable Liouvillian functions and rational functions. This generalizes the notion of elementary integration. For some specific foliations like $y = \ln x$

and $y = x^\alpha$, we will present a complete algorithm without a priori bound, and prove that with a telescoper, an expression using special functions always exists.

Clemens Raab (Johann Radon Institute for Computational and Applied Mathematics (RICAM), Austrian Academy of Sciences, Austria - visio-conference)
Reduction approach for polynomial solutions of Risch differential equations.

`Wednesday, March 27, 9:30-10:30`

Computing antiderivatives in differential fields, as done by the Risch algorithm and its variants, typically also requires solving inhomogeneous linear first-order ODEs in given differential fields (which are called Risch differential equations in this context). Specialized algorithms have been developed for them in certain types of differential fields. In 1990, Norman proposed a general approach for solutions of special form that is based on the completion of reduction systems. We present a refinement of his completion process that terminates in more instances. Even in some cases when the algorithm does not terminate, we can describe infinite reduction systems that are complete. Moreover, we discuss how such reduction systems give rise to adapted degree bounds for the solution. In fact, the techniques discussed can also be applied to find polynomial solutions of arbitrary inhomogeneous linear PDEs with polynomial coefficients. This is joint work with Hao Du.

2 Contributed talks abstracts

Sonia L. Rueda (Universidad Politécnica de Madrid, Spain)
Computing spectral curves for third order ODOs.

`Monday, March 25, 15:00-15:40`

Spectral curves are algebraic curves associated to commutative subalgebras of rings of ordinary differential operators (ODOs). Their origin is linked to the Korteweg-de Vries equation and to seminal works on commuting ODOs by I. Schur and, Burchnall and Chaundy. They allow the solvability of the spectral problem $Ly = \lambda y$, for an algebraic parameter λ and an algebro-geometric ODO L, whose centralizer is known to be the affine ring of a abstract spectral curve Γ.

In this talk, differential resultants we will used to effectively compute the defining ideal of the spectral curve Γ, defined by the centralizer of a third order differential operator L, with coefficients in an arbitrary differential field of zero characteristic. For this purpose, defining ideals of planar spectral curves associated to commuting pairs are described as radicals of differential elimination ideals. Our results establish a new framework appropriate to develop a Picard-Vessiot theory for spectral problems.

I am presenting joint work with Maria-Ángeles Zurro, as part of the project "Algorithmic Differential Algebra and Integrability" (ADAI), from the Spanish MICINN, PID2021-

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Camille Pinto (Sorbonne University - Inria Paris, France)
Effective characterization of evaluation ideals of the ring of integro-differential operators.

`Monday, March 25, 15:45-16:25`

In this talk, I will provide a step forward to the development of an algorithmic study of linear systems of polynomial ordinary integro-differential equations over a field $\mathbb{k}$ of characteristic zero. Such a study can be achieved by developing a constructive proof of the coherence property of the ring $\mathbb{I}_1(\mathbb{k})$ of linear ordinary integro-differential operators with coefficients in $\mathbb{k}[t]$. To do that, the finiteness of the intersection of two finitely generated ideals has to be algorithmically studied. Three cases have to considered: the case where the two ideals are defined by evaluations, the case where only one is defined by evaluations, and finally the case where none is generated by evaluations. In this talk, I will provide an explicit characterization of the intersection of two finitely generated ideals defined by evaluation operators. To handle the second case, a key result is the fact that the ideals generated by evaluations are semisimple $\mathbb{I}_1(\mathbb{k})$-modules. I will provide an algorithmic proof of this result. In particular, I will show how a finite set of generators, defined by "pure" evaluations, can be obtained, which characterizes the class of finitely generated evaluation ideals of $\mathbb{I}_1(\mathbb{k})$ as finitely generated $\mathbb{k}[t]$-modules. This is a joint work with Thomas Cluzeau and Alban Quadrat.

Thomas Dreyfus (CNRS, Université de Bourgogne, France)
Representability of G-functions as rational functions in hypergeometric series.

`Monday, March 25, 17:00-17:40`

In this talk we explain why a G function is not necessarily a polynomial in hypergeometric functions. This is a joint work with Tanguy Rivoal.

Lucas Legrand (Université de Limoges, XLIM, France)
Gröbner bases over polytopal affinoid algebras.

`Monday, March 25, 17:45-18:25`

In this talk, I will present a Gröbner bases theory for polytopal affinoid algebras. These are specific affinoid algebras connecting rigid analytic geometry and tropical geometry, as introduced by Einsiedler, Kapranov, and Lind. The theory extends the recent advancements by Caruso et al. on Gröbner bases for Tate algebras and incorporates the earlier work of Pauer et al. on Gröbner bases for Laurent polynomials. I will present effective algorithms for computing Gröbner bases in this context. In collaboration with Barkatou Moulay and Vaccon Tristan.

Adya Musson-Leymarie (Université de Limoges, XLIM, France)
Confluence for topological rewriting systems.

`Tuesday, March 26, 11:00-11:40`

Computations in multivariate formal power series make use of the concept of *standard bases* of ideals in an analogous manner as *Gröbner bases* are used for ideals in multivariate polynomials rings to algorithmically solve problems such as ideal membership, amongst other problems. Extending the classical rewriting theory to a topological setting, such as the complete metric space of formal power series, it has been shown that the property of being a standard basis is characterised by a certain confluence property of the rewriting relation. Using the fact that ideals of formal power series are topologically closed, we proved that this confluence property is equivalent to a generally stronger confluence property. This latter property is studied by computer scientists in the context of infinitary term rewriting and infinitary lambda-calculus. This result falls into the preliminary research towards the ultimate goal of applying topological rewriting theory to the study of formal solutions of partial differential equations. This is a joint work with Cyrille Chenavier and Thomas Cluzeau.

Florian Fürnsinn (University of Vienna, Austria)
Algebraicity of Hypergeometric Functions with Arbitrary Parameters.

`Tuesday, March 26, 16:30-17:10`

The classification of algebraic hypergeometric functions is a classical problem and early results go back to Schwarz in the late 19th century. In this talk I will give an overview on the history of the question and then present a complete criterion for algebraicity of (generalized) hypergeometric functions with no restriction on their set of parameters. It relies on the interlacing criteria of Christol (1986) and Beukers–Heckman (1989), which I will present as well, but allows arbitrary complex parameters with possibly integral differences. I will showcase the result on various examples. This talk is based on joint work with Sergey Yurkevich.

Daniel Augot (Inria Saclay-Île de France and LIX CNRS, France)
A differential equation for decoding derivative codes a.k.a Hermite interpolation with outliers.

`Tuesday, March 26, 17:15-17:55`

Decoding Reed-Solomon codes can be simply restated with polynomials as follows. Let (x_i, y_i), $i = 1, \ldots, n$, be given, and $0 \leq k, t \leq n$ auxiliary parameters, find all the polynomials $f(X)$ of degree less than k such that $f(x_i) = y_i$ for at least t values of i. This could be seen as a Lagrange interpolation problem with outliers.

For $t = \lfloor \frac{n+k}{2} \rfloor$ there is at most one solution, which can be found with many classical algorithms, and for $t = \lfloor \sqrt{n(k-1)} \rfloor$ there is as small number of solutions, which can be found with the Guruswami-Sudan algorithm.

Strangely enough, the situation is not solved when considering the same problem for Hermite interpolation with outliers. It actually boils down to solving a differential equation, but there is the issue that initial conditions are not well known. A quite small t can be reached, with a controlled number of solutions, provided that the problem is set over a finite field. There is no algorithm for the infinite field case.

This is embarrassing, since, for instance, the Guruswami-Sudan decoding algorithm is field agnostic. In this talk, the differential equations will be introduced and how the problem is solved over a finite field.

Over a generic, non finite, field, our contribution is that the problem can be treated, albeit for a restricted range of the parameters.

Joint work with Alain Couvreur, Emmanuel Hallouin, Thierry Henocq and Marc Perret.

Sergei Abramov (Federal Research Center "Computer Science and Control" of the Russian Academy of Sciences, Moscow, Russia)

On linear recurrent equations having infinite sequences in the role of coefficients.

`Wednesday, March 27, 11:00-11:40`

For a linear difference equation with the coefficients being computable sequences, we establish algorithmic undecidability of the problem of determining the dimension of the solution space including the case when some additional prior information on the dimension is available. This is a joint work with Gleb Pogudin (LIX, CNRS, Ecole polytechnique, Institute Polytechnique de Paris, France).