Polarization tensor approach for Casimir effect

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Outline

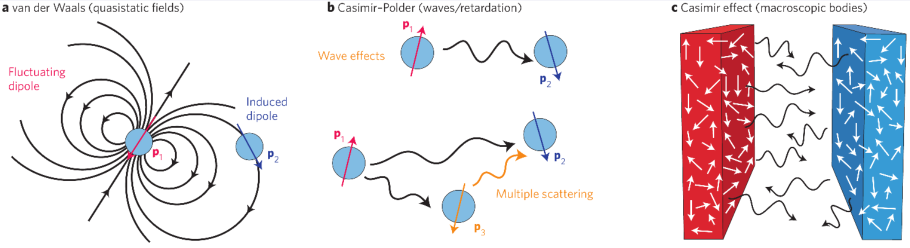
- The Casimir effect. Very short history
- Effective action
- Effective action for graphene
- The Casimir energy. Scattering approach
- Casimir-Polder force and torque for anisotropic molecules
- Stack of planes with tensorial conductivity
- Moving graphene

Based on the papers

1. NK, Vassilevich, arXiv:2402.06972, Sent in PRB (2024)
2. Antezza, Emelianova and NK, Nanotechnology, **35**, 235001, (2024)
3. NK, Emelianova, Physics, **6**, 148-163, (2024)
4. Emelianova, NK, and Kashapov, PRB, **107**, 235405, (2023)
5. NK, Emelianova, Universe, **7**, 70 (2021)
6. Bordag, Fialkovsky, NK, and Vassilevich, PRB, **104**, 195431, (2021)
7. Antezza, Fialkovsky, and NK, PRB, **102**, 195422 (2020)
8. Emelianova, Fialkovsky, and NK, Mod. Phys. Lett. A**35**, 2040012 (2020)
9. Rodriguez-Lopez, Popescu, Fialkovsky, NK, and Woods, Commun. Mater., **1**, 14 (2020)
10. NK, Emelianova, Int. J. Mod. Phys. A**34**, 1950008 (2019)
11. Fialkovsky, NK, and Vassilevich, PRB **97**, 165432 (2018)
12. NK, Kashapov and Woods, 2D Mater. **5**, 035032 (2018)

The Casimir and Casimir–Polder effects

Van der Waals, Casimir–Polder and Casimir forces[†]



a A fluctuating dipole $\mathbf{p}_1$ induces a fluctuating electromagnetic dipole field, which in turn induces a fluctuating dipole $\mathbf{p}_2$ on a nearby particle, leading to van der Waals forces between the particles

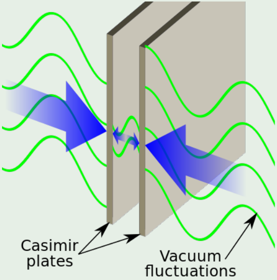
b When the particle spacing is large, retardation/wave effects modify the interaction, leading to Casimir–Polder forces. When more than two particles interact, the non-additive field interactions lead to a breakdown of the pairwise force laws

c In situations consisting of macroscopic bodies, the interaction between the many fluctuating dipoles present within the bodies leads to Casimir forces.

[†] Rodriguez, Capasso, and Johnson, Nature Photonics, **5**, 211 (2011)

The Casimir and Casimir–Polder effects

The Casimir force



The force is attractive because the mode density in free space is larger than that between the plates.

The Casimir effect

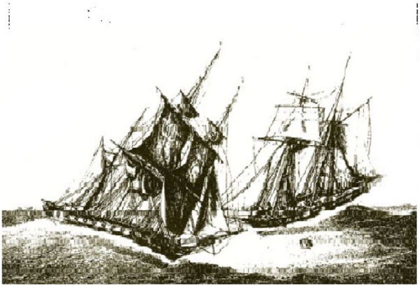


Fig. 1. Two ships roll heavily on a long swell and there is no more wind to damp their rolling. In this situation a strange force, "une certaine force attractive," will pull the two ships toward each other. From P. C. Caussée: "the Mariners Album," early 19th century.

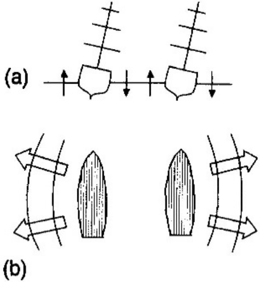


Fig. 3. (a) Two ships at close quarters roll on a long swell. (b) They re-emit the absorbed power as secondary waves.

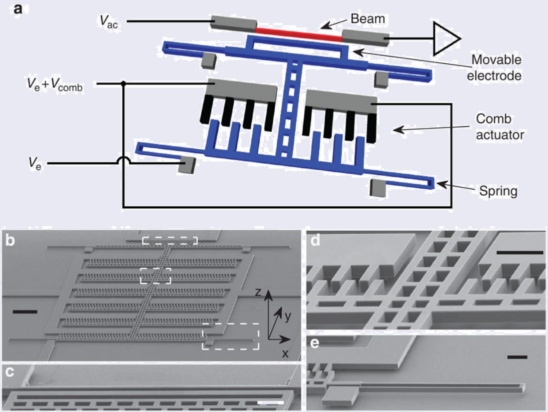
Nowadays, realized in white noise acoustic experiments†.

†A. Larraza B. Denardo, Phys. Lett. A, 248 (1998), 151-155

J.-C. Jaskula et al., Phys. Rev. Lett. 109 (2012) 220401

The Casimir and Casimir–Polder effects[†]

The Casimir force

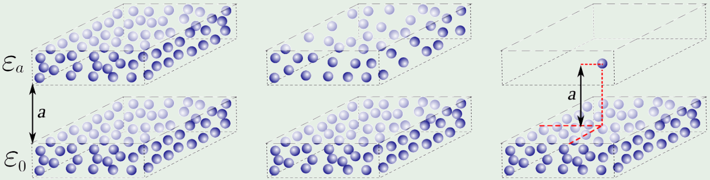


Scale bars, 50 μm (b) and 10 μm (c-e).

[†]Zou, J., Marcet, Z., Rodriguez, A. et al. Nature Commun **4**, 1845 (2013)

The Casimir and Casimir–Polder force

The Casimir–Polder force. The Lifshitz rarefying procedure[†]



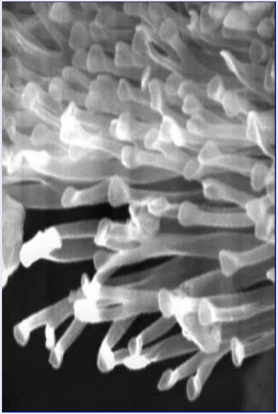
$$\varepsilon_a = 1 + 4\pi N\alpha,\ N \to 0$$

$$\mathcal{E}_{CP} = -\lim_{N\to 0} \frac{1}{N} \frac{\partial \mathcal{E}_C}{\partial a}$$

[†] E. M. Lifshitz, Sov. Phys. JETP **2**, 73 (1956)

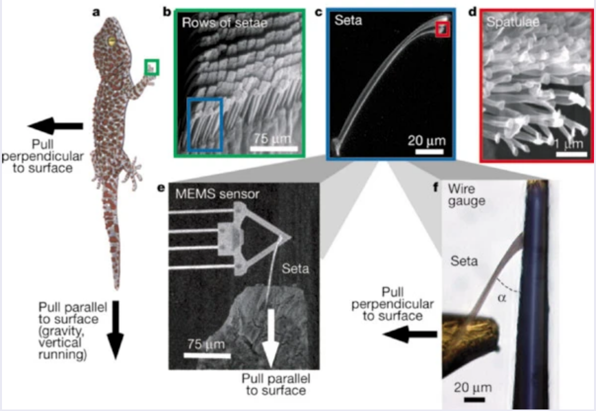
The Casimir and Casimir–Polder force

The Casimir force. Gecko vertical walk†



†Autumn, Liang, Hsieh, et al, Nature, **405**, 681 (2000)

The Casimir force. Gecko vertical walk†



The foot of a Tokay gecko has about 5.000 setae/mm^2 and can produce 10 N of adhesive force with approximately 100 mm^2 of pad area. Weight of a Tokay gecko is approx. 50 to 150 grams.

The Casimir and Casimir–Polder effects

Casimir force for perfectly conducting planes

$$P_{\text{Cas}} = -\frac{\hbar c \pi^2}{240 d^4} = -\frac{1.3\times 10^{-27}}{d^4}\text{Nm}^2 = -\frac{0.013}{d^4(\mu m)}\frac{\text{dyn}}{\text{cm}^2}$$

The Lifshitz approach

$$\mathcal{F}(a,T) = -\frac{k_B T}{2\pi}\sum_{l=0}^{\infty}{}' k_\perp \mathrm{d}k_\perp \left\{\ln\left[1 - r_{\text{I}}^{\text{tm}} r_{\text{II}}^{\text{tm}} e^{-2aq_l}\right] + \ln\left[1 - r_{\text{I}}^{\text{te}} r_{\text{II}}^{\text{te}} e^{-2aq_l}\right]\right\}$$

$$P = -\frac{\partial \mathcal{F}}{\partial a},\ ([\mathcal{F}(a,T)] = \left[\text{J}/\text{m}^2\right],\ [P] = \left[\text{J}/\text{m}^3\right])$$

$$q_l = \sqrt{k_\perp^2 + \xi_l^2}, \xi_l = 2\pi l \frac{k_B T}{\hbar} - \text{Matsubara frequences}$$

Effective action

Effective action expansion

$$S_{\text{eff}}[A,m] = N_{\text{f}} \ln\det \not{D}_A$$
$$= N_{\text{f}} \ln\det \not{D}_0 + N_{\text{f}} \operatorname{tr}\left(\not{D}_0^{-1} \not{A}\right) + \frac{1}{2} N_{\text{f}} \operatorname{tr}\left(\not{D}_0^{-1} \not{A} \not{D}_0^{-1} \not{A}\right) + \dots$$

$$\not{D}_A = \mathrm{i}\,(\not{\nabla} + \mathrm{i}e\not{A}) + m$$

N_{f} is the number of two-component fermion species. For graphene $N_{\text{f}} = N_A + N_B = 4$ – sum over sub-lattices A and B.

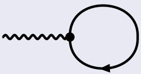
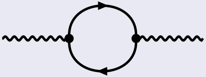
Effective action of the second order

The first term has to be subtracted, the second is tadpole and the third is effective action in the second order $(D = 3, 4)$

$$S^{(2)}_{\text{eff}}[A,m] = \frac{1}{2} \int \mathrm{d}^D x\, \mathrm{d}^D y\, A_i(x) \Pi^{ij}(x;y) A_j(y)$$

Effective action

Diagrams

$S_{\text{eff}}[A, m] =$  $+$  $+ \cdots$

Polarization tensor

$$\Pi^{ij}(p) = \mathrm{i}e^2 \int \frac{\mathrm{d}^D k}{(2\pi)^D} \operatorname{tr}\left(\gamma^i \not{D}_0^{-1}(k) \gamma^j \not{D}_0^{-1}(k-p)\right)$$

$$\not{D}_0^{-1}(k) = -\frac{(\gamma k) + m}{k^2 - m^2}$$

Effective action

Total action, $D = 3+1$ (with bulk)

$$S = S_{\mathrm{M}} + S^{(2)}_{\mathrm{eff}}[A,m],\ S_{\mathrm{M}} = -\frac{1}{4}\int \mathrm{d}^4 x F_{ij}F^{ij}$$

$$S^{(2)}_{\mathrm{eff}}[A,m] = \frac{1}{2}\int \mathrm{d}^4 x\, \mathrm{d}^4 y A_i(x)\Pi^{ij}(x;y)A_j(y)$$

Maxwell field in a media

$$S = \frac{1}{2}\int \frac{\mathrm{d}^4 p}{(2\pi)^4}\left[\varepsilon \vec{E}(-p)\cdot\vec{E}(p) - \mu^{-1}\vec{E}(-p)\cdot\vec{E}(p)\right]$$

Dielectric permittivity ε and magnetic permeability μ are expressed in terms of polarization tensor $-\Pi^{00}$ and $\mathrm{tr}(\Pi) = \Pi^i_i$. For Dirac materials (with bulk): Bordag, Fialkovsky, NK, and Vassilevich, PRB, **104**, 195431, (2021)

Effective action

Total action, $D = 2+1$. Plane $x^3 = const$

$$S_M = -\frac{1}{4}\int \mathrm{d}^4 x F_{ij} F^{ij}$$

$$S^{(2)}_{\text{eff}}[A,m] = \frac{1}{2}\int \mathrm{d}^3 x\, \mathrm{d}^3 y A_i(x) \Pi^{ij}(x-y) A_j(y)$$

$$\frac{\delta S}{\delta A} = 0 \Rightarrow \partial_j F^{ij} + \delta(x^3 - a)\int \mathrm{d}^3 y \Pi^{ij}(x-y) A_j(y) = 0$$

Boundary conditions $x^3 = a$, $\vec{n} = (0,0,1)$

$$[\vec{H}] \times \vec{n} = \vec{J},\ [\vec{E}] \cdot \vec{n} = -\rho,\ J^a = \sigma^{ab} E_b$$

$$\sigma^{ab} = \frac{\Pi^{ab}}{\mathrm{i}\omega},\ \rho = \frac{\Pi^{0a} E_a}{\mathrm{i}\omega}.$$

Effective action, graphene

Dirac electron $D = 2+1$ ($m < 0.1$ eV)

$$\not{D}_A = \mathrm{i}\,(\not{\nabla} + \mathrm{i}e\not{A}) + m = \mathrm{i}\tilde{\gamma}^i\,(\partial_i + \mathrm{i}eA_i) + m$$

$$\tilde{\gamma}^0 = \gamma^0,\ \tilde{\gamma}^1 = v_F\gamma^1,\ \tilde{\gamma}^2 = v_F\gamma^2.$$

Algebra of gamma matrices ($c \to v_F \sim c/300$)

$$\tilde{\gamma}^i\tilde{\gamma}^j + \tilde{\gamma}^j\tilde{\gamma}^i = 2\tilde{\mathrm{g}}^{ij}$$

$$\mathrm{tr}(\tilde{\gamma}^i\tilde{\gamma}^j) = 8\tilde{\mathrm{g}}^{ij},\ \mathrm{tr}(\tilde{\gamma}^i\tilde{\gamma}^j\tilde{\gamma}^l) = 0$$

$$\tilde{\mathrm{g}}^{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -v_F^2 & 0 \\ 0 & 0 & -v_F^2 \end{pmatrix}$$

Effective action, graphene

The Lorentz invariance is broken

$$\mathcal{A} = -\frac{1}{4}\int \mathrm{d}^4 x F^2 + \int \mathrm{d}^3 x \bar{\psi}\left[\tilde{\gamma}^n(\mathrm{i}\partial_n - eA_n) - m\right]\psi|_{z=a}$$

$$\mathcal{A}' = -\frac{1}{4}\int \mathrm{d}^4 x' F'^2 + \int \mathrm{d}^3 x' \bar{\psi}'\left[\tilde{\Gamma}^n(\mathrm{i}\partial_{n'} - eA'_n) - m\right]\psi'|_{z=a}$$

Lorentz boost

$$\tilde{\Gamma}^{n'} = v_F\gamma^n + u^n(1 - v_F)\not{u} \neq \tilde{\gamma}^n,\ [u^n] = (u^0, u^1, u^2)$$

Boost $K \to K'$: $x' = \Lambda x$, $\psi'(x') = S\psi(x)$, $\bar{\psi}'(x') = \bar{\psi}(x)S^{-1}$

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\boldsymbol{v} \\ -\gamma\boldsymbol{v} & \mathbf{I} + (\gamma - 1)\frac{\boldsymbol{v}\otimes\boldsymbol{v}}{\boldsymbol{v}^2} \end{pmatrix},\ S = e^{-\frac{\mathrm{i}}{4}\sigma^{ik}\lambda_{ik}},$$

$$\sigma^{ik} = \frac{\mathrm{i}}{2}[\gamma^i, \gamma^k],\ [\lambda_{ik}] = \begin{pmatrix} 0 & -\boldsymbol{v} \\ \boldsymbol{v} & 0 \end{pmatrix},\ \boldsymbol{v} = (v^1, v^2).$$

Effective action, graphene

Polarization tensor†, $T=0$, $\mu=0$, $\vec{B}=\vec{E}=0$

$$\Pi^{mn} = \frac{\alpha}{v_F^2}\left[\psi\left(\tilde{g}^{mn} - \frac{p^m p^n}{p^2}\right) + \mathrm{i} v_F^2 \phi \varepsilon^{mnl} p_l\right]$$

$$\psi = (N_A + N_B)\left[1 - \left(\frac{p}{2m} + \frac{2m}{p}\right)\operatorname{arcth}\left(\frac{p}{2m}\right)\right]$$

$$\phi = (N_A - N_B)\left[\frac{2m}{p}\operatorname{arcth}\left(\frac{p}{2m}\right) - 1\right]$$

ϕ is parity anomaly‡: $j^i = \frac{e}{8\pi}\varepsilon^{ijk}F_{jk}\,\mathrm{sgn}(m)$. $\phi = 0$ for graphene♯.

† Bordag, Fialkovsky, Gitman, and Vassilevich, PRB, **80**, 245406 (2009)

‡ Redlich, PRD, **29**, 2366 (1984)

♯ Semenoff, PRL, **53**, 2449 (1984)

Conductivity tensor

$$\sigma^{ab} = \frac{\Pi^{ab}}{\mathrm{i}\omega} \Longrightarrow \boldsymbol{\sigma} = \begin{pmatrix} \sigma_{\mathrm{tm}} & 0 \\ 0 & \sigma_{\mathrm{te}} \end{pmatrix}$$

$$\sigma_{\mathrm{tm}} = \frac{\mathrm{i}\omega}{\boldsymbol{p}^2}\Pi_{00},\ \sigma_{\mathrm{te}} = \frac{1}{\mathrm{i}\omega}\left(\Pi_n^n + \frac{\omega^2 - \boldsymbol{p}^2}{\boldsymbol{p}^2}\Pi_{00}\right)$$

Reflection coefficients

We can not use standard formulas which connect reflection coefficients with dielectric properties in the Lifshitz formula. Reasons:

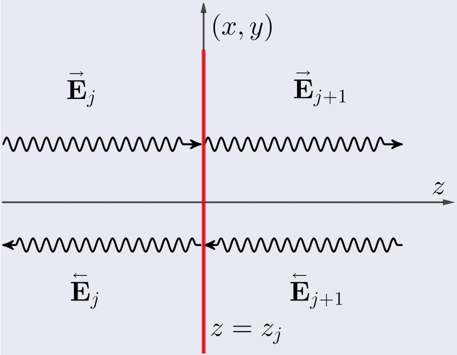
- Graphene is $2D$ material
- Reflection coefficients $\Longrightarrow$ reflection matrices

One way is the scattering matrix approach†.

† Reynaud, Canaguier-Durand, Messina, Lambrecht, and Maia Neto, Int. J. Mod. Phys. A **25**, 2201(2010)

The Casimir energy

Scattering matrix approach



$$\begin{pmatrix} \overleftarrow{\mathbf{E}}_j \\ \vec{\mathbf{E}}_{j+1} \end{pmatrix} = \mathcal{S}_j \begin{pmatrix} \vec{\mathbf{E}}_j \\ \overleftarrow{\mathbf{E}}_{j+1} \end{pmatrix}, \ \mathcal{S}_j = \begin{pmatrix} \boldsymbol{r}_j & \boldsymbol{t}'_j \\ \boldsymbol{t}_j & \boldsymbol{r}'_j \end{pmatrix}$$

Transmission and reflection of the EM waves.

The Casimir energy

The Casimir energy[†] ($k_E \equiv \sqrt{\xi^2 + k_\perp^2}, \omega = \mathrm{i}\xi$)

$$\mathcal{E} = -\frac{1}{4\pi \mathrm{i}} \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \int_0^\infty \ln\det \mathcal{S}(k_1, k_2, k_3) \frac{k_3 \mathrm{d}k_3}{\sqrt{\boldsymbol{k}^2 + k_3^2}}$$

$\omega = \mathrm{i}\xi$

$$\mathcal{E} = \frac{1}{4\pi} \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \int_{-\infty}^\infty \mathrm{d}\xi \ln\det\left(1 - e^{-2ak_E} \boldsymbol{r}'_{\mathrm{I}}(\mathrm{i}k_E)\boldsymbol{r}_{\mathrm{II}}(\mathrm{i}k_E)\right)$$

In general

$$\boldsymbol{r}'_{\mathrm{I}}\boldsymbol{r}_{\mathrm{II}} \neq \begin{pmatrix} r_{\mathrm{I}}^{\mathrm{te}} r_{\mathrm{II}}^{\mathrm{te}} & 0 \\ 0 & r_{\mathrm{I}}^{\mathrm{tm}} r_{\mathrm{II}}^{\mathrm{tm}} \end{pmatrix} \Longrightarrow \mathcal{E} \neq \mathcal{E}_{\mathrm{te}} + \mathcal{E}_{\mathrm{tm}}.$$

† Fialkovsky, NK, and Vassilevich, PRB **97**, 165432 (2018))

The Casimir energy

Conductive plane[†] $(\boldsymbol{\eta} = 2\pi\boldsymbol{\sigma}, k_3 = \sqrt{\omega^2 - \boldsymbol{k}^2})$

$$\boldsymbol{r} = \boldsymbol{r}' = -\frac{\omega^2\boldsymbol{\eta} - \boldsymbol{k}\otimes\boldsymbol{k}\boldsymbol{\eta} + \mathbf{I}\omega k_3 \det\boldsymbol{\eta}}{\omega^2 \operatorname{tr}\boldsymbol{\eta} - \boldsymbol{k}\boldsymbol{k}\boldsymbol{\eta} + \omega k_3(1+\det\boldsymbol{\eta})}, \ \boldsymbol{t} = \mathbf{I} + \boldsymbol{r}.$$

For veritable graphene with $\vec{B} = T = \mu = \Gamma = 0$

$$\boldsymbol{\sigma} = A\mathbf{I} + B\boldsymbol{k}\otimes\boldsymbol{k}, \ \sigma_{\rm te} = A, \ \sigma_{\rm tm} = A + \boldsymbol{k}^2 B.$$

[†] Fialkovsky, NK, and Vassilevich, PRB **97**, 165432 (2018))

The Casimir energy

$$\Pi^{ij}(p) = \mathrm{i}e^2 \int \frac{\mathrm{d}^D k}{(2\pi)^D} \operatorname{tr}\left(\gamma^i \not{D}^{-1}(k)\gamma^j \not{D}^{-1}(k-p)\right)$$

Extensions

- Non-zero temperature[†] T (Matsubara frequencies):

$$\int \mathrm{d}k_0 f(k_0) \Rightarrow 2\pi\mathrm{i}T \sum_{k=-\infty}^{\infty} f(2\pi\mathrm{i}T(k+1/2)),\ p_0 \to 2\pi\mathrm{i}nT$$

- Chemical potential[†] μ: $\partial_0 \to \partial_0 - \mathrm{i}\mu$, $k_0 \to k_0 + \mu$
- Impurities[‡] Γ: $k_0 + \mathrm{i}\Gamma \operatorname{sgn} k_0$

[†] Fialkovsky, Marachevsky, Vassilevich, PRB **84**, 035446 (2011))

[‡] Fialkovsky, Vassilevich, EPJ **B85**, 384 (2012); NK and Vassilevich, arXiv:2402.06972, submitted in PRB

The Casimir energy

$$\Pi^{ij}(p) = \mathrm{i}e^2 \int \frac{\mathrm{d}^D k}{(2\pi)^D} \mathrm{tr}\left(\gamma^i D\!\!\!/^{-1}(k)\gamma^j D\!\!\!/^{-1}(k-p)\right)$$

Extensions

- External EMF F_{ik}: $\partial_k \to \partial_k + \mathrm{i}eA_k$ (Faraday rotation in graphene[♮])
- Strained graphene[†] $v_F \to v_F^{ab} = v_F\left[\delta^{ab} - \frac{\beta}{4}\left(2u^{ab} + \delta^{ab}u_c^c\right)\right]$
- Casimir effect for doped graphene[‡]

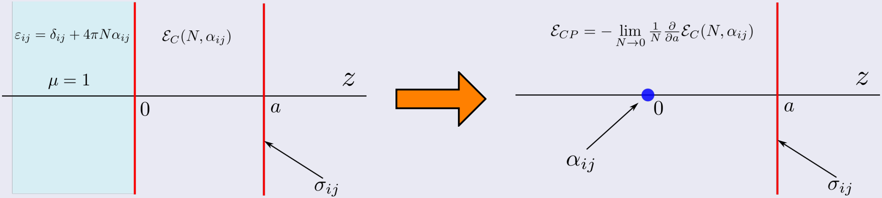
♮ Fialkovsky, and Vassilevich, EPJB, **85**, 384, (2012)

† Bordag, Fialkovsky and Vassilevich, PLA **381**, 2439 (2017)

‡ Bordag, Fialkovsky and Vassilevich, PRB **93**, 075414(2016)

Casimir–Polder force for anisotropic molecules[†]

The basic idea



$$\mathcal{E}_{\rm CP} = \iint \frac{d^2 k}{4\pi^2} \int_0^\infty \frac{d\xi}{\sqrt{\xi^2 + \boldsymbol{k}^2}} e^{-2a\sqrt{\xi^2+\boldsymbol{k}^2}} r^{ab} \left[\xi^2 \alpha_{ba} + \alpha_{ca} k^c k_b + \alpha_{33} k_a k_b\right]$$

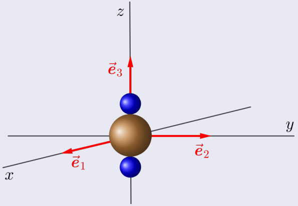
$\alpha_{ab} = \alpha_{ab}(\mathrm{i}\xi)$ and r^{ab} is the reflection matrix of the plane.

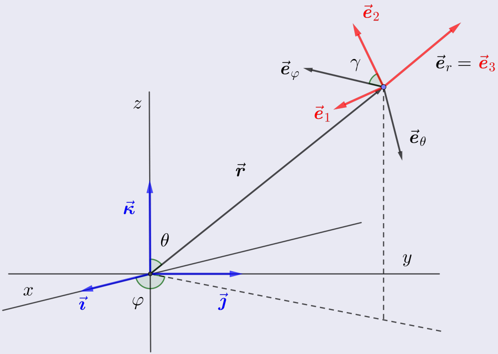
[†]Antezza, Fialkovsky, NK, PRB **102**, 195422 (2020))

Casimir–Polder force for anisotropic molecules

Orientation of molecule

The orientation of molecule is described by 3 angles θ, φ, γ. The vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3$ are the eigenvectors of polarization tensor α_{mn}. The conductive plane is (x, y)-plane.





Casimir–Polder force for anisotropic molecules

The general form of tensor conductivity

$$\boldsymbol{\sigma} = X\mathbf{I} + Y\frac{\boldsymbol{k}\otimes\boldsymbol{k}}{\boldsymbol{k}^2} + Z\boldsymbol{\epsilon} \qquad (Z \text{ is the Hall conductivity})$$

Casimir–Polder energy. $\alpha_{ij}(\omega) = \alpha_{ij}(\mathrm{i}\xi)$

$$\boxed{\mathcal{E}_{\mathrm{CP}} = \mathcal{E}^0_{\mathrm{CP}} + \Delta\mathcal{E}_{\mathrm{CP}}(\theta,\varphi,\gamma)} \quad \Delta\mathcal{E}_{\mathrm{CP}} = -\sin^2\theta(\mathcal{G}_{12}\cos^2\gamma + \mathcal{G}_{23})^{\dagger}$$

$$\mathcal{E}^0_{\mathrm{CP}} = \iint_0^\infty \mathrm{d}k\mathrm{d}\xi\, e^{-2a\sqrt{\xi^2+\boldsymbol{k}^2}}((\alpha_{11}+\alpha_{22})A(k,\xi,\boldsymbol{\sigma}) + \alpha_{33}B(k,\xi,\boldsymbol{\sigma}))$$

$$\mathcal{G}_{ij} = \iint_0^\infty \mathrm{d}k\mathrm{d}\xi\, e^{-2a\sqrt{\xi^2+\boldsymbol{k}^2}}(\alpha_{ii}-\alpha_{jj})G(k,\xi,\boldsymbol{\sigma})$$

The angle γ gives contribution in the case $\alpha_{11} \neq \alpha_{22}$, only.

† $\sin^2\theta$ was observed in Thiyam, Parashar, et al, PRA **92**, 052704 (2015)

Casimir–Polder force for anisotropic molecules

Perfect metal: $\boldsymbol{\sigma} \to \infty$

$$\mathcal{E}_{\rm CP}^{\rm id} = \mathcal{E}_{\rm CP}^{0} + \Delta\mathcal{E}_{\rm CP}\Big|_{\boldsymbol{\sigma}\to\infty}$$

$$\mathcal{E}_{\rm CP}^{0} = -\frac{1}{32\pi a^4}\int_0^\infty {\rm d}z e^{-z}\left\{{\rm tr}\,\alpha(1+z+z^2) + \alpha_{33}(1+z-z^2)\right\}$$

$$\Delta\mathcal{E}_{\rm CP} = -\frac{\sin^2\theta}{32\pi a^4}\int_0^\infty {\rm d}z e^{-z}(\alpha_{22}-\alpha_{33})(1+z-z^2)$$

$$\mathcal{E}_{\rm CP}^{\rm id}\Big|_{a\to\infty} = \mathcal{E}_{\rm CP}^{\infty} = -\frac{{\rm tr}\,\alpha(0)}{8\pi a^4},\ (\alpha(0) \to {\rm tr}\,\alpha(0)/3)$$

$$\alpha_{ij}(\omega) = \alpha_{ij}({\rm i}z/2a)$$

Casimir torque for anisotropic molecules

Casimir torque

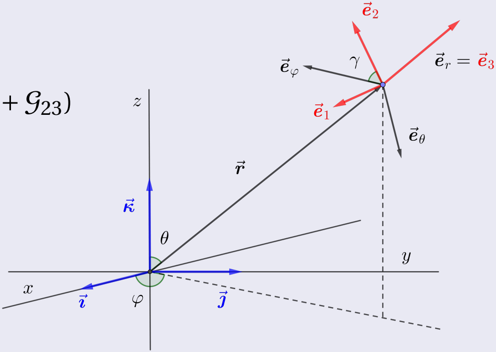
$$\mathcal{E}_{\rm CP} = \mathcal{E}^0_{\rm CP} + \Delta\mathcal{E}_{\rm CP}(\theta,\varphi,\gamma)$$

$$M_\theta = -\partial_\theta \mathcal{E}_{\rm CP} = \sin 2\theta(\mathcal{G}_{12}\cos^2\gamma + \mathcal{G}_{23})$$

$$M_\varphi = -\partial_\varphi \mathcal{E}_{\rm CP} = 0$$

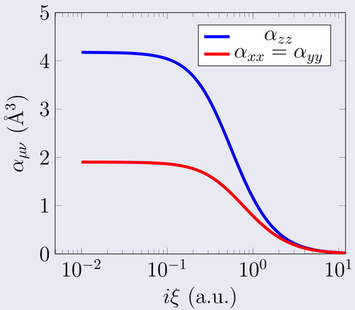
$$M_\gamma = -\partial_\gamma \mathcal{E}_{\rm CP} = \sin^2\theta \sin 2\gamma \mathcal{G}_{12}$$

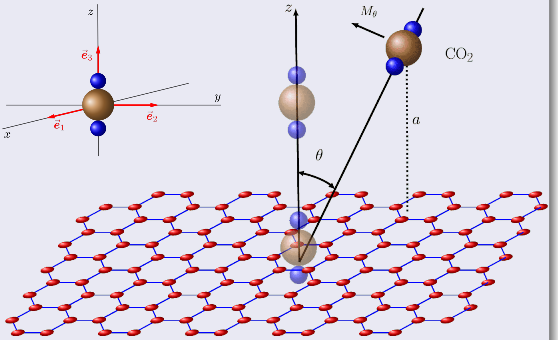
Zero torque with minima energy $\mathcal{E}_{\rm CP}$ describes the equilibrium states.



CO_2: Graphene

Casimir torque for CO_2





$$\mathcal{E}_{CP} = \mathcal{E}^0_{CP} - \sin^2\theta\,\mathcal{G}_{23}, M_\theta = \sin 2\theta\,\mathcal{G}_{23},\ M_\varphi = M_\gamma = 0$$

$\boldsymbol{M} = 0 \Leftrightarrow$ extrema of energy $\mathcal{E}^e_{CP} = (\mathcal{E}^0_{CP}, \mathcal{E}^0_{CP} - \mathcal{G}_{23}, \mathcal{E}^0_{CP})$ for $\theta_e = \left(0, \frac{\pi}{2}, \pi\right)$.
$\theta = 0, \pi \Leftrightarrow CO_2 \perp$ to plane; $\theta = \pi/2 \Leftrightarrow CO_2 \parallel$ to plane.

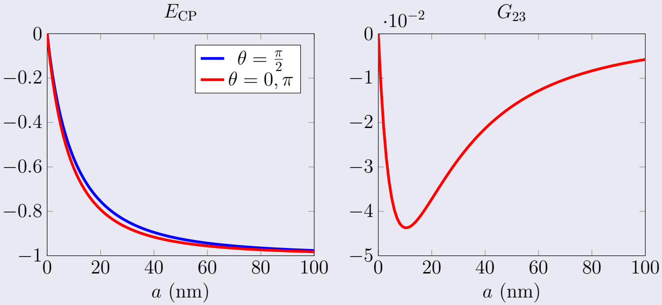
The Casimir torque M_θ tends to rotate molecule to direction axis z and as the result molecules can go through the membrane.

CO_2: Plane with constant Hall conductivity: $\boldsymbol{\sigma} = Z\boldsymbol{\epsilon}$

$$\mathcal{E}_{\mathrm{CP}} = \mathcal{E}^0_{\mathrm{CP}} - \sin^2\theta\,\mathcal{G}_{23} = \frac{Z^2}{1+Z^2}\mathcal{E}^{\mathrm{id}}_{\mathrm{CP}}$$

Constant Hall conductivity: $\boldsymbol{\sigma} = Z\boldsymbol{\epsilon}$

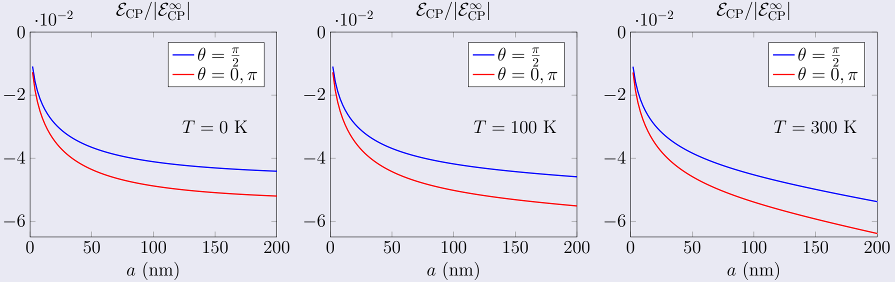
Position of CO_2 $\perp$ to plane is preferable



$$E_{\mathrm{CP}} = \frac{1+Z^2}{Z^2}\frac{\mathcal{E}_{\mathrm{CP}}}{|\mathcal{E}^{\infty}_{\mathrm{CP}}|},\ G_{\mathrm{CP}} = \frac{1+Z^2}{Z^2}\frac{\mathcal{G}_{23}}{|\mathcal{E}^{\infty}_{\mathrm{CP}}|}, \mathcal{E}^{\infty}_{\mathrm{CP}} = -\frac{\mathrm{tr}\,\alpha(0)}{8\pi a^4}\ \left(= \mathcal{E}^{\mathrm{id}}_{\mathrm{CP}}\Big|_{a\to\infty}\right)$$

CO_2: Graphene

Casimir torque for CO_2

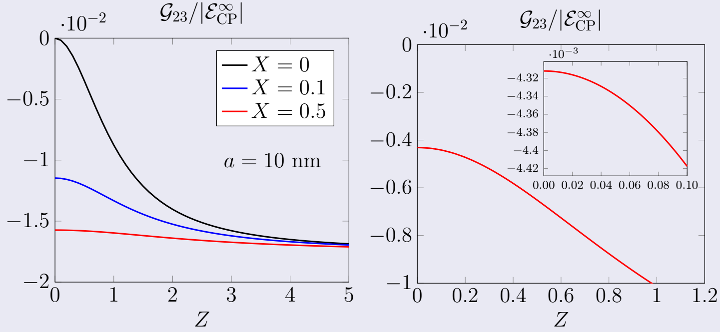


The Casimir torque M_θ tends to rotate molecule to direction axis z and as the result molecules can go through the membrane

CO_2: Graphene

$$M_\theta = \sin(2\theta)\mathcal{G}_{23}, \quad \boldsymbol{\sigma} = X\mathbf{I} + Y\frac{\boldsymbol{k}\otimes\boldsymbol{k}}{\boldsymbol{k}^2} + Z\boldsymbol{\epsilon}$$

Casimir torque for CO_2. The influence of the Hall conductivity

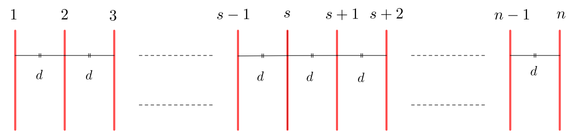


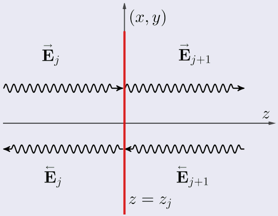
Left panel: A plane with constant dc conductivity ($Y = 0$). Right panel: Graphene.

The torque is shown to be enhanced by enlarging the Hall conductivity contribution of the conductivity tensor of the membrane

Stack of planes with tensorial conductivity†

Equidistant stack of planes





Scattering problem

$$\begin{pmatrix} \overleftarrow{\mathbf{E}}_1 \\ \vec{\mathbf{E}}_{n+1} \end{pmatrix} = \mathcal{S} \begin{pmatrix} \vec{\mathbf{E}}_1 \\ \overleftarrow{\mathbf{E}}_{n+1} \end{pmatrix}, \begin{pmatrix} \overleftarrow{\mathbf{E}}_j \\ \vec{\mathbf{E}}_{j+1} \end{pmatrix} = \mathcal{S}_j \begin{pmatrix} \vec{\mathbf{E}}_j \\ \overleftarrow{\mathbf{E}}_{j+1} \end{pmatrix}, \mathcal{S} \neq \prod_j \mathcal{S}_j$$

† Emelianova, NK, and Kashapov, PRD, **107**, 235405, (2023)

Stack of planes with tensorial conductivity

Obtaining $\mathcal{S}$-matrix

One represents the set of connected scattering equations in the form

$$\begin{bmatrix} \mathbf{A}_1 & \mathbf{C}_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \mathbf{A}_2 & \mathbf{C}_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \mathbf{A}_3 & \mathbf{C}_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \mathbf{A}_{n+1} & \mathbf{C}_{n+1} \end{bmatrix} \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \\ \vdots \\ \mathbf{E}_{n+1} \end{bmatrix} = 0$$

$$\mathbf{A}_i = \begin{pmatrix} \boldsymbol{r}_i & -\mathbf{I} \\ \boldsymbol{t}_i & 0 \end{pmatrix}, \mathbf{C}_i = \begin{pmatrix} 0 & \boldsymbol{t}'_i \\ -\mathbf{I} & \boldsymbol{r}'_i \end{pmatrix}, \mathbf{E}_i = \begin{pmatrix} \vec{\mathbf{E}}_i \\ \overleftarrow{\mathbf{E}}_i \end{pmatrix}.$$

Stack of planes with tensorial conductivity

$$\text{total stack} \to \mathcal{S} = \begin{pmatrix} \mathbf{R} & \mathbf{T}' \\ \mathbf{T} & \mathbf{R}' \end{pmatrix}, \ \mathcal{S}_j = \begin{pmatrix} \boldsymbol{r}_j & \boldsymbol{t}'_j \\ \boldsymbol{t}_j & \boldsymbol{r}'_j \end{pmatrix} \leftarrow \text{specific plane}$$

$\mathcal{S}$-matrix for stack of n planes

$$\mathcal{S} = \begin{pmatrix} -\mathbf{K}_{22}^{-1}\mathbf{K}_{21} & (-1)^{n+1}\mathbf{K}_{22}^{-1} \\ (-1)^{n+1}\left(\mathbf{K}_{11} - \mathbf{K}_{12}\mathbf{K}_{22}^{-1}\mathbf{K}_{21}\right) & \mathbf{K}_{12}\mathbf{K}_{22}^{-1} \end{pmatrix}, \ \mathbf{K} = \begin{pmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} \\ \mathbf{K}_{21} & \mathbf{K}_{22} \end{pmatrix}$$

$$\text{recurrent relations} \to \mathbf{K}_{n+1} = \mathbf{B}_{n+1}\mathbf{K}_n, \ \mathbf{B}_i = \mathbf{C}_i^{-1}\mathbf{A}_i$$

The Casimir energy

$$\mathcal{E}_n = \Re \iint \frac{\mathrm{d}^2 k}{(2\pi)^3} \int_0^\infty \mathrm{d}\xi \ln\det \boldsymbol{D}_n$$

$$\boldsymbol{D}_n = (-1)^n \boldsymbol{t}'_n \mathbf{K}_{22} \boldsymbol{t}'_1 \boldsymbol{t}'_2 \cdots \boldsymbol{t}'_{n-1} = \mathbf{I} + \ldots, \ \boldsymbol{D}_1 = \boldsymbol{D}_0 = \mathbf{I}$$

Stack of planes with tensorial conductivity

Solving the recurrent relations

$$\mathbf{K}_{n+1} = \mathbf{B}_{n+1}\mathbf{K}_n \Rightarrow \boldsymbol{D}_{n+2} = \boldsymbol{u}\boldsymbol{D}_{n+1} + \boldsymbol{v}\boldsymbol{D}_n$$

$$\boldsymbol{u} = \mathbf{I} + e^{-2k_E d}(\mathbf{I} + 2\boldsymbol{r}), \boldsymbol{v} = -e^{-2k_E d}(\mathbf{I} + \boldsymbol{r})^2$$

To solve the recurrent relations one uses a generation function method extended for matrix-valued coefficients

$$\mathbf{G} = \sum_{s=0}^{\infty} \boldsymbol{D}_s z^s = \boldsymbol{D}_0 + (\boldsymbol{D}_1 - \boldsymbol{u}\boldsymbol{D}_0)z + \boldsymbol{u}\mathbf{G}z + \boldsymbol{v}\mathbf{G}z^2$$

$$\Downarrow$$

$$\mathbf{G} = \frac{\boldsymbol{D}_0 + (\boldsymbol{D}_1 - \boldsymbol{u}\boldsymbol{D}_0)z}{\mathbf{I} - \boldsymbol{u}z - \boldsymbol{v}z^2} = (\boldsymbol{D}_0 + (\boldsymbol{D}_1 - \boldsymbol{u}\boldsymbol{D}_0)z)\sum_{s=0}^{\infty} \mathbf{M}_s z^s = \sum_{s=0}^{\infty} z^s \boldsymbol{D}_s$$

Stack of planes with tensorial conductivity

Solving the recurrent relation

$$\boldsymbol{D}_n = \mathbf{M}_n - \mathbf{M}_{n-1} e^{-2k_E d} (\mathbf{I} + 2\boldsymbol{r})$$

$$\mathbf{M}_s = \sum_{l=0}^{\left[\frac{s}{2}\right]} \boldsymbol{u}^{s-2l} \boldsymbol{v}^l C_l^{s-l}.$$

The matrix $\boldsymbol{r}$ maybe diagonalized with eigenvalues $r_{\rm te}$ and $r_{\rm tm}$.

$$\boldsymbol{r} = \begin{pmatrix} r_{\rm te} & 0 \\ 0 & r_{\rm tm} \end{pmatrix} \Rightarrow \boldsymbol{D}_n = \begin{pmatrix} D_n^{\rm te} & 0 \\ 0 & D_n^{\rm tm} \end{pmatrix}$$

$$\Downarrow$$

$$\mathcal{E}_n = \mathcal{E}_n^{\rm te} + \mathcal{E}_n^{\rm tm} = \iint \frac{\mathrm{d}^2 k}{(2\pi)^3} \int_0^\infty \mathrm{d}\xi \left(\ln D_n^{\rm te} + \ln D_n^{\rm tm} \right),$$

Stack of planes with tensorial conductivity

The energy: Solving the recurrent relation

$$D_n^{\mathrm{x}} = \frac{e^{-(n-1)(k_E d)} f_{\mathrm{x}}^{2-n}}{(1+t_{\mathrm{x}})^n} \left[\frac{1+t_{\mathrm{x}}}{f_{\mathrm{x}}} \frac{1-f_{\mathrm{x}}^{2n}}{1-f_{\mathrm{x}}^2} - e^{-z} \frac{1-f_{\mathrm{x}}^{2(n-1)}}{1-f_{\mathrm{x}}^2} \right]^{\dagger}, \quad t_{\mathrm{x}} = -\frac{r_{\mathrm{x}}}{1+r_{\mathrm{x}}}$$

$$f_{\mathrm{x}} = \sqrt{(\cosh(k_E d) + t_{\mathrm{x}} \sinh(k_E d))^2 - 1} + \cosh(k_E d) + t_{\mathrm{x}} \sinh(k_E d), \mathrm{x} = \mathrm{te}, \mathrm{tm}$$

The matrix $\boldsymbol{r}$ maybe diagonalized with eigenvalues r_{te} and r_{tm}.

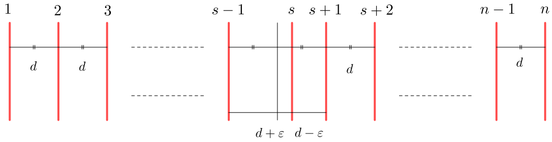
$$\boldsymbol{r} = \begin{pmatrix} r_{\mathrm{te}} & 0 \\ 0 & r_{\mathrm{tm}} \end{pmatrix} \Rightarrow \boldsymbol{D}_n = \begin{pmatrix} D_n^{\mathrm{te}} & 0 \\ 0 & D_n^{\mathrm{tm}} \end{pmatrix}$$

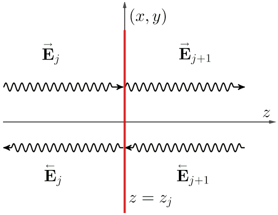
$$\Downarrow$$

$$\mathcal{E}_n = \mathcal{E}_n^{\mathrm{te}} + \mathcal{E}_n^{\mathrm{tm}} = \iint \frac{\mathrm{d}^2 k}{(2\pi)^3} \int_0^\infty \mathrm{d}\xi \left(\ln D_n^{\mathrm{te}} + \ln D_n^{\mathrm{tm}} \right),$$

† NK, Kashapov and Woods, PRD **92**, 045002 (2015)

Stack of planes with tensorial conductivity





The force: Solving the recurrent relation

$$F_{s,n} = -\lim_{\varepsilon\to 0}\frac{E_n(\varepsilon) - E_n(0)}{\varepsilon} = -\Re \sum_{\mathrm{x}=\mathrm{te,tm}} \iint \frac{\mathrm{d}^2 k}{(2\pi)^3}\int_0^\infty \mathrm{d}\xi\, D_n^{\mathrm{x}\,-1} D_n^{\mathrm{x}\,\prime}.$$

$$D_n^{\mathrm{x}} = \frac{e^{-(n-1)(k_E d)} f_{\mathrm{x}}^{2-n}}{(1+t_{\mathrm{x}})^n}\left[\frac{1+t_{\mathrm{x}}}{f_{\mathrm{x}}}\frac{1-f_{\mathrm{x}}^{2n}}{1-f_{\mathrm{x}}^2} - e^{-z}\frac{1-f_{\mathrm{x}}^{2(n-1)}}{1-f_{\mathrm{x}}^2}\right],\ \mathrm{x} = \mathrm{te,tm}$$

$$f_{\mathrm{x}} = \sqrt{(\cosh(k_E d) + t_{\mathrm{x}}\sinh(k_E d))^2 - 1} + \cosh(k_E d) + t_{\mathrm{x}}\sinh(k_E d)$$

Stack of planes with tensorial conductivity

$\lim_{n\to\infty} \mathcal{E}_n \to \infty$. The energy for unit plane $\overline{\mathcal{E}} = \lim_{n\to\infty} \mathcal{E}_n/n$ is finite. The force acting on the plane s in the stack, $F_s = \lim_{n\to\infty} F_{s,n}$ is also finite.

Infinite planes in stack: $n \to \infty$

$$\overline{\mathcal{E}} = \iint \frac{\mathrm{d}^2 k}{(2\pi)^3} \int_0^\infty \mathrm{d}\xi \left(\ln D^{\mathrm{te}} + \ln D^{\mathrm{tm}}\right)$$

$$F_s = -\Re \sum_{\mathrm{x}=\mathrm{te},\mathrm{tm}} \iint \frac{\mathrm{d}^2 k}{(2\pi)^3} \int_0^\infty \mathrm{d}\xi\, G_s^x$$

$$D^{\mathrm{x}} = \lim_{n\to\infty} \sqrt[n]{D_n^{\mathrm{x}}} = \frac{e^{-z} f_{\mathrm{x}}}{1+t_{\mathrm{x}}}$$

$$G_s^{\mathrm{x}} = \lim_{n\to\infty} D_n^{\mathrm{x}\,-1} D_n^{\mathrm{x}\,\prime} = \frac{2z t_{\mathrm{x}}^2 f_{\mathrm{x}}^{2(1-s)}}{1 - e^z f_{\mathrm{x}}(1+t_{\mathrm{x}})}$$

Stack of graphenes

$d \to 0$ and $d \to \infty$

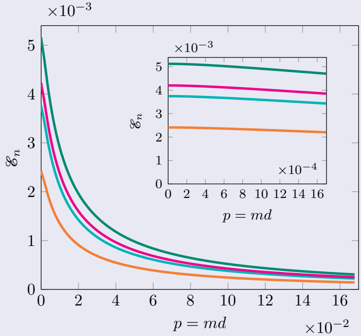
$$\mathcal{E}_n|_{d\to 0} \sim \frac{1}{d^3},\ \mathcal{E}_n|_{d\to\infty} = -\frac{\pi^2 e^2 (n-1)}{450 d^3 (md)^2} \sim \frac{1}{m^2 d^5}.$$

[†]Two graphene $\mathcal{E}_{d\to\infty} \sim \frac{1}{m^2 d^5}$; graphene/perfect metal $\mathcal{E}_{d\to\infty} \sim \frac{1}{md^4}$.

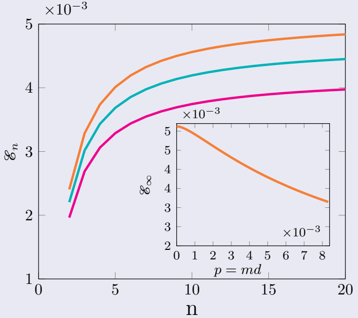
[†]Bordag, Fialkovsky, Gitman, Vassilevich, PRB, **80**, 245406 (2009)

Stack of graphenes

The Casimir energy per unit graphene

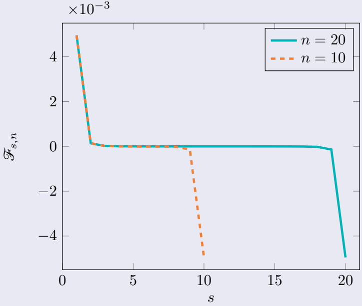


$m = 0.1$ eV, $n = 2, 4, 6, \infty$ from the bottom upwards. The insert shows the figure's neighborhood of origin

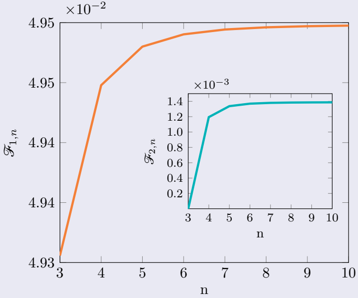


$m = 0.1$ eV, $d = d_c, 10d_c, 20d_c$ from the top down, where $d_c = 0.3345$ nm is inter-plane distance in graphite.

The Casimir force for plane s



The Casimir force $\mathcal{F}_{s,n}$ acting on the graphene s in the stack of 10 and 20 planes.



The Casimir force $\mathcal{F}_{s,n}$ acting on the first and second graphene in the stack

The binding energy per unit atom

For graphite interplane separation $d_c = 0.3345$ nm, we obtain the following energy per unit plane in an infinite stack of graphene $\overline{E}_\infty = 59.23$ erg/cm^2. The binding energy $E_{ib} = \overline{E}_\infty / d_c \rho_c$, where $\rho_c = 2.23$ g/cm^3 is the graphite mass density. For these values, we obtain $E_{ib} = 9.9$ meV/atom, which is by 10% smaller than for constant conductivity case[1]. From the first principle, the cohesion energies are $24-26$ meV/atom[2] and 24 meV/atom[3]. The experimental data gives cohesion 35 ± 10, 15 meV/atom[4] and 61 ± 5 meV/atom[5]. Most likely, the Casimir energy gives an essential contribution to the binding energy.

[1] NK, Kashapov, and Woods, PRD, **92**, 045001 (2015)

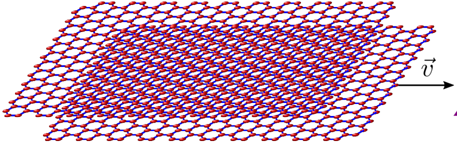
[2] Schabel and Martins, PRB, **46**, 7185 (1992)

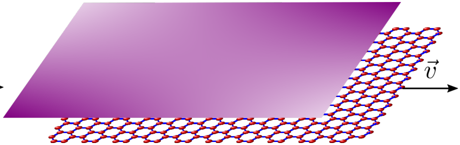
[3] Rydberg, Dion, et al, PRL, **91**, 126402 (2003)

[4] Benedict, Chopra, et al, Chem. Phys. Lett., **286**, 490 (1998)

[5] Zacharia, Ulbricht, and Hertel, PRB., **69**, 155406 (2004)

Lateral moving graphene[†]





The conductivity of moving graphene

The conductivity tensor in the co-moving frame K' is defined by relation $J'^i = \sigma'^{ij} E'_j$ and $\sigma'^{ij} = \Pi'^{ij}/\mathrm{i}\omega'$. In laboratory frame K: σ^{ij} – ?

$$K: \begin{cases} [A_\mu(k)] & = 0, \\ [F^{z\nu}(k)] & = \Pi^{\nu\rho}(k) A_\rho(k), \end{cases} \quad K': \begin{cases} [A'_\mu(k)] & = 0, \\ [F'^{z\nu}(k)] & = \Pi'^{\nu\rho}(k) A'_\rho(k), \end{cases}$$

$$\Lambda = \begin{pmatrix} u^0 & -\boldsymbol{u} \\ -\boldsymbol{u} & \mathbf{I} + \frac{\boldsymbol{u}\otimes\boldsymbol{u}}{u^0+1} \end{pmatrix}$$

† Antezza, Emelianova and NK, Nanotechnology, **35**, 235001, (2024)

NK, Emelianova, Physics, **6**, 148-163, (2024)

The conductivity of moving graphene

$$\boldsymbol{\sigma} = i_1 \mathbf{I} + i_2 \boldsymbol{k} \otimes \boldsymbol{k} + i_3 (\boldsymbol{k} \otimes \boldsymbol{v} + \boldsymbol{v} \otimes \boldsymbol{k}),\ i_n = i_n(\omega, \boldsymbol{k}, \boldsymbol{v})$$

Real frequency representation

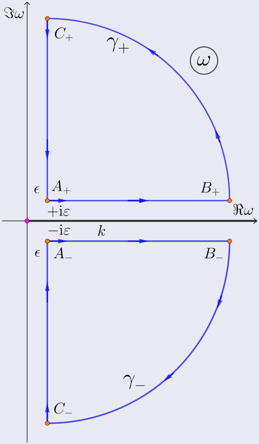
$$\mathcal{E}_C = -\frac{1}{2\mathrm{i}} \iint \frac{\mathrm{d}^2 k}{(2\pi)^3} (I_- - I_+),\ \mathcal{P} = \iint \frac{\mathrm{d}^2 k}{(2\pi)^3} (J_- + J_+),$$

$$I_\pm = \int_0^\infty \mathrm{d}\omega \ln \det \left[\mathbf{I} - e^{\pm 2\mathrm{i} a k_3} \mathcal{R}(\pm k_3) \right]$$

$$J_\pm = \int_0^\infty \mathrm{d}\omega\, k_3 \frac{e^{\pm 2\mathrm{i} a k_3} \left(\operatorname{tr} \mathcal{R}(\pm k_3) - 2 e^{\pm 2\mathrm{i} a k_3} \det \mathcal{R}(\pm k_3) \right)}{\det \left[\mathbf{I} - e^{\pm 2\mathrm{i} a k_3} \mathcal{R}(\pm k_3) \right]}.$$

Here $\mathcal{R}(\pm k_3) = \boldsymbol{r}'_{\mathrm{I}}(\pm k_3) \boldsymbol{r}_{\mathrm{II}}(\pm k_3)$, and $k_3 = \sqrt{\omega^2 - \boldsymbol{k}^2}$.

The conductivity of moving graphene



Casimir friction ∥

Normal Casimir ⊥

$$I_\pm : \int_0^\infty d\omega \Rightarrow \oint_{\gamma_\pm} - \int_{C_\pm A_\pm} - \int_{B_\pm C_\pm} \Rightarrow \sum_{i=1}^{3} \mathcal{E}_i,$$

Here, $\epsilon, \varepsilon \to 0$, $R \to \infty$, $\mathcal{E}_3 \to 0$.

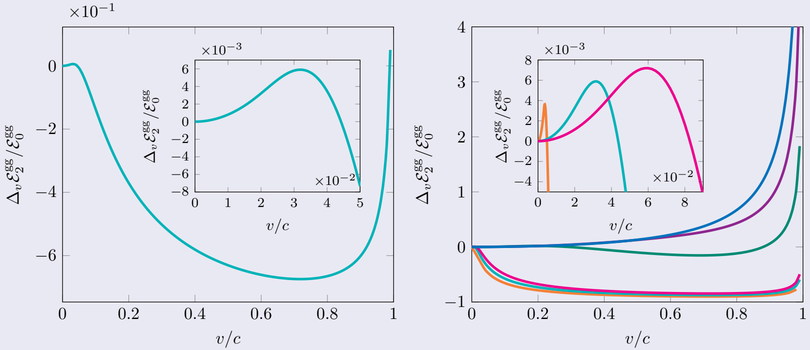
$\mathcal{E}_2 = \mathcal{E}_\perp$ – the normal Casimir effect (⊥ to planes)

$\mathcal{E}_1 = \mathcal{E}_\parallel$ – the Casimir friction (∥ to planes)

Lateral moving graphene†

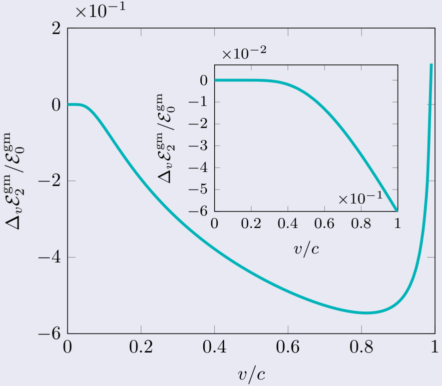
For graphene $v_F \approx 1/300 \approx 10^6$m/s and situation $v \ll v_F$ takes place in laboratory. For $m = 0, v = v_F$ one has the upper boundary $\Delta_v\mathcal{E}_\perp^{gg}/\mathcal{E}_0^{gg} = 0.33\%$.

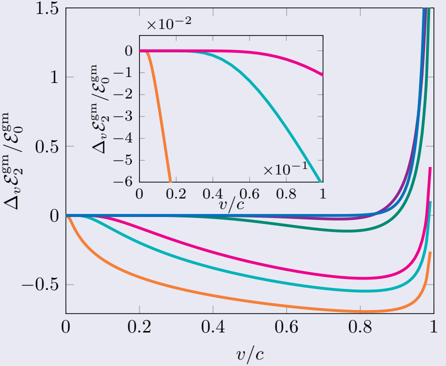
Normal Casimir force. Graphene/graphene: $\Delta_v\mathcal{E}_\perp^{gg} = \mathcal{E}_\perp^{gg} - \mathcal{E}_0^{gg}$



Here $v_F = c/300$. Left panel: $ma = 0.05$ ($m = 0.1$ eV, $a = 100$ nm). The quadratic behavior occurs, $\Delta_v\mathcal{E}_\perp^{gg} \sim v^2/c^2$ for small velocities $v \ll v_F$. Right panel: $ma = 0, 0.05, 0.1, 0.3, 0.9, 2$ (from the bottom upwards). The maxima occurred for critical velocity $v_{cr}^{gg} = v_F + \frac{1}{2}(ma)$.

Normal Casimir force. Perfect metal/graphene: $\Delta_v \mathcal{E}_\perp^{\rm gm} = \mathcal{E}_\perp^{\rm gm} - \mathcal{E}_0^{\rm gm}$





Here $v_F = c/300$. Left panel: $ma = 0.05$ ($m = 0.1$ eV, $a = 100$ nm). The correction, $\Delta_v \mathcal{E}_\perp^{\rm gg} \sim 0$ for small velocities $v \ll v_F$. Right panel: $ma = 0, 0.05, 0.1, 0.3, 0.9, 2$ (from the bottom upwards).

Lateral moving graphene[†]

The Casimir friction

$$\mathcal{F}_\parallel = \iint \frac{d^2 k}{(2\pi)^3} \left\{ \oint_{\gamma_-} d\omega k_3 \left[\frac{1}{1 - e^{-2iak_3} r_{\rm tm}^2(-k_3)} + \frac{1}{1 - e^{-2iak_3} r_{\rm te}^2(-k_3)} \right] + \oint_{\gamma_+} d\omega k_3 \left[\frac{1}{1 - e^{2iak_3} r_{\rm tm}^2(k_3)} + \frac{1}{1 - e^{2iak_3} r_{\rm te}^2(k_3)} \right] \right\}, \ k_3 = \sqrt{\omega^2 - k^2}$$

Here, $r_{\rm tm}^2(\pm k_3), r_{\rm te}^2(\pm k_3)$ are eigenvalues

$$\boldsymbol{r}_{\rm I}'(\pm k_3)\boldsymbol{r}_{\rm II}(\pm k_3) \Rightarrow \begin{pmatrix} r_{\rm tm}^2(\pm k_3) & 0 \\ 0 & r_{\rm te}^2(\pm k_3) \end{pmatrix}.$$

The Casimir friction

The contributions to these integrals come from the complex roots ω

$$1 - e^{\pm 2iak_3} r_{\rm tm}^2(\pm k_3) = 0,\ 1 - e^{\pm 2iak_3} r_{\rm te}^2(\pm k_3) = 0.$$

The roots exist if and only if

$$\left| r_{\rm tm}^2(\pm k_3) \right| > 1, \left| r_{\rm te}^2(\pm k_3) \right| > 1.$$

For $v = 0$, there are no roots[†]

$$\left| r_{\rm tm}^2(\pm k_3) \right| < 1, \left| r_{\rm te}^2(\pm k_3) \right| < 1.$$

[†] E. M. Lifshitz, Sov. Phys. JETP **2**, 73 (1956)

Lateral moving graphene[†]

Mixing between positive and negative frequencies of the spectrum is the origin of the Casimir friction[†].

The threshold for Casimir friction

$v = 0$, the spectrum $\tilde{k}^2 = \omega^2 - v_F^2 k^2 = 0 \Rightarrow \omega_0^\pm / k = \pm v_F$.
$v > 0$, the spectrum $\tilde{k}'^2 = \omega'^2 - v_F^2 k'^2 = 0 \Rightarrow \omega_v^\pm$:

$$(\omega_v^\pm)^2 - v_F^2 k^2 + \frac{1 - v_F^2}{1 - v^2}((\omega_v^\pm)^2 v^2 + (\boldsymbol{k}\boldsymbol{v})^2 - 2\omega_v^\pm (\boldsymbol{k}\boldsymbol{v})) = 0,$$

$\omega_v^+ < 0$ or $\omega_v^- > 0 \Leftrightarrow \cos^2\varphi > 1 - \frac{v^2 - v_F^2}{v^2(1 - v_F^2)} \Leftrightarrow v > v_F$[‡]

[†] M. F. Maghrebi, R. Golestanian, and M. Kardar, PRA **88**, 042509 (2013)
[‡] M. B. Farias, C. D. Fosco, F. C. Lombardo, and F. D. Mazzitelli, PRD **95**, 065012 (2017)

Thank you for your attention!