Line arrangements, operators and elliptic modular surfaces

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What is it about ?

Partly with L. Kühne (Bielefeld, Germany)

Main results

- A construction of the elliptic modular surface $\Xi_1(n)_{/\mathbb{C}}$, $n \geq 7$ as a realization space of line arrangements.
- Definition and use of operators acting on line arrangements, and obtention of some unexpected dynamical systems.

Vocabulary

- The elliptic modular surfaces $\Xi_1(n)_{/\mathbb{C}}$ will be later described. This is the universal curve over the modular curve $X_1(n)$, which is an important curve in arithmetic geometry.
- "realization space of line arrangements" = parameter space of some line arrangements having the same incidences / combinatorics.

Tool used: Matroids, and their realization space, and classical geometry.

We start with a partial overview on line arrangements.

Notations, Definitions of line or point arrangements

- We work over $\mathbb{C}$. Lines and points are in the projective plane $\mathbb{P}^2_{\mathbb{C}}$.

- $\mathcal{C} = \ell_1 + \cdots + \ell_n$ a line arrangement: this is a finite union of distinct lines in $\mathbb{P}^2_{\mathbb{C}}$ (here there is a labelling is by $\{1, \dots, n\}$; the labelling may be important in the following).

- $P = p_1 + \cdots + p_n$ a point arrangement, this is a finite union of distinct points in $\mathbb{P}^2_{\mathbb{C}}$.

- $\mathcal{D}$ the dual operator

$$\ell = \{ax + by + cz = 0\} \xleftrightarrow{\mathcal{D}} p = (a : b : c) \in \mathbb{P}^2$$

By duality, understanding line arrangements is the same as understanding point arrangements.

A brief tour on Line or Point arrangements

Line or points arrangements are studied or used e.g. in the following fields

- The orchard-planting problem which asks for the maximum number of lines each containing exactly 3 points of a set of real points.
- Dirac-Motzkin Conjecture: on the number of lines containing exactly two points of a given set of points.
- (Topology) Zarisky pairs i.e. two line arrangements $\mathcal{C}_1$, $\mathcal{C}_2$ which have the same combinatorics, but such that the topology of the pairs $(\mathbb{P}^2, \mathcal{C}_1)$ and $(\mathbb{P}^2, \mathcal{C}_2)$ are different.
- (Algebraic Geometry) Construction by Hirzebruch of ball quotient surfaces as covering of the plane branched over some specific line arrangement.
- Terao's Conjecture: the topology and the geometry of a free line arrangement is determined by its combinatorics.
- The classification of the simplicial line arrangements or the complex line arrangements without double point are open problems.

The remarkable Hirzebruch inequality (1983)

To a line arrangement $\mathcal{C} = \ell_1 + \cdots + \ell_n$ and an integer m, Hirzebruch associate a surface $X = X(\mathcal{C}, m)$ which is the minimal desingularization of a Galois $(\mathbb{Z}/m\mathbb{Z})^{n-1}$-cover of $\mathbb{P}^2$ branched over $\mathcal{C}$ to the order m.

Let r be an integer. By definition, a r-point of $\mathcal{C}$ is a point where exactly r lines of $\mathcal{C}$ meet.

We denote by $t_r = t_r(\mathcal{C})$ the number of r-points of $\mathcal{C}$.

Hirzebruch computed the two Chern numbers c_1^2, c_2 of X as functions of n, m and the sums $f_0 = \sum t_r,\ f_1 = \sum rtr$.

When $n \geq 6$ and $t_n = t_{n-1} = t_{n-2}$ (which means in most of the cases) and for $m \geq 3$, the surface X is minimal of general type. One can therefore apply Miyaoka-Yau inequality $c_1^2 \leq 3c_2$, from which Hirzebruch obtain the following non-trivial inequality

$$t_2 + t_3 \geq n + \sum_{r \geq 4} (r-4) t_r \tag{I}$$

In particular that shows that a non-trivial complex line arrangement has always some double or triple points. No other proof of (I) is known.

Melchior Inequality (1941)

Before Hirzebruch inequality, Melchior obtained in 1941 the following inequality for a real line arrangement which is not a pencil:

$$t_2 \geq 3 + \sum_{r \geq 3} (r-3) t_r$$

That shows that a non-trivial real line arrangement has always some double points, and in fact al least 3; Melchior's proof is topological.

Examples of nice line arrangements: Ceva

The Ceva(n) line arrangements

$$Ceva(n): \ (x^n - y^n)(x^n - z^n)(y^n - z^n) = 0$$

with $3n$ lines, n^2 triple points and 3 n-points, and no other singularities for $n \geq 3$.

The n^2 triple points and $3n$ lines form what is called a

$$((n^2)_3, (3n)_n)$$

configuration: each line contain n triple points, through each points there are 3 lines. There are very few other examples of line arrangement with only triple or higher singularities.

The case $n = 3$ is the only known complex line arrangement which has only triple point singularities (12 such).

When $n = 3$, the surface $X = X(Ceva(3), 4)$ that Hirzebruch associates to $Ceva(3)$ is a ball-quotient surface: one has $c_1^2 = 3c_2$ (a rare surface). That was the original aim of Hirzebruch for studying line arrangements.

Examples of nice line arrangements: simplicial line arrangements

An arrangement of real lines $\mathcal{C}$ is called simplicial if all the polygons cut out by the lines are triangles.
There are three known infinite families of simplicial line arrangements and some sporadic ones:

- Quasi-trivial: the union of $n \leq 2$ lines passing through the same point and a line not containing that point,
- The polygonal arrangements: the union of the n-gon and its n-lines of symmetries,
- The extended polygonal arrangements: the union of the n-gon and the line at infinity, when n is even,
- Around one hundred of sporadic line arrangement, the most complicated having 37 lines.

Grünbaum spent years tracking the sporadic lines arrangements and gave a conjectural list. Recently Michael Cuntz found one more example.

Examples of nice line arrangements: simplicial

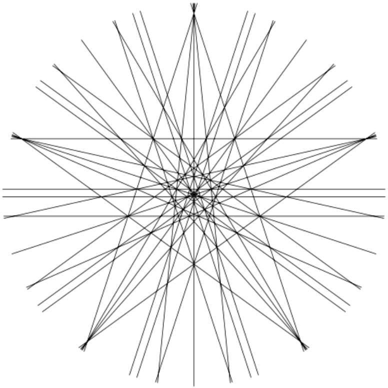

FIGURE 1. The “new” simplicial arrangement of rank three with 35 hyperplanes (from two different perspectives).

Picture from the paper ”A Greedy algorithm to compute arrangements of lines in the projective plane”, 2020, by M. Cuntz.

Operators acting on line or point arrangements

Motivations for the introduction of operators acting on line or point arrangements:

The aim was to construct new interesting line arrangements from known one, by taking their images by some operators.

For example, interesting or nice means Simplicial, or Free line arrangements, or line arrangements without double points.

Operators acting on line or point arrangements

Let M, N be two sub-sets of integers ≥ 2.
As above, let $\mathcal{C} = \ell_1 + \cdots + \ell_n$ be a line arrangement, and
$P = p_1 + \cdots + p_n$ be a point arrangement.
For an integer m, a m-point of $\mathcal{C}$ is a point where exactly m lines of $\mathcal{C}$ meet.
Dually: A m-rich line of P is a line containing exactly m points of P.
Define the point operator

$$\mathcal{P}_M(\mathcal{C}) = \cup_{m \in M} \{ p \in \mathbb{P}^2 \,|\, p \text{ is a } m\text{-point of } \mathcal{C} \}$$

Define the line operator

$$\mathcal{L}_M(P) = \cup_{m \in M} \{ \ell \text{ line in } \mathbb{P}^2 \,|\, \ell \text{ is } m\text{-rich of point of } P \}$$

(one has $\mathcal{D} \circ \mathcal{P}_M \circ \mathcal{D} = \mathcal{L}_M$). Then we define the operators

$$\Lambda_{M,N} = \mathcal{L}_N \circ \mathcal{P}_M \text{ and } \psi_{M,N} = \mathcal{P}_N \circ \mathcal{L}_M.$$

$\Lambda_{M,N}$ maps a line arrangement to a line arrangement.

Operators acting on line or point arrangements

When $M = \{m\}$ or $N = \{n\}$, we drop the brackets, thus for example $\Lambda_{M,N} = \Lambda_{m,n}$.

Example.

Let $k \geq 2$ be an integer. The operator

$$\Lambda_{2,k}$$

takes into input a line arrangement $\mathcal{C}$ and returns the set of lines in the plane which contain exactly k of the double points of $\mathcal{C}$ (a double point is a 2-point).

Remark.

In general, if Λ is an operator and $\mathcal{C}$ is a 'random' line arrangement, one has $\Lambda(\mathcal{C}) = \emptyset$.

Examples of use of the operator $\Lambda_{2,3}$

Pascal's Hexagon Theorem (weak form). Let H be a hexagon such that the set of vertices $p_1, \ldots, p_6$ of H are six generic points of an irreducible conic. Then $\Lambda_{2,3}(H)$ is a line.

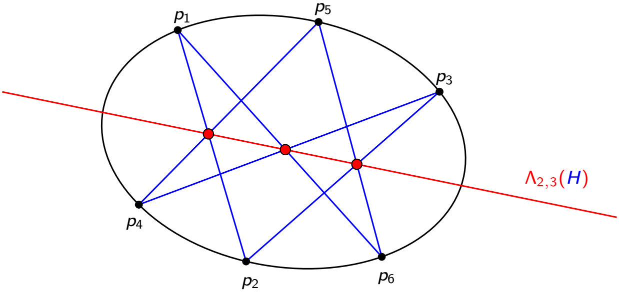

Figure: Pascal's Hexagon Theorem

Operators acting on line or point arrangements

Theorem

(XR) There exist line arrangements, which I called 'unassuming', with 6 *lines and only double points singularities* (15 *of these) forming (up to projective equivalence) a one dimensional family* $U \subset \mathbb{P}^1$, *such that the following holds:*

- *The image by* $\Lambda_{2,3}$ *of a generic unassuming arrangement is again an unassuming arrangement.*
- *The operator* $\Lambda_{2,3}$ *acts on* $U \subset \mathbb{P}^1$ *though the map* $z \to z^{-2}$.
- *For any* n, *there exists periodic unassuming arrangements with period* n.

Discovery Method

A known set of 150 interesting line arrangements, ranging from 7 to 50 lines, was used. A computer program was instructed to randomly select one of these line arrangements. From this arrangement, a random subset $\mathcal{C}$ of 6 lines was chosen. The transformation $\Lambda_{2,3}$ was then applied. If any of the sets $\Lambda_{2,3}(\mathcal{C})$, $\Lambda_{2,3}\Lambda_{2,3}(\mathcal{C})$, or $\Lambda_{2,3}\Lambda_{2,3}\Lambda_{2,3}(\mathcal{C})$ did not consist of exactly 6 lines, the attempt was rejected, and the process restarted.

Geogebra : a geometrical construction

That way, I obtained a first example which, as I realized later, belongs to a one-dimensional family.
A few months after finishing my paper on these arrangements, I figured out how to construct them, and their images by $\Lambda_{2,3}$, geometrically.

Geometric construction using Geogebra

Aim: to find other examples of a family of line arrangements and an operator that acts on that family.
This is where I asked Lukas Kühne to help me.

Matroids and their Realization Space

Definition. Let I be a finite set, (the elements of I are called atoms). A matroid M (of rank 3) with atoms in I is a set of sub-sets of order 3 of I called non-bases. The bases i.e. the remaining sub-sets of order 3 must satisfy the following properties:
a) The set of bases in non-empty
b) If A, B are two distinct bases and $a \in A \setminus B$, then there exists $b \in B$ such that

$$A \setminus \{a\} \cup \{b\}$$

is a base.
There are many equivalent definitions of matroids. In my opinion 'non-bases' are more interesting than 'bases', that is why I use the set of non-bases as a definition of a matrid.

Matroids and their Realization Space

Definition. Let I be a finite set, (the elements of I are called atoms). A matroid M (of rank 3) with atoms in I is a set of sub-sets of order 3 of I subject to the above restrictions, here is an example:

Example. If $\mathcal{C} = \sum_{i \in I} \ell_i$ is an arrangement of lines indexed by I, the matroid associated to $\mathcal{C}$ is the set

$$M(\mathcal{C}) := \{\{i, j, k\} \subset I \,|\, \ell_i,\, \ell_j,\, \ell_k \text{ meet at a common point}\}.$$

The matroid $M(\mathcal{C})$ encodes how the lines meet, the combinatorics of $\mathcal{C}$.

Definition. A line arrangement $\mathcal{C}$ such that $M(\mathcal{C}) = M$ is said a realization of the matroid M.

Proposition

Let M be a matroid. There exists a scheme $\mathcal{S}(M) \hookrightarrow (\mathbb{P}^2)^I$ parametrizing the realizations of M.

We will see an example.

Matroids and their Realization Space

Dually, if $P = \{p_i \mid i \in I\}$ is a point arrangement, one defines the matroid

$$M(P) := \{\{i, j, k\} \subset I \mid p_i,\ p_j,\ p_k \text{ are aligned}\}.$$

That matroid describes the combinatorics of P.

One has

$$M(\mathcal{D}(P)) = M(P).$$

Matroids and their Realization Space

Consider $\gamma \in PGL_3(\mathbb{C})$. Define $\gamma\mathcal{C} := \sum_{\ell \in \mathcal{C}} \gamma(\ell)$. Then $\mathcal{C}$ and $\gamma\mathcal{C}$ have the same combinatorics, which means

$$M(\mathcal{C}) = M(\gamma\mathcal{C}),$$

thus it is natural to consider line arrangement up to projective equivalence.

Proposition

Let M be a matroid. There exists a scheme $\mathcal{R}(M)$ parametrizing orbits $[\mathcal{C}]$ under $PGL_3(\mathbb{C})$ of realizations $\mathcal{C}$ of M.

Definition. The scheme $\mathcal{R}(M)$ is called the realization space of M.

Remark. The operators Λ and $\gamma \in PGL_3$ are such that

$$\gamma\Lambda = \Lambda\gamma,$$

one may therefore apply Λ to an orbit and define $\Lambda([\mathcal{C}]) := [\Lambda(\mathcal{C})]$.

Definition of the matroid M_n

Let $n \geq 7$ be an integer. Define the set of atoms by

$$I := \mathbb{Z}/n\mathbb{Z} \sqcup \mathbb{Z}/n\mathbb{Z}'.$$

This is the disjoint union of two copies of $\mathbb{Z}/n\mathbb{Z}$.

Define the matroid:

$$M_n := \{\{i,j;k'\} \mid i,j \in \mathbb{Z}/n\mathbb{Z}, i \neq j,\ k' \in \mathbb{Z}/n\mathbb{Z}' \text{ such that } i+j+k' = 0\}.$$

Geometrically, one has (almost) such configuration by considering the regular n-gon $\mathcal{C}_0^r$ and its n lines of symmetries $\mathcal{C}_1^r$.

The next slide discusses the case $n = 7$, but the analog situtation holds for any $n \geq 7$.

Definition of the matroid M_n

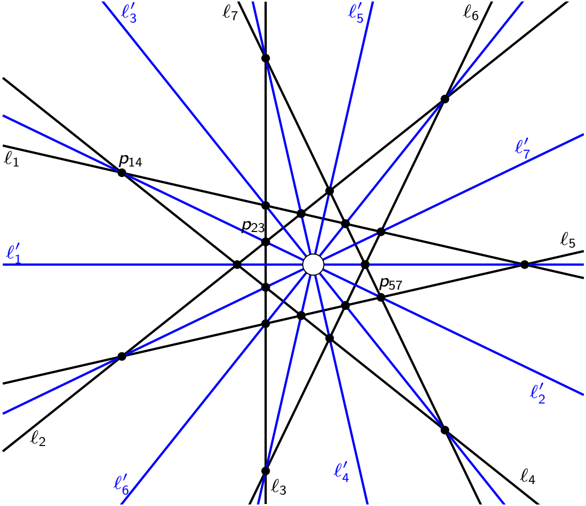

Figure: A line arrangement (almost) realizing the matroid M_7. In black the regular 7-gon $\mathcal{C}_0^r = \{\ell_1, \dots, \ell_7\}$, in blue $\mathcal{C}_1^r = \{\ell_1', \dots, \ell_7'\}$ the line of sym.

Definition of the operators

Definition. Let $n \geq 7$ be an integer. Let us define the operator Λ according if n is odd or even:

- Suppose that n is odd, $n = 2k+1$, then

$$\Lambda := \Lambda_{2,k}$$

- Suppose that n is even, $n = 2k$, then

$$\Lambda := \Lambda_{2,\{k-1,k\}}$$

It is not difficult to check that for the regular n-gon $\mathcal{C}_0^r$, one has

$$\Lambda(\mathcal{C}_0^r) = \mathcal{C}_1^r$$

where $\mathcal{C}_1^r$ are the lines of symmetries. One uses different kind of operators according when n is odd or even, because in the even case, there are somme parallel lines on the regular n-gon.
In case $n = 7$, then $\Lambda = \Lambda_{2,3}$.

Action of $\Lambda_{2,3}$ on M_7

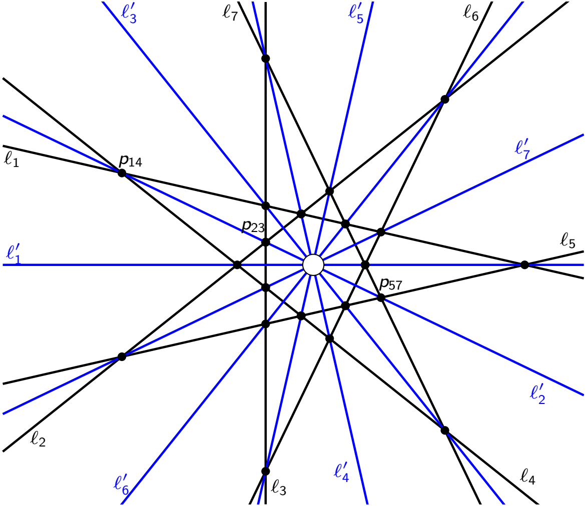

Figure: Each line of symmetry contains 3 double points, one has: $\Lambda_{2,3}(\mathcal{C}_0^r) = \mathcal{C}_1^r$.

Main result

One has $\Lambda(\mathcal{C}_0^r) = \mathcal{C}_1^r$ but $\Lambda(\mathcal{C}_1^r) = \emptyset$. The line arrangement $\mathcal{C}_0^r + \mathcal{C}_1^r$ is **not** a realization of M_n because of the central singularity. But the incidences between $\mathcal{C}_0^r$ and $\mathcal{C}_1^r$ are the incidences described by the matroid M_n.
A realization of M_n is a union $\mathcal{C} = \mathcal{C}_0 + \mathcal{C}_1$, where $\mathcal{C}_0 = \sum_{i \in \mathbb{Z}/n\mathbb{Z}} \ell_i$ and $\mathcal{C}_1 = \sum_{i' \in \mathbb{Z}/n\mathbb{Z}'} \ell_{i'}$, and by construction of the matroid M_n, one has also $\Lambda(\mathcal{C}_0) = \mathcal{C}_1$. One may see $\mathcal{C}$ as a deformation of $\mathcal{C}_0^r + \mathcal{C}_1^r$, but such that the central singularity disappears.

Theorem

(Lukas Kühne, X.R.) Let $\mathcal{C} = \mathcal{C}_0 + \mathcal{C}_1$ be a realization of M_n.

- *$\mathcal{C}_2 := \Lambda(\mathcal{C}_1)$ is a union of n-lines.*
- *There exists a labelling of $\mathcal{C}_2$ by $\mathbb{Z}/n\mathbb{Z}'$ such that $\mathcal{C}_1 + \mathcal{C}_2$ is a realization of M_n ($\mathcal{C}_1$ being naturally labeled by $\mathbb{Z}/n\mathbb{Z}$).*
- *The realization space $\mathcal{R}_n := \mathcal{R}(M_n)$ is birational to the modular elliptic surface $\Xi_1(n)$.*
- *The operator Λ acts on $\mathcal{R}_n$, and therefore on $\Xi_1(n)$; that action on $\Xi_1(n)$ is by the multiplication by -2 map.*

Main result

Theorem

(Lukas Kühne, X.R.) Let $\mathcal{C} = \mathcal{C}_0 + \mathcal{C}_1$ *be a realization of* M_n*.*

- *There exists a labelling of* $\mathcal{C}_2 := \Lambda(\mathcal{C}_1)$ *by* $\mathbb{Z}/n\mathbb{Z}'$ *such that* $\mathcal{C}_1 + \mathcal{C}_2$ *is a realization of* M_n*.*
- *The realization space* $\mathcal{R}_n := \mathcal{R}(M_n)$ *is birational to the modular elliptic surface* $\Xi_1(n)$*.*
- *The operator* Λ *acts on* $\mathcal{R}_n$*, and therefore on* $\Xi_1(n)$*; that action on* $\Xi_1(n)$ *is by the multiplication by* -2 *map.*

Let $X_1(n)$ be the modular curve which parametrizes the pairs (E, t) up to isomorphisms, where E is an elliptic curve and t be a point of order n on E. The elliptic modular surface $\Xi_1(n)$ is the universal family over $X_1(n)$, meaning that there exists a fibration $\Xi_1(n) \to X_1(n)$ such that the fiber over (E, t) is (isomorphic to) E. We will consider a point in that fiber as a triple (E, t, p) with (E, t) as above and p a point of E.
The multiplication by -2-map is $(E, t, p) \to (E, t, -2p)$.

That results was unexpected to us; it gives a connection between the elliptic modular surfaces and line arrangements, moreover some dynamical system which have a geometric interpretation.

In the next half of my talk, I will give some elements of the proof. There will be two parts using completely different tools: one part for the case $n = 7$, using explicit equations of the lines and realization space, and then a general proof which works for any $n \geq 7$ using classical geometry (Chasles Lemma) recently revisited by Green and Tao.

First case: $n = 7$

Before knowing the matroids $M_7, M_8, \dots$, we first found a matroid $M_7' := \{\{1,2,10\},\{1,3,12\},\{1,4,14\},\{1,5,9\},\{1,6,11\},\{1,7,13\},$ etc... with atoms the set $I = \{1,\dots,14\}$, and such that

Proposition

(L. Kühne, XR). Let $\mathcal{C} = \mathcal{C}_0 + \mathcal{C}_1$ be a realization of M_7', with $\mathcal{C}_0 = \ell_1 + \cdots + \ell_7$ and $\mathcal{C}_1 = \ell_8 + \cdots + \ell_{14}$. Define $\mathcal{C}_2 := \Lambda_{2,3}(\mathcal{C}_1)$. Then:

- *One has $\Lambda_{2,3}(\mathcal{C}_0) = \mathcal{C}_1$. There exists a labelling of $\mathcal{C}_1 + \mathcal{C}_2$ so that it is a realization of M_7' (in particular $\mathcal{C}_2$ is the union of 7 lines).*
- *$\mathcal{R}(M_7')$ is birational to the surface $\Xi_1(7)$.*
- *The degree of the operator $\Lambda_{2,3}$ acting on $\mathcal{R}(M_7')$ is 4.*

We later generalized that matroid M_7' and we obtained in that way M_7, M_8 etc... with M_7 and M_7' being isomorphic, thus describing the same line arrangements, up to re-labelling the lines.
Let us discuss the proof of that Proposition before coming to the other cases, because it has its own interest. For examples one sees the explicit equations of the realization space, and one has an explicit descriptions of the normal vectors of the line arrangements.

First case: $n = 7$

Define the 3×14 matrix

$$A := \begin{pmatrix} x_{1,1} & \cdots & x_{1,14} \\ \vdots & & \vdots \\ x_{3,1} & \cdots & x_{3,14} \end{pmatrix}$$

where for fixed i, the $(x_{1,i} : x_{2,i} : x_{3,i})$ are homogeneous coordinates on a copy of $\mathbb{P}^2$.
For $s := \{i, j, k\}$ an order 3 sub-set of $\{1, \dots, 14\}$, define A_s to be the determinant of the 3×3 sub-matrix of A with the columns in s.
The variety $\mathcal{S}(M_7')$ parametrizing the realizations of M_7' is simply

$$\mathcal{S}(M_7') := V(\langle A_s \,|\, s \in M_7' \rangle) \setminus \cup_{s \notin M_7'} V(A_s) \hookrightarrow (\mathbb{P}^2)^{14}.$$

A point $P = (p_1, \dots, p_{14})$ (with $p_i \in \mathbb{P}^2$) is a point realization of M_7', and any point realization is a point of $\mathcal{S}(M_7')$. In order to have a line realization take the dual. If $\mathcal{C} = \ell_1 + \cdots + \ell_{14}$ is a realization M_7', the meeting of three lines ℓ_i, ℓ_j, ℓ_k at one point is expressed by the vanishing of the determinant of the 3×3 matrix formed by the 3 normal vectors to the 3 lines.

First case $n = 7$

The matroid M_7' does **not** contain any of the following triples

$$\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\}.$$

That means that any realization of M_7' is projectively equivalent to a unique realization such that the first four points (or normal vectors if we consider lines) are the canonical base

$$(1:0:0),(0:1:0),(0:0:1),(1:1:1)$$

Thus the realization space $\mathcal{R}(M_7') = \mathcal{S}(M_7')/PGL_3$ is simply the intersection of $\mathcal{S}(M_7')$ with the equations defining the above four points.

Using OSCAR/JULIA, we computed a model of the realization space $\mathcal{R}(M_7')$ reducing the number of variables: it is is an open sub-scheme of the quartic surface Y in $\mathbb{A}^3$:

$$Y : x_1^2x_2^2+x_1^2x_2x_3-x_1^2x_2-x_1x_2^2x_3-x_1x_2^2-x_1x_2x_3^2+x_1x_2x_3+x_1x_2-x_2x_3^2+x_3^2=0.$$

First case $n = 7$

For a point generic (x_1, x_2, x_3) of the above quartic Y, OSCAR/JULIA also tells that the following seven points

$$\begin{aligned}&(1:0:0),\ (0:1:0),\ (0:0:1),\ (1:1:1),\\ &(-x_1x_2^2x_3 - x_1x_2x_3 - x_1x_2 - x_1 - x_2x_3 : x_1x_2 + x_1 : x_2),\\ &(-x_1x_2^2x_3 - x_1x_2x_3 - x_1x_2 - x_1 - x_2x_3 : x_1x_2 + x_1 + x_2^2x_3 + x_2x_3 :\\ &x_2^2x_3 + x_2x_3 + x_2),\\ &(-x_1x_2^2x_3 - x_1x_2x_3 - x_1x_2 - x_1 + x_3 : x_1x_2 + x_1 - x_2x_3 - x_3 :\\ &x_2^2x_3 + x_2x_3 + x_2)\end{aligned}$$

are the seven normal vectors to the seven first lines $\mathcal{C}_0$ of the generic realization of M_7'.
The seven other lines are $\mathcal{C}_1 := \Lambda_{2,3}(\mathcal{C}_0)$. Moreover one checks that there is a canonical labelling so that $\mathcal{C}_1 + \mathcal{C}_2$ is a realization of M_7', where $\mathcal{C}_2 := \Lambda_{2,3}(\mathcal{C}_1)$.
By taking the unique line arrangement in the orbit $[\mathcal{C}_1 + \mathcal{C}_2]$ such that the first four normal vectors are the canonical basis, one obtains the point $\Lambda_{2,3}([\mathcal{C}_1 + \mathcal{C}_2])$ in Y, and therefore we know the action of $\Lambda_{2,3}$ on Y.

First case $n = 7$

The projective closure $\tilde{Y} \hookrightarrow \mathbb{P}^2$ of Y is quartic surface with ADE singularities. The minimal resolution X of $\tilde{Y}$ is a K3 surface. It is easy to find irreducible divisors and their intersections on X, it turns out that the Néron-Severi group has discriminant -7 and rank 20. This is sufficient to identify X as being (isomorphic to) the elliptic modular surface $\Xi_1(7)$.

We then study the degree of the action of $\Lambda_{2,3}$ on X by using a 2-form $\omega = \frac{dx_2 \wedge dx_3}{\partial f / \partial x_1}$: an explicit computation gives that

$$\Lambda_{2,3}^* \omega = -2\omega,$$

so that the volume form $\omega\bar{\omega}$ is multiplied by 4, thus the degree of $\Lambda_{2,3}$ is 4. Here f is the equation of Y.

We then realized that one may also construct the matroids M_8, M_9 etc... After proving that $\mathcal{R}_8$ has similar properties than $\mathcal{R}_7$ and was birational to $\Xi_1(8)$, we searched for a general proof for any $n \geq 7$.

Main result to be proved, general case $n \geq 7$

Let us now sketch the proof of our main result, recalled here:

Theorem

(Lukas Kühne, X.R.) Let $\mathcal{C} = \mathcal{C}_0 + \mathcal{C}_1$ be a realization of M_n.

- *There exists a labelling of $\mathcal{C}_2 := \Lambda(\mathcal{C}_1)$ by $\mathbb{Z}/n\mathbb{Z}'$ such that $\mathcal{C}_1 + \mathcal{C}_2$ is a realization of M_n.*
- *The realization space $\mathcal{R}_n := \mathcal{R}(M_n)$ is birational to the modular elliptic surface $\Xi_1(n)$.*
- *The operator Λ acts on $\mathcal{R}_n$, and therefore on $\Xi_1(n)$; that action on $\Xi_1(n)$ is by the multiplication by -2 map.*

• $X_1(n)$ is the modular curve which parametrizes the pairs (E, t) up to isomorphisms, where E is an elliptic curve and t be a point of order n on E.

• $\Xi_1(n)$ is the universal family over $X_1(n)$: $\exists$ an elliptic fibration $\Xi_1(n) \to X_1(n)$ such that the fiber over (E, t) is (isomorphic to) E.
We consider a point in that fiber as a triple (E, t, p) with (E, t) as above and p a point of E.

Sketch of the proof: the general case

Let $E \hookrightarrow \mathbb{P}^2$ be a smooth cubic curve, with neutral element 0 a flex. Let t be a point of order n on E and let p be a point on E such that $6np \neq 0$. Define the point arrangements

$$P_0 = (p + mt)_{m \in \mathbb{Z}/n\mathbb{Z}}$$

and

$$P_1 = (-2p + mt)_{m \in \mathbb{Z}/n\mathbb{Z}}.$$

For $i \neq j$ in $\mathbb{Z}/n\mathbb{Z}$, the line containing the points $p + it, p + jt$ cuts the cubic curve E at a third point which is $-(2p + (i + j)t)$ by definition of the group law.
In the dual settings, that means that

$$\mathcal{C} := \mathcal{D}(P_0) + \mathcal{D}(P_1)$$

is a realization of M_n.
Then the action of Λ on $\mathcal{C}$ is clear: Define $\psi = \mathcal{D} \circ \Lambda \circ \mathcal{D}$; using the incidences, one obtain that

$$\psi(P_0) = P_1,$$

and therefore, replacing p by $-2p$, one gets that $P_2 := \psi(P_1)$ is an arrangement of n points, and $\mathcal{D}(P_1) + \mathcal{D}(P_2)$ is a realization of M_n.

Sketch of the proof

An amusing consequence of equality $\psi(P_0) = P_1$ is that one my obtain geometrically the point $-2p$ without taking the tangent to p.

From the previous slides, given a point (E, t, p) of $\Xi_1(n)$, we define the line arrangement $D(P_0) + D(P_1)$. The class

$$\Gamma(E, t, p) := [\mathcal{D}(P_0) + \mathcal{D}(P_1)]$$

of that line arrangement modulo the action of $PGL_3(\mathbb{C})$ does not depend on the choice of the embedding of E into $\mathbb{P}^2$ such that 0 is a flex.

We thus defined a rational map

$$\Gamma : \Xi_1(n) - \to \mathcal{R}_n$$

Lemma

The rational map Γ *has degree* 9 *onto its image: one has* $\Gamma(E, t, p) = \Gamma(E', t', p')$ *if and only if* $(E, t) \simeq (E', t')$ *and, by that isomorphism,* $p - p'$ *is* 3*-torsion.*

Sketch of the proof

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if and only if $(E,t)\simeq(E',t')$ *and, by that isomorphism,* $p-p'$ *is* 3*-torsion.*

Proof of the Lemma. Suppose $\Gamma(E,t,p)=\Gamma(E',t',p')$ (the equality is equality between orbits). That means that there is a projective transformation τ that sends a set of $2n>9$ element onto the other set. Thus the two elliptic curve containing these two sets respectively must be isomorphic by τ because of Bézout's Lemma. Hence, we can identify E and E'.

It is well-known that, for E generic, the group of projective transformations of $\mathbb{P}^2$ preserving E has order 18 and is generated by the maps inducing the multiplication map $[-1]$ and the group T_3 of translations by order 3 torsion elements. From that only the group T_3 survives, which means that $p-p'$ is 3-torsion.

Sketch of the proof

Denote T_3 the group of translation by order 3 elements acting on $\Xi_1(n)$ fiber by fiber. It has order 9 and the quotient $\Xi_1(n)/T_3$ is simply $\Xi_1(n)$. The following diagram of rational maps

$$\begin{tikzcd}
\Xi_1(n) \arrow[d] \arrow[dr, "\Gamma"] & \\
\Xi_1(n)/T_3 = \Xi_1(n) \arrow[r] & \mathcal{R}_n
\end{tikzcd}$$

is commutative. We thus prove that

Lemma

The map Γ induces a rational map between $\Xi_1(n)$ and $\mathcal{R}_n$ which is 1-to-1 onto its image.

Sketch of the proof

It remains to show that the rational map Γ is generically onto. We will continue to work on the dual settings, i.e., using point arrangements. Let us therefore consider a point realization $P := P_0 + P_1$ of M_n, with

$$P_0 = (p_i)_{i \in \mathbb{Z}/n\mathbb{Z}} \text{ and } P_1 = (q_{i'})_{i' \in \mathbb{Z}/n\mathbb{Z}'}$$

an let us prove that there exists a cubic curve passing through the $2n$ points.

Consider $i, j, k' \in \mathbb{Z}/n\mathbb{Z}$, $i \neq j$. The points p_i, p_j and $q_{k'}$ verify the following property

$$\text{The points } p_i, p_j \text{ and } q_{k'} \text{ are aligned if and only if } i + j + k' = 0. \quad \text{(P)}$$

Chasles and Green-Tao

Lemma

(Chasles's Lemma) A cubic curve containing 8 *points among the* 9 *intersection points of two cubic curves necessarily contains the ninth.*

Let I, J, K be three finite intervals in $\mathbb{Z}$. A triangular grid is a collection of distinct points $\{p_i,\ i \in I\}$, $\{q_j, j \in J\}$, $\{r_k, k \in K\}$ such that: $\forall i \in I, j \in J, k \in K$ with $i + j + k = 0$, the points p_i, q_j, r_k are aligned. By repeated use of Chasles's Lemma, Green and Tao obtain the following

Lemma

(Green Tao 2013) Let $m > 2$ *be an integer. Suppose that* $I := \{2, \dots, m-2\}$, $J := \{-m, \dots, -1\}$ *and* $K := \{1, \dots, m\}$ *and that* $p_i,\ i \in I,\ q_j, j \in J,\ r_k, k \in K$ *form a triangular grid. Then all points* p_i, p_j, p_k *lie on a single cubic curve* γ.

They needed that Lemma for their work on the orchard-planting problem, which asks for the maximum number of lines through exactly 3 points of a set of points P in the real plane. When $|P|$ is large, the maximal examples come from points arrangements on cubic curves.

Sketch of the proof

In our situation, we recall that for $i, j, k' \in \mathbb{Z}/n\mathbb{Z}$, $i \neq j$, the points p_i, p_j and $q_{k'}$ verify the property

The points p_i, p_j and $q_{k'}$ are aligned if and only if $i + j + k = 0$. (P)

We apply Green-Tao's Lemma to suitable grids formed by the points in the point arrangement $P_0 + P_1$, with

$$P_0 = (p_i)_{i \in \mathbb{Z}/n\mathbb{Z}} \text{ and } P_1 = (q_i')_{i' \in \mathbb{Z}/n\mathbb{Z}}$$

and conclude that there is a cubic curve γ passing through these points. That cubic curve may not be smooth. We do a case-by-case analysis, and obtain that γ can only be smooth, or a nodal cubic curve.
In the later case, the smooth locus γ^* has group structure $\mathbb{G}_m$: there exists an isomorphism $f : \mathbb{C}^* \to \gamma^*$ such that $f(a), f(b), f(c)$ are on a line if and only if $abc = 1$.
Example The points arrangement

$$(f(a\zeta^m))_{m \in \mathbb{Z}/n\mathbb{Z}} + (f(a^{-2}\zeta^m))_{m \in \mathbb{Z}/n\mathbb{Z}}$$

is an explicit example of a point realization of M_n, where ζ is a primitive n-th root of unity, and $a \in \mathbb{C}^*$.

Thank you !