Rigidity of complex projective manifolds

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Table of Contents

1. Deformation rigidity problems for homogeneous spaces
2. Deformation rigidity problems for quasi-homogeneous varieties
3. VMRT

Deformation rigidity problems

Conjecture ('58 Kodaira-Spencer)

Let $\pi\colon \mathcal{X} \to \Delta$ be a smooth projective morphism from complex manifold $\mathcal{X}$ to the unit disc $\Delta \subset \mathbb{C}$. Suppose for any nonzero $t \in \Delta$, the fiber $\mathcal{X}_t = \pi^{-1}(t)$ is biholomorphic to $\mathbb{P}^n$. Then the central fiber $\mathcal{X}_0$ is biholomorphic to $\mathbb{P}^n$.

The conjecture asserts that $\mathbb{P}^n$ is globally rigid.
The case of $\mathbb{P}^3$ was firstly studied by Peternell['85 '86].

Theorem (Hirzebruch, Kodaira, Novikov and Yau / '89 Siu)

$\mathbb{P}^n$ is globally rigid for all n.

Deformation rigidity problems

Let X be a Hermitian symmetric manifold of compact type, that is, a symmetric space G/K, where G is a compact Lie group and G/K admits a complex structure.

Example

The Grassmannian $Gr(k, n+1)$ with the $G = SU_{n+1}$ action is an irreducible Hermitian symmetric manifold of the compact type. In particular, $Gr(k, n+1) = SU_{n+1}/S(U_k \times U_{n-k-1})$.

Example

The smooth hyperquadric of dimension n

$$\mathbf{Q}^n := \{z_0^2 + \cdots + z_{n+1}^2 = 0\} \subset \mathbb{P}^{n+1}$$

with the complex orthogonal group $G = SO_{n+2}$ is an irreducible Hermitian symmetric manifold of the compact type. $\mathbf{Q}^n = SO_{n+2}/(SO_n \times SO_2)$.

Deformation rigidity problems

Conjecture ('95 Hwang)

Let $\pi\colon \mathcal{X} \to \Delta$ be a smooth projective morphism from complex manifold $\mathcal{X}$ to the unit disc $\Delta \subset \mathbb{C}$. Suppose for any nonzero $t \in \Delta$, the fiber $\mathcal{X}_t = \pi^{-1}(t)$ is biholomorphic to a Hermitian symmetric manifold of compact type X. Then the central fiber $\mathcal{X}_0$ is biholomorphic to X.

The case of $\mathbf{Q}^3 \subset \mathbb{P}^4$ was studied by Nakamura ['88].

Theorem ('95 Hwang)

$\mathbf{Q}^n \subset \mathbb{P}^{n+1}$ *for $n \geq 3$ is globally rigid.*

Theorem ('98 Hwang and Mok)

Every irreducible Hermitian symmetric manifold of the compact type is globally rigid.

Rational homogeneous manifold

Definition

A rational homogeneous manifold is a projective manifold X which is homogeneous under a complex Lie group, and rational.

Let G be a complex semisimple Lie group, a manifold with group structure which dose not have non-trivial normal abelian subgroups.
A Borel subgroup B is a maximal closed connected solvable subgroup of G which is unique upto cojugate.

Definition

A subgroup P of G is called a parabolic subgroup of G if $B \subset P$ for some Borel subgroup B.

Since G acts on a rational homogeneous manifold X, B acts on X. There is a B-fixed point $x \in X$ by the Borel-fixed point theorem. Since X is homogeneous, $G.x = X$ with isotropy subgroup P which contains B. Hence, X is isomorphic to G/P.

Rational homogeneous manifold

Example

The projective space $\mathbb{P}^n$ with the complex special linear group $G = SL_{n+1}$ action is a rational homogeneous manifold of type (A_n, α_1). Moreover, the Grassmannian $Gr(k, n+1)$ with the $G = SL_{n+1}$ action is of type (A_n, α_k).

Example

The smooth hyperquadric $\mathbf{Q}^n$ of dimension n with the complex orthogonal group $G = SO_{n+2}$ is a rational homogeneous manifold of type (B_l, α_1) or (D_l, α_1) where $2l - 1 = n$ or $2(l-1) = n$

Example

Irreducible Hermitian symmetric manifolds of the compact type. (A_l, α_k), (B_l, α_1), (D_l, α_1), (D_l, α_l), (C_l, α_l), (E_6, α_1) and (E_7, α_7).

Example

Let q be a non-degenerate symmetric bilinear form on $\mathbb{C}^n$. The orthogonal Grassmannian with the complex orthogonal group $G = SO_n$ action is

$$OG(k,n) := \{\text{k-dimensional isotropic subspace with respect to } q \text{ in } \mathbb{C}^n)\}.$$

We note this by (B_l, α_k) or (D_l, α_k) where $2l+1 = n$ or $2l = n$ resp.

Example

Let w be a non-degenerate anti-symmetric bilinear form on $\mathbb{C}^n$. The sympletic Grassmannian with the complex sympletic group $G = Sp_n$ action is defined as

$$G_w(k,n) := \{\text{k-dimensional isotropic subspace with respect to } w \text{ in } \mathbb{C}^n)\}.$$

We note this by (C_l, α_k) where $2l = n$.

Deformation rigidity problems

Let X be a homogeneous complex manifold of dimension $2n+1$ with a holomorphic contact form. Assume that Picard number of X is one. Then there exists exactly one such manifold corresponding to the class of simple Lie groups (B_l, α_2) for $l \geq 3$, (D_l, α_2) for $l \geq 4$ and the five exceptional types (E_6, α_2), (E_7, α_2), (E_8, α_2), (F_4, α_1), (G_2, α_2). These are irreducible rational homogeneous manifolds with Picard number one.

Theorem ('97 Hwang)

Let X is the homogeneous complex contact manifold with Picard number one except the orthogonal isotropic Grassmannian $OG(2,7)$. Then X is globally rigid.

The counter example is known by Pasquier and Perrin (2010)

Deformation rigidity problems

Conjecture ('97 Hwang)

For an irreducible rational homogeneous manifolds $X = G/P$ with Picard number one, the global deformation rigidity is true.

Theorem ('02 '04 '05 Hwang and Mok)

A rational homogeneous manifold G/P with Picard number one, different from the orthogonal isotropic Grassmannian $OG(2,7)$, is globally rigid.

Deformation rigidity problems for quasi-homogeneous varieties

Quasi-homogeneous variety

Definition

Let G be a complex reductive group. A G-variety X is called quasi-homogeneous if X is a complete projective variety with an open dense G-orbit.

Definition (Spherical variety)

Let G be a connected reductive algebraic group over $\mathbb{C}$. A normal variety X equipped with an action of G is called *spherical variety* if a Borel subgroup B of G acts on X with an open dense orbit.

Remark

A complete projective spherical G-variety is quasi-homogeneous.

Quasi-homogeneous variety

Definition (horospherical variety)

Let G be an reductive algebraic group. The homogeneous space G/H is called a *horospherical* if H contains an unipotent radical of a Borel subgroup of G. A G-variety X is called a *horospherical variety* if G acts with an open orbit of horospherical homogeneous space G/H.

Definition (symmetric variety)

Let G be an algebraic reductive group. Let θ be an involution and G^θ be the θ-fixed subgroup by the involution. Let H be the subgroup such that $(G^\theta)^0 \subset H \subset N_G(G^\theta)$ where ^{0} denote the neutral component. Then we call G/H the *symmetric homogeneous space*. A G-variety X is called a *symmetric variety* if G acts with an open orbit which is a symmetric homogeneous space G/H.

Ridigity of quasi-homogeneous variety

Question

Is quasi-homogeneous variety X (locally/globally) rigid? Could we find further counter examples of the global rigidity problem?

Conjecture (Hwang, Hong, Park and -)

Let $\pi\colon \mathcal{X} \to \Delta$ be a smooth projective morphism from $\mathcal{X}$ to the unit disc $\Delta \subset \mathbb{C}$. Suppose for any nonzero $t \in \Delta$, the fiber $\mathcal{X}_t = \pi^{-1}(t)$ is biholomorphic to a horospherical variety or symmetric variety X with Picard number one, then the central fiber $\mathcal{X}_0$ is biholomorphic to X, but with few exceptions.

Ridigity of quasi-homogeneous variety

Theorem('09 Pasquier)

Let X be a projective horospherical G-manifold of Picard number one. If X is nonhomogeneous, it is of rank one and its automorphism group is a connected non-reductive linear algebraic group. Moreover, X is uniquely determined by its two closed G-orbits Y and Z, isomorphic to rational homogeneous manifolds G/P^{α_i} and G/P^{α_j}, respectively, where (G, α_i, α_j) is one of the following triples:

(1) $X^1(n) := (B_n, \alpha_{n-1}, \alpha_n)$ with $n \geq 3$;
(2) $X^2 := (B_3, \alpha_1, \alpha_3)$;
(3) $X^3(n, k) := (C_n, \alpha_k, \alpha_{k-1})$ with $n \geq k \geq 2$;
(4) $X^4 := (F_4, \alpha_2, \alpha_3)$;
(5) $X^5 := (G_2, \alpha_2, \alpha_1)$.

Ridigity of quasi-homogeneous variety

Theorem ('10 Pasquier and Perrin)

A smooth projective horospherical variety of Picard number one is locally rigid except X^5.

Theorem ('16 Park)

The odd Lagrangian Grassmannian $X^3(n,n)$ *is globally rigid.*

Theorem ('21 Hwang and Li)

The odd-symplectic Grassmannian $X^3(n,k)$ *is globally rigid.*

Ridigity of quasi-homogeneous variety

Theorem ('22 Hwang and Li)

The G_2 horospherical variety of Picard number one X^5 is globally rigid.

Let $\pi\colon \mathcal{X} \to \Delta$ be a smooth projective morphism from $\mathcal{X}$ to the unit disc $\Delta \subset \mathbb{C}$. Suppose for any nonzero $t \in \Delta$, the fiber $\mathcal{X}_t = \pi^{-1}(t)$ is biholomorphic to orthogonal Grassmannian $OG(2,7)$, then the central fiber $\mathcal{X}_0$ is biholomorphic to either $OG(2,7)$ or X^5.

Theorem (working in progress, Hong and K.)

Let X be $X^1(n)$ with $n > 3$, X^2, or X^4. Then, X is globally rigid.

A smooth projective horospherical variety of Picard number one is globally rigid, except $X^1(3)$.

Ridigity of quasi-homogeneous variety

Let G be a semisimple and simply connected Lie group and $G/H = G/G^{\theta}$.

Theorem ('10 Ruzzi)

Let X be a non-homogeneous projective symmetric manifold with Picard number one. Then the restricted root system has type either G_2 or A_2. This symmetric manifolds X is an unique equivariant completion of G/G^{θ}, where G/G^{θ} is one of the following;

- $G_2/(SL_2 \times SL_2)$ *and* $G_2/(G_2 \times G_2)$ *for* G_2*-type*
- SL_3/SO_3 *(AI),* SL_3*,* SL_6/Sp_6 *(AII) and* E_6/F_4 *(EII) for* A_2*-type*

Ridigity of quasi-homogeneous variety

Theorem ('18, Manivel '19 Park and -.)

The Cayley Grassmannian is locally rigid.

Theorem ('19 Park and -. / '23 Fu, Li and Chen)

Every non-homogeneous projective symmetric manifold with Picard number one of type A_2 is locally rigid and globally rigid.

Theorem (21, Manivel)

The double Cayley Grassmannian is locally rigid.

Theorem ('24+ K.- and Park)

The double Cayley Grassmannian is globally rigid.

The global rigidity problem of the Cayley Grassmannian is still open.

VMRT

Variety of minimal rational tangents

Let X be a smooth projective variety.

Definition

An irreducible family $\mathcal{K}$ of irreducible rational curves on X is called *a covering family* if there exist a curve $C \in \mathcal{K}$ passing through a general point $x \in X$.

Definition

A covering family $\mathcal{K}$ is a *minimal* covering family of irreducible rational curves if the subfamily $\mathcal{K}_x$ of curves through x is proper for a general point $x \in X$

Variety of minimal rational tangents

Definition

For a covering family $\mathcal{K}$ on X, a rational map $\tau_x\colon \mathcal{K}_x \to \mathbb{P}(T_xX)$ that maps every curve $C \in K_x$ which is smooth at x, to its tangent line. This is called tangential map. The closure of the image of τ_x is denoted by $\mathcal{C}_x$ and called *variety of tangents of $\mathcal{K}$ at x*.

Theorem ('04 Mok and Hwang)

If $\mathcal{K}$ is minimal, then the normalization τ_x^n of τ_x

$$\tau_x^n\colon \mathcal{K}_x^n \to \mathbb{P}(T_xX)$$

is birational morphism to its image $\mathcal{C}_x$.

Definition

The image $\mathcal{C}_x$ of τ_x^n is called *variety of minimal rational tangents(VMRT) of $\mathcal{K}$ at x*.

Variety of minimal rational tangents

Example

For $X = \mathbb{P}^n$, the VMRT $\mathcal{C}_x$ is $\mathbb{P}^{n-1}$.

Example

For $X = \mathbf{Q^n}$, the VMRT $\mathcal{C}_x$ is $\mathbf{Q}^{n-2}$.

Example

For $X = Gr(k, n)$, the VMRT $\mathcal{C}_x$ is $\mathbb{P}^{k-1} \times \mathbb{P}^{n-k-1}$.

Let $X = G/P^\alpha$ be a rational homogeneous space associated with a long simple root α. Then $\mathcal{C}_o$ is $L^\alpha.v_\alpha$, where $o \in X$ is the base point, L^α is the Levi subgroup of P^α, and $v_o \in span(\mathcal{C}_o)$ is the highest weight vector as L^α-representation.

Variety of minimal rational tangents

Sketch of the proofs.

Let $\pi\colon \mathcal{X} \to \Delta$ is a smooth projective morphism. Assume that $\mathcal{X}_t$ is biholomorphic to a given model X for $t \neq 0$.

Step 1.(Deforamtion rigidity of VMRT) For a generic section σ_t of $\pi\colon \mathcal{X} \to \Delta$, show that the variety of minimal rational tangents $\mathcal{C}_{\sigma_0}(\mathcal{X}_0) \subset \mathbb{P}T_{\sigma_0}\mathcal{X}_0$ are projectively equivalent to the variety of minimal rational tangents $\mathcal{C}_{\sigma_t}(\mathcal{X}_t) \subset \mathbb{P}T_{\sigma_t}\mathcal{X}_t$ for $t \neq 0$.

Step 2. (Specialization/Characterization) From the equivalence of variety of minimal rational tangents $\mathcal{C}_{\sigma_0}(\mathcal{X}_0) \subset \mathbb{P}T_{\sigma_0}\mathcal{X}_0$ with the model $\mathcal{C}_x(X) \subset \mathbb{P}T_xX$ for generic $x \in X$, we show that X_0 is biholomorphic to X.

Varieties of minimal rational tangents

Assume that $X = G/P$ is not a hermitian symmetric nor a contact manifold and assume that P is a maximal parabolic subgroup of G associated with a long simple root. For $\pi\colon \mathcal{X} \to \Delta$, we assume that $\mathcal{X}_t$ is biholomorphic to X for $t \neq 0$ and let σ_t be a generic section of $\pi\colon \mathcal{X} \to \Delta$.

Step 1-1. show that $\mathcal{K}_{\sigma_t} \to \Delta$ is the trivial family.
Step 1-2. show that the variety of minimal rational tangents $\mathcal{C}_{\sigma_0}(\mathcal{X}_0) \subset \mathbb{P}T_{\sigma_0}\mathcal{X}_0$ is projectively equivalent to $\mathcal{C}_0 \subset \mathbb{P}T_oX$.

Step 2-1. From the equivalence of variety of minimal rational tangents $\mathcal{C}_{\sigma_0}(\mathcal{X}_0) \subset \mathbb{P}T_{\sigma_0}\mathcal{X}_0$ with $\mathcal{C}_0 \subset \mathbb{P}T_oX$, we have a symbol-algebra isomorphic to the nilpotent algebra $\bigoplus_{i>0} \mathfrak{g}_i$.
Step 2-2. By the works of Tanaka and Yamaguchi, we can construct a flat Cartan Connection on $\mathcal{X}_t$ near $t = 0$. This give us the local equivalence from a open subset of $\mathcal{X}_0$ to G/P.
Step 2-3. From the fact that $aut(\mathcal{X}_o) = \mathfrak{g}$, we get the conclusion.

Varieties of minimal rational tangents

Assume that X is either horospherical variety or symmetric variety of Picard number one.

For step 1, we use the known global rigidity and a method that is similar to that of Hermitian symmetric manifold of compact type (and also of the rational homogeneous manifold with Picard number one).

For non-homogeneous horospherical variety X of Picard number one, we need "Characterization" in Step 2-2.

For non-homogeneous symmetric variety X of Picard number one, we need "a prolongation method" using infinitesimal automorphism of VMRT for step 2 which firstly developed for G/P of short root cases by Hwang and Mok.

Step 3. We need to consider the boundary.

Varieties of minimal rational tangents

Question

Which quasi-homogeneous variety is locally and globally rigid? Can we replace Picard number one by Fano? Could we find further counter examples of the global rigidity problem? A VMRT also has a crucial role for the global rigidity of such a quasi-homogeneous variety?

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