# Rigidity of compact complex manifolds

Michel BRION

Institut Fourier, Université Grenoble Alpes

26 September 2024

Talk at the French-Korean Webinar on Algebraic and Analytic Geometry

# Families

## Definition

A **family of compact complex manifolds** is a holomorphic map of connected complex spaces $f : \mathcal{X} \to S$ which is proper, surjective, and submersive with connected fibers.

Then the fibers $\mathcal{X}_s$, $s \in S$, are (connected) compact complex manifolds.

The simplest example of a family is the projection $X \times S \to S$, where $X$ is a compact complex manifold. This is the **trivial family** with fiber $X$.

More generally, a family $f : \mathcal{X} \to S$ is **locally trivial** if every $s \in S$ has an open neighborhood $U = U_s$ such that the restriction $f^{-1}(U) \to U$ is the trivial family $\mathcal{X}_s \times U \to U$. Then all fibers $\mathcal{X}_s$ are biholomorphic.

Conversely, every family $f : \mathcal{X} \to S$ with biholomorphic fibers is locally trivial if $S$ is reduced, by a theorem of Fischer and Grauert.

# Families (continued)

By a theorem of Ehresmann, every family of compact complex manifolds $f : \mathcal{X} \to S$ is a locally trivial fibration of differentiable manifolds.

In particular, all the fibers $\mathcal{X}_s$ are diffeomorphic. But they are generally not biholomorphic :

## Example

Consider the family of all smooth curves of degree 3 in the projective plane $\mathbb{P}^2$ : the space of homogeneous equations of such curves is

$$S = \{f \in \mathbb{C}[x,y,z]_3 \quad | \quad D(f) \neq 0\}$$

where $D$ denotes the discriminant, we have

$$\mathcal{X} = \{(x:y:z), f) \in \mathbb{P}^2 \times S \quad | \quad f(x,y,z) = 0\},$$

and the map $f : \mathcal{X} \to S$ is the projection.
The fibers of $f$ are elliptic curves. They are all diffeomorphic to $\mathbb{C}/\mathbb{Z} + i\mathbb{Z}$, but not biholomorphic.

# Notions of rigidity

Let $X$ be a compact complex manifold.

### Definition

A **deformation** of $X$ over a connected complex space $S$ is a family of compact complex manifolds $f : \mathcal{X} \to S$, equipped with a point $s_0 \in S$ such that the fiber $\mathcal{X}_0$ is biholomorphic to $X$.

We then write $f : (\mathcal{X}, X) \to (S, s_0)$, and $\mathcal{X}_0 \simeq X$ for simplicity.

### Definition

We say that $X$ is **locally rigid** if each deformation $f : (\mathcal{X}, X) \to (S, s_0)$ is locally trivial at $s_0$.

If $S$ is reduced, this is equivalent to the condition that $\mathcal{X}_s \simeq X$ for any $s$ in a neighborhood of $s_0$.

### Definition

We say that $X$ is **globally rigid** if for each family $f : \mathcal{X} \to \Delta$ such that $\mathcal{X}_s \simeq X$ for all $s \neq 0$, we have $\mathcal{X}_0 \simeq X$.

# A local rigidity criterion

## Theorem

*Let $X$ be a compact complex manifold with tangent bundle $T_X$.*

- *If $H^1(X, T_X) = 0$ then $X$ is locally rigid.*
- *Conversely, if $H^1(X, T_X) \neq 0$ and $H^2(X, T_X) = 0$ then $X$ is not locally rigid.*

The idea of the proof is that $H^1(X, T_X)$ parameterizes the infinitesimal deformations of $X$, and the obstructions to lifting such deformations to global ones live in $H^2(X, T_X)$.

## Example

Let $X$ be a compact Riemann surface of genus $g$.
Then $H^2(X, T_X) = 0$. Moreover, $H^1(X, T_X) = H^0\left(X, (\Omega^1_X)^{\otimes 2}\right)^*$ by Serre duality. Thus, $H^1(X, T_X) \neq 0$ if $g \geq 1$ and hence $X$ is not locally rigid under this assumption.

If $g = 0$ then $X \simeq \mathbb{P}^1$ and this identifies $T_X$ with $\mathcal{O}_{\mathbb{P}^1}(2)$. So $X$ is locally rigid. It is also globally rigid, since the genus is constant in families.

# Rigidity of rational homogeneous manifolds

- The projective space $\mathbb{P}^n$ is locally rigid.
  Indeed, the exact sequence

  $$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n+1)} \longrightarrow T_{\mathbb{P}^n} \longrightarrow 0$$

  yields that $H^1(\mathbb{P}^n, T_{P^n}) = 0$.
- As a consequence, every product of projective spaces is locally rigid.
- More generally, let $X$ be a rational homogeneous manifold, i.e., a homogeneous space $G/P$ where $G$ is a complex semisimple Lie group, and $P \subset G$ a parabolic subgroup. Then $H^1(X, T_X) = 0$ by a theorem of Bott, so $X$ is locally rigid as well.
- We will see in this talk that $\mathbb{P}^n$ is globally rigid, but $\mathbb{P}^n \times \mathbb{P}^n$ is not.
- The global rigidity of rational homogeneous manifolds will be discussed in the next talk.

# A general construction of deformations

Consider a compact complex manifold $Y$, and an exact sequence of complex vector bundles

$$0 \longrightarrow E_1 \longrightarrow E \longrightarrow E_2 \longrightarrow 0$$

on $Y$. Let

$$X = \mathbb{P}(E_1 \oplus E_2)$$

(the projectivization of the vector bundle $E_1 \oplus E_2$). Then there exists a deformation

$$f : (\mathcal{X}, X) \longrightarrow (\mathbb{C}, 0)$$

such that $\mathcal{X}_s \simeq \mathbb{P}(E)$ for all $s \neq 0$.

Indeed, the pushout of the above exact sequence by the multiplication map $E_1 \to E_1$, $v \mapsto sv$ yields a vector bundle $\mathcal{E} \to Y \times \mathbb{C}$ such that $\mathcal{E}_{Y \times s} \simeq E$ for all $s \neq 0$, and $\mathcal{E}_{Y \times 0} \simeq E_1 \oplus E_2$. We then take $\mathcal{X} = \mathbb{P}(\mathcal{E})$.

# Examples of deformations

1) Take $Y = \mathbb{P}^1$ with homogeneous coordinates $x_0, x_1$ and consider the exact sequence of vector bundles

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-2) \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \stackrel{(x_0,x_1)}{\longrightarrow} \mathcal{O}_{\mathbb{P}^1} \longrightarrow 0.$$

The above construction yields a family $f : \mathcal{X} \to \mathbb{C}$ with general fibers (over $\mathbb{C} \setminus 0$)

$$\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)) \simeq \mathbb{P}^1 \times \mathbb{P}^1$$

and central fiber (over 0)

$$\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)) \simeq \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}).$$

This is the Hirzebruch surface $\mathbb{F}_2$, which contains a unique compact complex curve with self-intersection $-2$. So $\mathbb{F}_2 \not\simeq \mathbb{P}^1 \times \mathbb{P}^1$.

We conclude that $\mathbb{P}^1 \times \mathbb{P}^1$ is not globally rigid.

# Examples of deformations (continued)

2) More generally, for any integer $n \geq 0$, we have an exact sequence of vector bundles over $\mathbb{P}^1$

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-n-2) \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-n-1) \xrightarrow{(x_0, x_1^{n+1})} \mathcal{O}_{\mathbb{P}^1} \longrightarrow 0$$

and hence a family over $\mathbb{C}$ with general fiber

$$\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-n-1)) \simeq \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(n) \oplus \mathcal{O}_{\mathbb{P}^1}) = \mathbb{F}_n$$

and central fiber

$$\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-n-2) \oplus \mathcal{O}_{\mathbb{P}^1}) \simeq \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(n+2) \oplus \mathcal{O}_{\mathbb{P}^1}) = \mathbb{F}_{n+2}.$$

These two surfaces are not biholomorphic, as every Hirzebruch surface $\mathbb{F}_n$ (where $n \geq 1$) admits a unique birational contraction and its exceptional divisor is a curve of self-intersection $-n$.

Thus, $\mathbb{F}_n$ is not globally rigid.

# A further example

3) Take $Y = \mathbb{P}^n$ and consider the Euler sequence

$$0 \longrightarrow \Omega^1_{\mathbb{P}^n} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus(n+1)} \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow 0.$$

This exact sequence of vector bundles yields a family of compact complex manifolds over $\mathbb{C}$ with general fiber

$$\mathbb{P}(\mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus(n+1)}) \simeq \mathbb{P}^n \times \mathbb{P}^n$$

and special fiber $\mathbb{P}(\Omega^1_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n})$. The latter is not biholomorphic to $\mathbb{P}^n \times \mathbb{P}^n$ as it admits a non-trivial birational contraction.

Indeed, there is a natural map from $\Omega^1_{\mathbb{P}^n}$ to $M_{n+1}$ (the space of complex matrices of size $(n+1) \times (n+1)$) which is linear on fibers and birational to its image, the variety of nilpotent matrices of rank at most 1. Moreover, this map can be compactified to a map $\mathbb{P}(\Omega^1_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}) \to \mathbb{P}(M_{n+1} \oplus \mathbb{C})$.

As a consequence, $\mathbb{P}^n \times \mathbb{P}^n$ is not globally rigid.

# A characterization of the projective space

The following result implies that $\mathbb{P}^n$ is globally rigid for deformations with Kähler central fiber :

## Theorem

*Let $X$ be a compact Kähler manifold homeomorphic to $\mathbb{P}^n$. Then $X \simeq \mathbb{P}^n$.*

This result is due to Hirzebruch and Kodaira, under additional assumptions which were later removed in work of Novikov and Yau. We sketch some steps in the proof :

The cohomology ring $H^*(X, \mathbb{Z})$ is isomorphic to $H^*(\mathbb{P}^n, \mathbb{Z}) \simeq \mathbb{Z}[t]/(t^{n+1})$, where $t$ has degree 2. Using Hodge theory, it follows that $H^p(X, \Omega_X^q) = 0$ for all $p \neq q$. In particular, $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$, and hence the Chern class map $\mathrm{Pic}(X) \to H^2(X, \mathbb{Z})$ is an isomorphism.

Let $g \in H^2(X, \mathbb{R})$ be the class of a Kähler form on $X$. By rescaling, we may assume that $g$ is a generator of $H^2(X, \mathbb{Z}) \simeq H^2(\mathbb{P}^n, \mathbb{Z}) \simeq \mathbb{Z}$. Thus, $X$ is projective algebraic, and has an ample line bundle $L$ with first Chern class $g$.

# Characterization of the projective space (continued)

An argument based on the Hirzebruch-Riemann-Roch theorem shows that $h^0(X, L) = n + 1$. This is the most technical step of the paper by Hirzebruch and Kodaira. It works either if $n$ is odd, or if $n$ is even and $c_1(X) \neq (n + 1)g$.

Under these assumptions, the proof is completed by the following result of Kobayashi and Ochiai :

Let $X$ be a projective manifold of dimension $n$, and $L$ an ample line bundle on $X$. If $h^0(X, L) \geq n + 1$ and $L^n = 1$ (where $L^n$ denotes the self-intersection number), then $(X, L) \simeq (\mathbb{P}^n, \mathcal{O}_{P^n}(1))$.

It remains to exclude the case where $n$ is even and $c_1(X) = (n + 1)g$. But then $X$ is of general type, and a result of Yau implies that $X$ is the quotient of the ball $\mathbb{B}^n$ by a discrete group of isometries. As $X$ is simply-connected and $\mathbb{B}^n$ is non-compact, this yields a contradiction.

The global rigidity of $\mathbb{P}^n$ (without assuming that the central fiber is Kähler) is due to Siu via very different arguments.

# Some references


- E. Brieskorn, *Ein Satz über die komplexen Quadriken*, Math. Ann. **155** (1964), 184–193.
- D. Greb, S. Kebekus, T. Peternell, *Miyaoka-Yau inequalities and the topological characterization of certain klt varieties*, C. R. Math. Acad. Sci. Paris **362** (2024), 141–157.
- F. Hirzebruch, K. Kodaira, *On the complex projective spaces*, J. Math. Pures Appl. **36** (1957), 201–216.
- S. Kobayashi, T. Ochiai, *Characterizations of complex projective spaces and hyperquadrics*, J. Math. Kyoto Univ. **13** (1973), 31–47.
- Y.-T. Siu, *Nondeformability of the projective space*, J. Reine Angewandte Math. **399** (1989), 208–219.
- S. T. Yau, *Calabi's conjecture and some new results in algebraic geometry*, Proc. Natl. Acad. Sci. USA **74** (1977), 1798–1799.