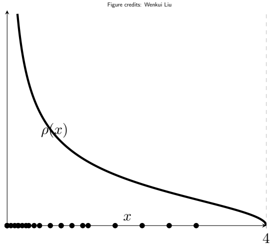
A universal interacting particle system in discrete random matrix theory

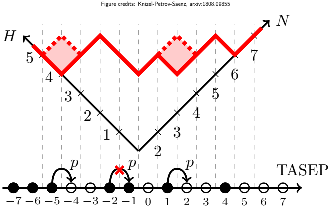
Roger Van Peski (KTH $\to$ Columbia)

PIICQ online seminar
17 June 2024

Based on https://arxiv.org/abs/2310.12275, https://arxiv.org/abs/2312.11702

Random matrices and interacting particle systems



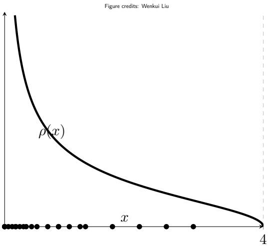


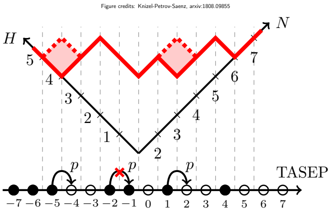
Random matrix theory over $\mathbb{R}, \mathbb{C}$

Discrete-space interacting particle systems (TASEP, ASEP, ...)

Many limit objects (e.g. Tracy-Widom) appear in both.

Random matrices and interacting particle systems





Random matrix theory over $\mathbb{R}, \mathbb{C}$

Discrete-space interacting particle systems (TASEP, ASEP, ...)

Many limit objects (e.g. Tracy-Widom) appear in both.

There is also discrete random matrix theory (over $\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}$, etc.):

On the singularity probability of discrete random matrices

Jean Bourgain[a], Van H. Vu[b,*,1], Philip Matchett Wood[b]

ON THE PROBABILITY THAT A RANDOM ±1-MATRIX IS SINGULAR

JEFF KAHN, JÁNOS KOMLÓS, AND ENDRE SZEMERÉDI

MODELING THE DISTRIBUTION OF RANKS, SELMER GROUPS, AND SHAFAREVICH–TATE GROUPS OF ELLIPTIC CURVES

MANJUL BHARGAVA, DANIEL M. KANE, HENDRIK W. LENSTRA JR., BJORN POONEN, AND ERIC RAINS

Today: interacting particle systems as *universal limits* in discrete RMT

Aperitif: $\mathbb{Z}/p\mathbb{Z}$ case

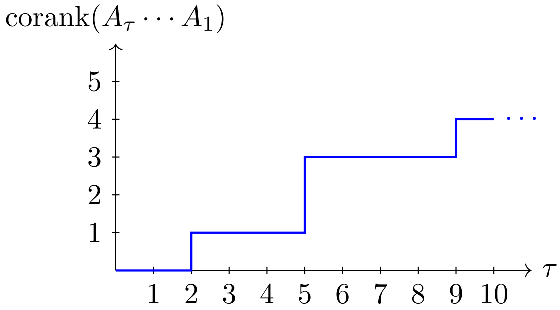
Random groups and random matrices

The reflecting Poisson sea and limit theorems

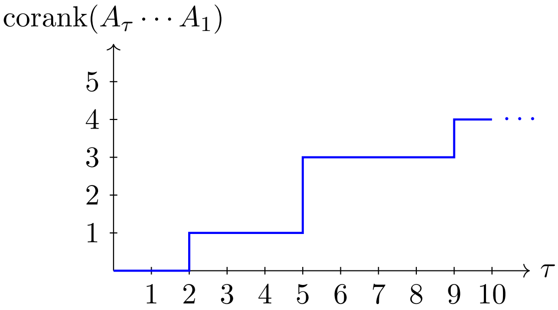
Integrability and symmetric functions

Aperitif: $\mathbb{Z}/p\mathbb{Z}$ case

Let $A_1, A_2, \ldots$ be iid uniform in $\mathrm{Mat}_N(\mathbb{Z}/p\mathbb{Z})$. What does the distribution of $\mathrm{rank}(A_\tau \cdots A_2 A_1)$ look like for large N and τ?



Let $A_1, A_2, \ldots$ be iid uniform in $\mathrm{Mat}_N(\mathbb{Z}/p\mathbb{Z})$. What does the distribution of $\mathrm{rank}(A_\tau \cdots A_2 A_1)$ look like for large N and τ?



Fact: $\mathrm{corank}(A_\tau \cdots A_1) \approx \log_p \tau$, finite limit fluctuations.

Question 1: What is the limit of

$$\mathrm{corank}(A_\tau \cdots A_1) - \log_p \tau$$

as $N, \tau \to \infty$?.

An intriguing random integer

Definition

For any $\chi \in \mathbb{R}_{>0}, t \in (0,1)$, $\mathcal{L}_{1,t,\chi}$ *is the* $\mathbb{Z}$*-valued random variable given by*

$$\Pr(\mathcal{L}_{1,t,\chi} = x) = \frac{1}{\prod_{i\geq 1}(1-t^i)} \sum_{j\geq 0} e^{-\chi t^{x-j}} \frac{(-1)^j t^{\binom{j}{2}}}{\prod_{i=1}^{j}(1-t^i)}$$

for any $x \in \mathbb{Z}$.

An intriguing random integer

Definition

For any $\chi \in \mathbb{R}_{>0}, t \in (0,1)$, $\mathcal{L}_{1,t,\chi}$ is the $\mathbb{Z}$-valued random variable given by

$$\Pr(\mathcal{L}_{1,t,\chi} = x) = \frac{1}{\prod_{i\geq 1}(1-t^i)} \sum_{j\geq 0} e^{-\chi t^{x-j}} \frac{(-1)^j t^{\binom{j}{2}}}{\prod_{i=1}^{j}(1-t^i)}$$

for any $x \in \mathbb{Z}$.

Equivalently, $X = \chi^{-1} t^{-\mathcal{L}_{1,t,\chi}}$ solves the indeterminate Stieltjes moment problem

$$\mathbb{E}[X^m] = \frac{t^{-\binom{m+1}{2}}(t;t)_m}{m!}, \qquad m = 0, 1, 2, \ldots$$

and is the unique solution supported on $\{\chi^{-1} t^n : n \in \mathbb{Z}\}$.

Want a limit of $\operatorname{corank}(A_\tau \cdots A_1) - \log_p \tau$, but this is not an integer.

Want a limit of $\operatorname{corank}(A_\tau \cdots A_1) - \log_p \tau$, but this is not an integer.

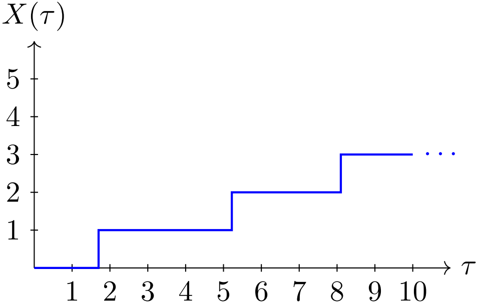
Theorem (VP '23, special case)

For each $N \geq 1$ take $A_1^{(N)}, A_2^{(N)}, \ldots$ iid uniform in $\operatorname{Mat}_N(\mathbb{Z}/p\mathbb{Z})$. Then as $N \to \infty$,

$$\operatorname{corank}(A_{\tau_N}^{(N)} \cdots A_1^{(N)}) - \operatorname{Int}(\log_p \tau_N + \zeta) \to \mathcal{L}_{1,p^{-1},p^{-\zeta}/(p-1)}$$

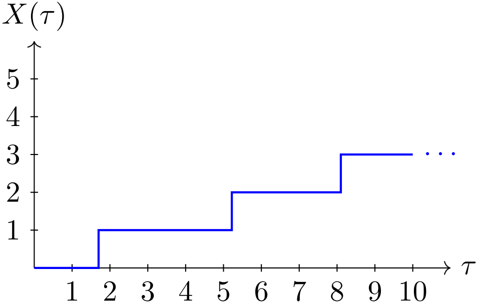
for any sequence $\tau_N, N \geq 1$ s.t. $1 \ll \tau_N \ll p^N$ and the fractional part $\{-\log_p \tau_N\}$ converges to some $\zeta \in [0,1]$.

Let $t \in (0,1)$ and $X(\tau), \tau \in \mathbb{R}_{\geq 0}$ be the $\mathbb{Z}_{\geq 0}$-valued process which jumps by 1, and waits at $x \in \mathbb{Z}_{\geq 0}$ for an $\mathrm{Exp}(t^x)$-distributed time.



Question 2: What are the limiting fluctuations of $X(\tau)$?

Let $t \in (0,1)$ and $X(\tau), \tau \in \mathbb{R}_{\geq 0}$ be the $\mathbb{Z}_{\geq 0}$-valued process which jumps by 1, and waits at $x \in \mathbb{Z}_{\geq 0}$ for an $\mathrm{Exp}(t^x)$-distributed time.



Question 2: What are the limiting fluctuations of $X(\tau)$?

Theorem (VP '23, special case)

$$X(\tau) - (\log_{t^{-1}} \tau + \zeta) \to \mathcal{L}_{1,t,t^{\zeta+1}/(1-t)}$$

in distribution as $\tau \to \infty$ *along the sequence* $\tau \in t^{-\mathbb{N}+\zeta}$.

Random groups and random matrices

In many contexts we want asymptotics of a (pseudo-)random finite abelian group G,

$$G = \bigoplus_{p \text{ prime}} G_p$$
$$G_p = \bigoplus_i \mathbb{Z}/p^{\lambda_i^{(p)}}\mathbb{Z},$$

such as:

- Arithmetic statistics—distributions of class groups of number fields, Tate-Shafarevich groups—(Bhargava, Cohen, Ellenberg, Kane, Lenstra Jr., Nguyen, Poonen, Rains, Sawin, Venkatesh, Westerland, Wood... '83-present),
- Sandpile groups of random graphs—(Clancy, Fulman, Kaplan, Koplewitz, Leake, Nguyen, Payne, Wood... '14-present),
- (co)homology groups of random chain complexes—(Kahle, Lutz, Meszaros, Newman, Parsons,...)

Random matrices produce random groups

The first such distribution (1983) was the *Cohen-Lenstra distribution* on abelian p-groups $G = \bigoplus_i \mathbb{Z}/p^{\lambda_i}\mathbb{Z}$ given by

$$\Pr(G) = \frac{\prod_{i \geq 1}(1 - 1/p^i)}{|\operatorname{Aut}(G)|},$$

observed empirically for p-parts of class groups of fields $\mathbb{Q}(\sqrt{-d})$.

Random matrices produce random groups

The first such distribution (1983) was the *Cohen-Lenstra distribution* on abelian p-groups $G = \bigoplus_i \mathbb{Z}/p^{\lambda_i}\mathbb{Z}$ given by

$$\Pr(G) = \frac{\prod_{i \geq 1}(1 - 1/p^i)}{|\operatorname{Aut}(G)|},$$

observed empirically for p-parts of class groups of fields $\mathbb{Q}(\sqrt{-d})$.
$A \in \operatorname{Mat}_N(\mathbb{Z})$ is a linear map $A : \mathbb{Z}^N \to \mathbb{Z}^N$,

$$\operatorname{cok}(A) := \mathbb{Z}^N / A\mathbb{Z}^N.$$

Random matrices produce random groups

The first such distribution (1983) was the *Cohen-Lenstra distribution* on abelian p-groups $G = \bigoplus_i \mathbb{Z}/p^{\lambda_i}\mathbb{Z}$ given by

$$\Pr(G) = \frac{\prod_{i \geq 1}(1 - 1/p^i)}{|\operatorname{Aut}(G)|},$$

observed empirically for p-parts of class groups of fields $\mathbb{Q}(\sqrt{-d})$.
$A \in \operatorname{Mat}_N(\mathbb{Z})$ is a linear map $A : \mathbb{Z}^N \to \mathbb{Z}^N$,

$$\operatorname{cok}(A) := \mathbb{Z}^N / A\mathbb{Z}^N.$$

For *random* A, $\operatorname{cok}(A)$ is a *random* abelian group.

Random matrices produce random groups

The first such distribution (1983) was the *Cohen-Lenstra distribution* on abelian p-groups $G = \bigoplus_i \mathbb{Z}/p^{\lambda_i}\mathbb{Z}$ given by

$$\Pr(G) = \frac{\prod_{i \geq 1}(1 - 1/p^i)}{|\operatorname{Aut}(G)|},$$

observed empirically for p-parts of class groups of fields $\mathbb{Q}(\sqrt{-d})$.
$A \in \operatorname{Mat}_N(\mathbb{Z})$ is a linear map $A : \mathbb{Z}^N \to \mathbb{Z}^N$,

$$\operatorname{cok}(A) := \mathbb{Z}^N / A\mathbb{Z}^N.$$

For *random* A, $\operatorname{cok}(A)$ is a *random* abelian group.

Can probe $\operatorname{cok}(A)$ by reducing mod p^k:

$$\operatorname{cok}(A \pmod{p^k}) \cong \operatorname{cok}(A)/p^k \operatorname{cok}(A).$$

Now $A \pmod{p^k} \in \operatorname{Mat}_N(\mathbb{Z}/p^k\mathbb{Z})$, a finite set.

Friedman-Washington: a random matrix explanation?

Theorem (Friedman-Washington 1987)

Let $A^{(N)} \in \mathrm{Mat}_N(\mathbb{Z}/p^k\mathbb{Z})$ *have iid uniform entries. Then as* $N \to \infty$, $\mathrm{cok}(A^{(N)})$ *limits to* G/p^kG, *where* G *has Cohen-Lenstra distribution* $\Pr(G) = \prod_{i \geq 1}(1-p^{-i})/|\mathrm{Aut}(G)|$.

Friedman-Washington: a random matrix explanation?

Theorem (Friedman-Washington 1987)

Let $A^{(N)} \in \mathrm{Mat}_N(\mathbb{Z}/p^k\mathbb{Z})$ *have iid uniform entries. Then as* $N \to \infty$, $\mathrm{cok}(A^{(N)})$ *limits to* G/p^kG, *where* G *has Cohen-Lenstra distribution* $\Pr(G) = \prod_{i \geq 1}(1 - p^{-i})/|\mathrm{Aut}(G)|$.

Fix a prime p. $\mathbb{Z}_p := \varprojlim \mathbb{Z}/p^k\mathbb{Z}$, concretely

$$\mathbb{Z}_p = \{a_0 + a_1 p + a_2 p^2 + \ldots : a_i \in \{0, \ldots, p-1\}\}.$$

Friedman-Washington: a random matrix explanation?

Theorem (Friedman-Washington 1987)

Let $A^{(N)} \in \mathrm{Mat}_N(\mathbb{Z}/p^k\mathbb{Z})$ *have iid uniform entries. Then as* $N \to \infty$, $\mathrm{cok}(A^{(N)})$ *limits to* G/p^kG*, where* G *has Cohen-Lenstra distribution* $\Pr(G) = \prod_{i \geq 1}(1-p^{-i})/|\mathrm{Aut}(G)|$.

Fix a prime p. $\mathbb{Z}_p := \varprojlim \mathbb{Z}/p^k\mathbb{Z}$, concretely

$$\mathbb{Z}_p = \{a_0 + a_1 p + a_2 p^2 + \ldots : a_i \in \{0, \ldots, p-1\}\}.$$

$(\mathbb{Z}_p, +)$ has Haar probability measure $\mu_{Haar}^{\mathbb{Z}_p}$, explicitly given by taking $a_i \in \{0, \ldots, p-1\}$ iid uniformly random, projects to uniform on $\mathbb{Z}/p^k\mathbb{Z}$ for any k.

Cohen-Lenstra universality and prime decoupling

Theorem (Friedman-Washington 1987, full version)

Let $A^{(N)} \in \mathrm{Mat}_N(\mathbb{Z}_p)$ have iid additive Haar entries. Then as $N \to \infty$, $\mathrm{cok}(A^{(N)})$ limits to the Cohen-Lenstra distribution $\Pr(G) = \prod_{i \geq 1}(1 - p^{-i})/|\mathrm{Aut}(G)|$.

Cohen-Lenstra universality and prime decoupling

Theorem (Friedman-Washington 1987, full version)

Let $A^{(N)} \in \mathrm{Mat}_N(\mathbb{Z}_p)$ have iid additive Haar entries. Then as $N \to \infty$, $\mathrm{cok}(A^{(N)})$ limits to the Cohen-Lenstra distribution $\Pr(G) = \prod_{i \geq 1}(1 - p^{-i})/|\mathrm{Aut}(G)|$.

For integer matrices A, cokernel is sum of its p-parts:

$$\mathrm{cok}(A) \cong \bigoplus_{p \text{ prime}} \mathrm{cok}(A)_p.$$

Theorem (Wood 2015)

Let $A^{(N)} \in \mathrm{Mat}_N(\mathbb{Z})$ have iid entries from any distribution which is nonconstant modulo p. Then $\mathrm{cok}(A^{(N)})_p$ converges to the Cohen-Lenstra distribution.

Cohen-Lenstra universality and prime decoupling

Theorem (Friedman-Washington 1987, full version)

Let $A^{(N)} \in \mathrm{Mat}_N(\mathbb{Z}_p)$ have iid additive Haar entries. Then as $N \to \infty$, $\mathrm{cok}(A^{(N)})$ limits to the Cohen-Lenstra distribution $\Pr(G) = \prod_{i \geq 1}(1 - p^{-i})/|\mathrm{Aut}(G)|$.

For integer matrices A, cokernel is sum of its p-parts:

$$\mathrm{cok}(A) \cong \bigoplus_{p \text{ prime}} \mathrm{cok}(A)_p.$$

Theorem (Wood 2015)

Let $A^{(N)} \in \mathrm{Mat}_N(\mathbb{Z})$ have iid entries from any distribution which is nonconstant modulo p. Then $\mathrm{cok}(A^{(N)})_p$ converges to the Cohen-Lenstra distribution.

"Primes decouple, $\mathbb{Z}_p$ case has complete information about $\mathbb{Z}$"

Groups and singular numbers

Proposition (Smith normal form)

For nonsingular $A \in \mathrm{Mat}_N(\mathbb{Z}_p)$*, there are* $U, V \in \mathrm{GL}_N(\mathbb{Z}_p)$ *for which*

$$UAV = \mathrm{diag}(p^{\lambda_1}, \ldots, p^{\lambda_N})$$

for singular numbers $\lambda_i \in \mathbb{Z}_{\geq 0}$ *(unique).*

(Like singular value decomposition, $\mathrm{GL}_N(\mathbb{Z}_p)$ replacing $O(N), U(N)$).

Groups and singular numbers

Proposition (Smith normal form)

For nonsingular $A \in \mathrm{Mat}_N(\mathbb{Z}_p)$, there are $U, V \in \mathrm{GL}_N(\mathbb{Z}_p)$ for which

$$UAV = \mathrm{diag}(p^{\lambda_1}, \ldots, p^{\lambda_N})$$

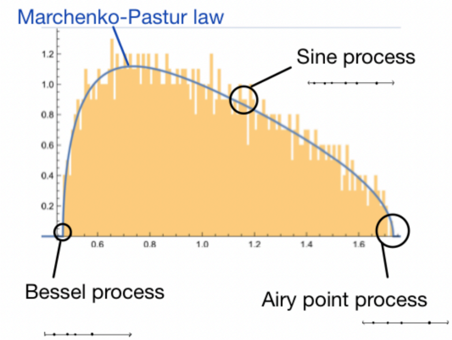
for singular numbers $\lambda_i \in \mathbb{Z}_{\geq 0}$ (unique).

(Like singular value decomposition, $\mathrm{GL}_N(\mathbb{Z}_p)$ replacing $O(N), U(N)$).

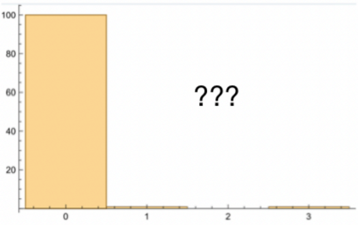
Write $\mathrm{SN}(A) = (\mathrm{SN}(A)_1, \ldots, \mathrm{SN}(A)_N) := (\lambda_1, \ldots, \lambda_N)$ above. Note

$$\mathrm{cok}(A) \cong \bigoplus_{1 \leq i \leq N} \mathbb{Z}/p^{\lambda_i}\mathbb{Z}$$

At a probabilistic level things look quite different



Histogram of singular values of a single $10^3 \times 10^4$ Ginibre (iid Gaussian) matrix



Histogram of singular numbers of a single 100×100 iid additive Haar matrix

The real/complex product process

Can study singular values of $A_\tau A_{\tau-1} \cdots A_1$ for A_i $N \times N$ random real/complex matrices, $\tau = 1, 2, \ldots$ [Bellman 1954].

The real/complex product process

Can study singular values of $A_\tau A_{\tau-1} \cdots A_1$ for A_i $N \times N$ random real/complex matrices, $\tau = 1, 2, \ldots$ [Bellman 1954].

For fixed N, the logarithms of singular values have Gaussian fluctuations as $\tau \to \infty$ ([Furstenberg-Kesten 1960], [Akemann-Burda-Kieburg 2014], [Liu-Wang-Wang 2018]).

The real/complex product process

Can study singular values of $A_\tau A_{\tau-1} \cdots A_1$ for A_i $N \times N$ random real/complex matrices, $\tau = 1, 2, \ldots$ [Bellman 1954].

For fixed N, the logarithms of singular values have Gaussian fluctuations as $\tau \to \infty$ ([Furstenberg-Kesten 1960], [Akemann-Burda-Kieburg 2014], [Liu-Wang-Wang 2018]).

When $N, \tau \to \infty, N/\tau \to c \in (0, \infty)$, the bulk (resp. soft edge) statistics are c-parametrized deformations of sine (resp. Airy) kernel ([Akemann-Burda-Kieburg 2018], [Liu-Wang-Wang 2018]).

The real/complex product process

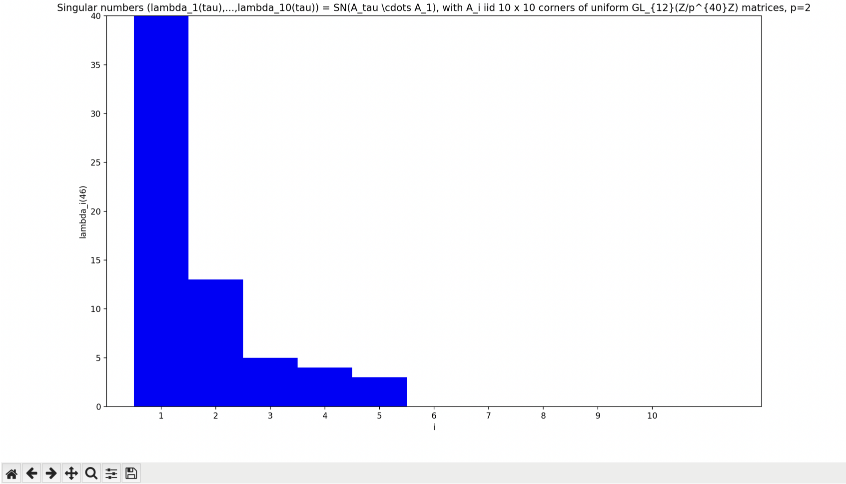
Can study singular values of $A_\tau A_{\tau-1} \cdots A_1$ for A_i $N \times N$ random real/complex matrices, $\tau = 1, 2, \ldots$ [Bellman 1954].

For fixed N, the logarithms of singular values have Gaussian fluctuations as $\tau \to \infty$ ([Furstenberg-Kesten 1960], [Akemann-Burda-Kieburg 2014], [Liu-Wang-Wang 2018]).

When $N, \tau \to \infty, N/\tau \to c \in (0, \infty)$, the bulk (resp. soft edge) statistics are c-parametrized deformations of sine (resp. Airy) kernel ([Akemann-Burda-Kieburg 2018], [Liu-Wang-Wang 2018]).

Question: How do singular numbers of matrix products behave over $\mathbb{Z}_p$?

Visualizing singular numbers

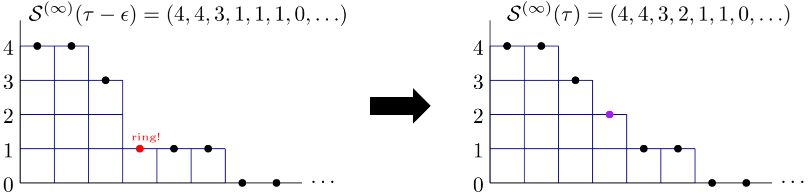


The reflecting Poisson sea and limit theorems

Particle positions $\mathcal{S}^{(\infty)}(\tau) = (\mathcal{S}^{(\infty)}(\tau)_1, \mathcal{S}^{(\infty)}(\tau)_2, \ldots), \tau \in \mathbb{R}_{\geq 0}$.

Starts at $\mathcal{S}^{(\infty)}(0) = (0, 0, \ldots)$.

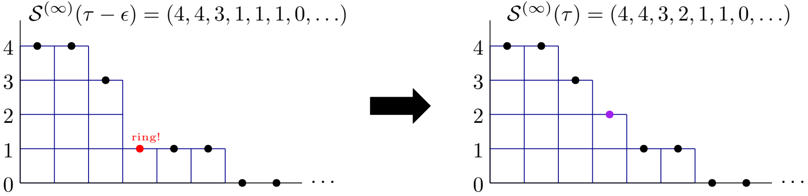
Indep. exp. clocks at $1, 2, \ldots$ of rates $t, t^2, \ldots$ control jumps.



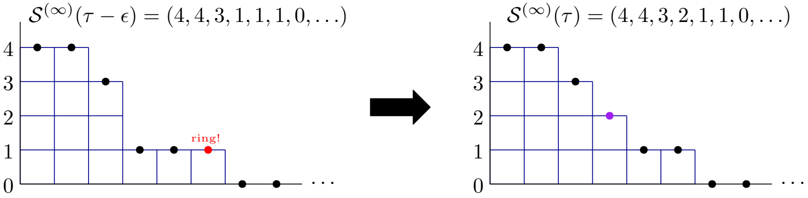
Particle positions $\mathcal{S}^{(\infty)}(\tau) = (\mathcal{S}^{(\infty)}(\tau)_1, \mathcal{S}^{(\infty)}(\tau)_2, \ldots), \tau \in \mathbb{R}_{\geq 0}$.

Starts at $\mathcal{S}^{(\infty)}(0) = (0, 0, \ldots)$.

Indep. exp. clocks at $1, 2, \ldots$ of rates $t, t^2, \ldots$ control jumps.



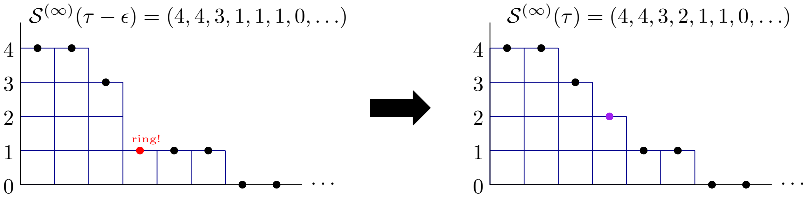
When particle is blocked, donates jump:



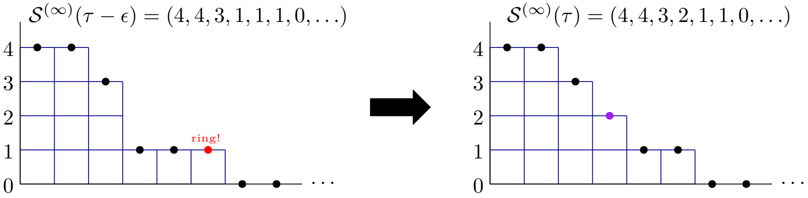
Particle positions $\mathcal{S}^{(\infty)}(\tau) = (\mathcal{S}^{(\infty)}(\tau)_1, \mathcal{S}^{(\infty)}(\tau)_2, \ldots), \tau \in \mathbb{R}_{\geq 0}$.

Starts at $\mathcal{S}^{(\infty)}(0) = (0, 0, \ldots)$.

Indep. exp. clocks at $1, 2, \ldots$ of rates $t, t^2, \ldots$ control jumps.

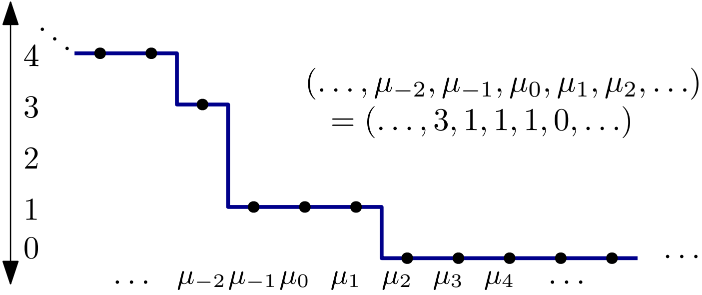


When particle is blocked, donates jump:



1^{st} row's length is $X(\frac{t}{1-t}\tau)$ (jump rate $t^{x+1} + t^{x+2} + \ldots = \frac{t}{1-t}t^x$).

The reflecting Poisson sea

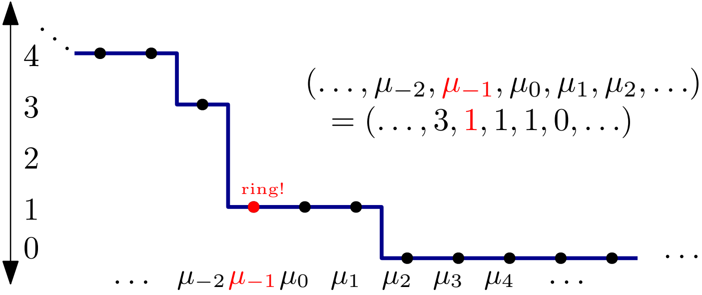


$$(\dots,\mu_{-2},\mu_{-1},\mu_0,\mu_1,\mu_2,\dots)$$
$$=(\dots,3,1,1,1,0,\dots)$$

Definition (VP 2023)

The reflecting Poisson sea $\mu(T)=(\dots,\mu_{-1}(T),\mu_0(T),\mu_1(T),\dots)$, $T\geq 0$ *is the continuous-time stochastic process with each* $\mu_i(T)$ *increasing by* 1 *according to rate-*t^i *exponential clock (independent of each other), donating move if blocked.*

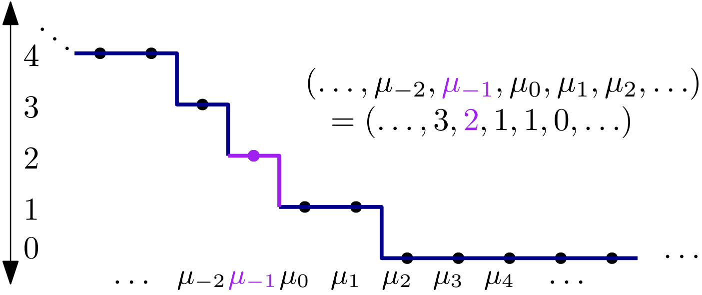
The reflecting Poisson sea



Definition (VP 2023)

The reflecting Poisson sea $\mu(T) = (\dots, \mu_{-1}(T), \mu_0(T), \mu_1(T), \dots)$, $T \geq 0$ *is the continuous-time stochastic process with each* $\mu_i(T)$ *increasing by* 1 *according to rate-*t^i *exponential clock (independent of each other), donating move if blocked.*

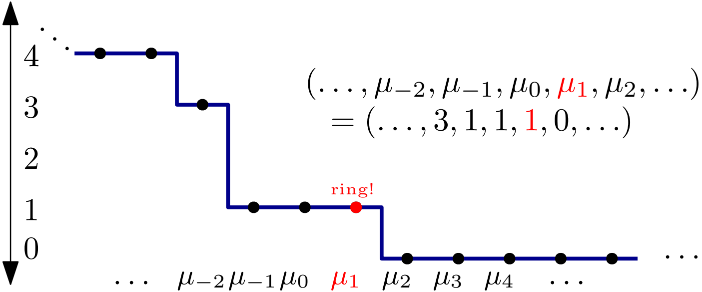
The reflecting Poisson sea



Definition (VP 2023)

The reflecting Poisson sea $\mu(T) = (\ldots, \mu_{-1}(T), \mu_0(T), \mu_1(T), \ldots)$, $T \geq 0$ *is the continuous-time stochastic process with each* $\mu_i(T)$ *increasing by* 1 *according to rate-*t^i *exponential clock (independent of each other), donating move if blocked.*

The reflecting Poisson sea

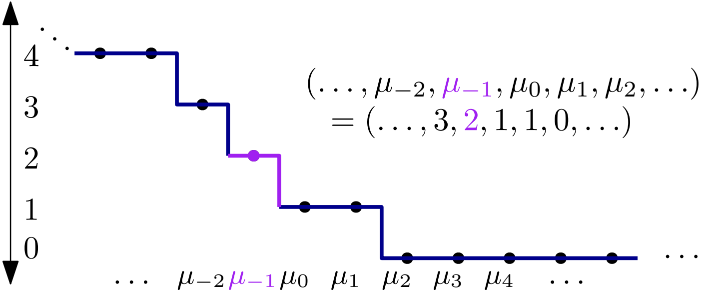


$$(\dots, \mu_{-2}, \mu_{-1}, \mu_0, \mu_1, \mu_2, \dots)$$
$$= (\dots, 3, 1, 1, 1, 0, \dots)$$

Definition (VP 2023)

The reflecting Poisson sea $\mu(T) = (\dots, \mu_{-1}(T), \mu_0(T), \mu_1(T), \dots)$, $T \geq 0$ *is the continuous-time stochastic process with each* $\mu_i(T)$ *increasing by* 1 *according to rate-*t^i *exponential clock (independent of each other), donating move if blocked.*

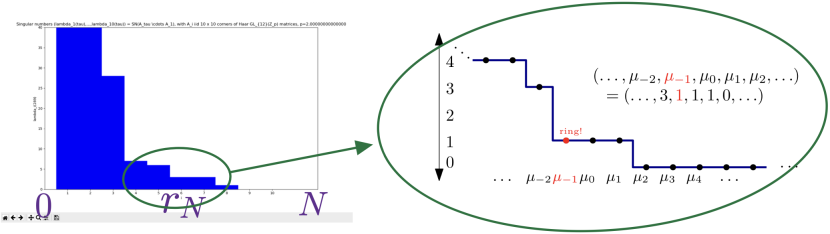
The reflecting Poisson sea



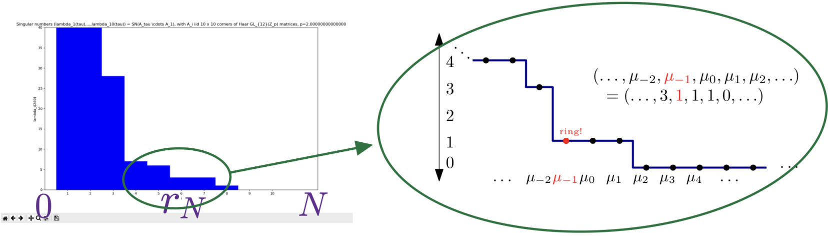
$$(\ldots, \mu_{-2}, \mu_{-1}, \mu_0, \mu_1, \mu_2, \ldots) = (\ldots, 3, 2, 1, 1, 0, \ldots)$$

Definition (VP 2023)

The reflecting Poisson sea $\mu(T) = (\ldots, \mu_{-1}(T), \mu_0(T), \mu_1(T), \ldots)$*,* $T \geq 0$ *is the continuous-time stochastic process with each* $\mu_i(T)$ *increasing by* 1 *according to rate-*t^i *exponential clock (independent of each other), donating move if blocked.*

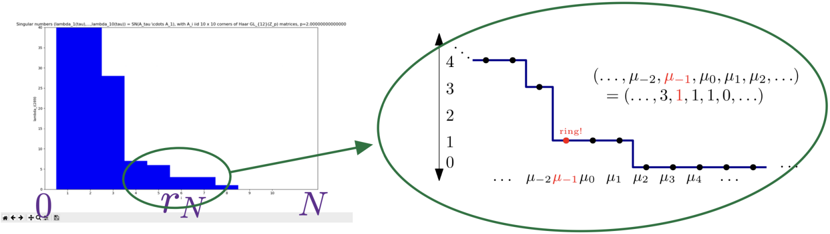


Singular numbers (lambda_1(tau),...,lambda_10(tau)) = SN(A_tau \cdots A_1), with A_i iid 10 x 10 corners of Haar GL_{12}(Z_p) matrices, p=2.00000000000000
lambda_i(289)
0
r_N
N
4
3
2
1
0
ring!
$(\dots, \mu_{-2}, \mu_{-1}, \mu_0, \mu_1, \mu_2, \dots)$
$= (\dots, 3, 1, 1, 1, 0, \dots)$
$\dots$ μ_{-2} μ_{-1} μ_0 μ_1 μ_2 μ_3 μ_4 $\dots$



Theorem (VP '23)

Fix $d \geq 1$. For each $N \geq 1$ let $A_i^{(N)} \in \mathrm{Mat}_N(\mathbb{Z}/p^d\mathbb{Z}), i \geq 1$ be iid Haar and $\Pi^{(N)}(\tau) := \mathrm{SN}(A_\tau^{(N)} \cdots A_1^{(N)})$.
Let $(r_N)_{N \geq 1}$ be such that (1) $r_N \to \infty$ and (2) $N - r_N \to \infty$. Then $(\dots, \Pi^{(N)}(\lfloor p^{r_N} T \rfloor)_{r_N - 1}, \Pi^{(N)}(\lfloor p^{r_N} T \rfloor)_{r_N}, \Pi^{(N)}(\lfloor p^{r_N} T \rfloor)_{r_N + 1}, \dots)$ converges to $(\dots, \mu_{-1}(T), \mu_0(T), \mu_1(T), \dots)$ (with $t = 1/p$) in finite-dimensional distribution across multiple times T.



Theorem (VP '23)

Fix $d \geq 1$. For each $N \geq 1$ let $A_i^{(N)} \in \mathrm{Mat}_N(\mathbb{Z}/p^d\mathbb{Z}), i \geq 1$ be iid Haar and $\Pi^{(N)}(\tau) := \mathrm{SN}(A_\tau^{(N)} \cdots A_1^{(N)})$.
Let $(r_N)_{N \geq 1}$ be such that (1) $r_N \to \infty$ and (2) $N - r_N \to \infty$. Then $(\ldots, \Pi^{(N)}(\lfloor p^{r_N} T \rfloor)_{r_N - 1}, \Pi^{(N)}(\lfloor p^{r_N} T \rfloor)_{r_N}, \Pi^{(N)}(\lfloor p^{r_N} T \rfloor)_{r_N+1}, \ldots)$ converges to $(\ldots, \mu_{-1}(T), \mu_0(T), \mu_1(T), \ldots)$ (with $t = 1/p$) in finite-dimensional distribution across multiple times T.

- The joint distribution of first d rows of $\mu(T)$ at time T is an explicit random variable $\mathcal{L}_{d,t,tT/(1-t)}$.
- Universality? [VP23], [Nguyen-VP '24+]

Integrability and symmetric functions

A word on proofs

Convergence to $\mu(T)$ at a fixed T uses symmetric function theory.

A word on proofs

Convergence to $\mu(T)$ at a fixed T uses symmetric function theory.

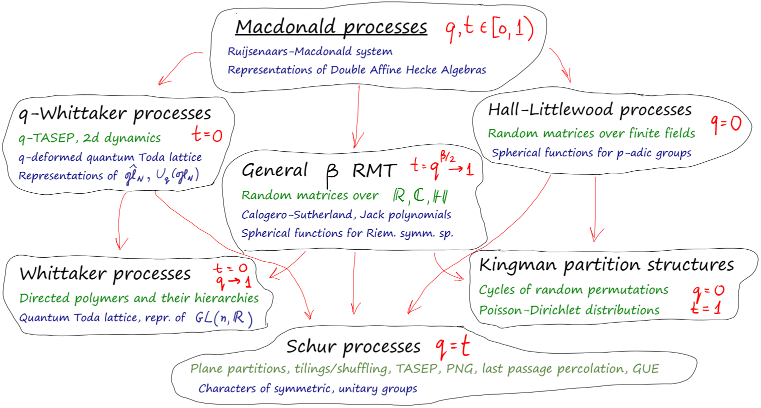
Given convergence of fixed-time marginals, explicit linear-algebraic arguments show multi-time convergence to $\mu(T)$ (robust, universal for generic $\mathrm{GL}_N(\mathbb{Z}_p)$-invariant distributions).

Macdonald processes [Borodin-Corwin '11]

Macdonald polynomials $P_\lambda(x_1, \ldots, x_n; q, t)$ indexed by *integer partitions* $\lambda = (\lambda_1 \geq \ldots \geq \lambda_n \geq 0)$ are symmetric polynomials in $x_1, \ldots, x_n$ with two parameters q, t.

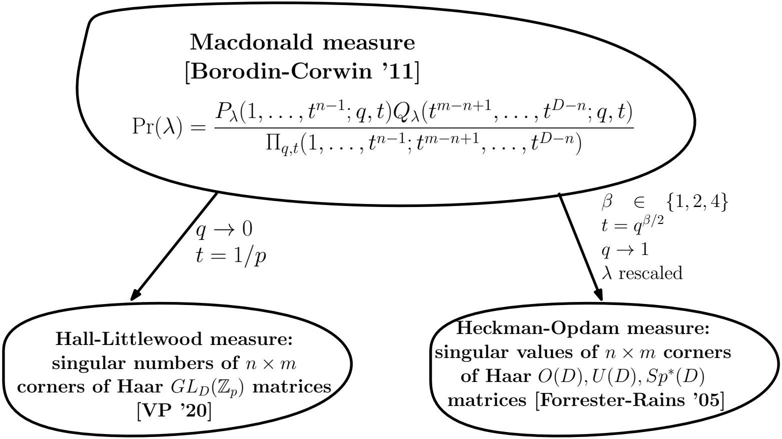
Macdonald processes [Borodin-Corwin '11]

Macdonald polynomials $P_\lambda(x_1, \ldots, x_n; q, t)$ indexed by *integer partitions* $\lambda = (\lambda_1 \geq \ldots \geq \lambda_n \geq 0)$ are symmetric polynomials in $x_1, \ldots, x_n$ with two parameters q, t.



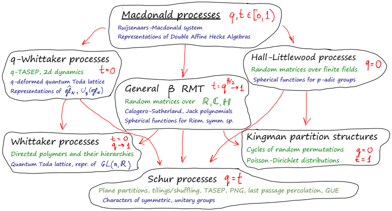
(Figure credits: A. Borodin, ICM 2014 slides)

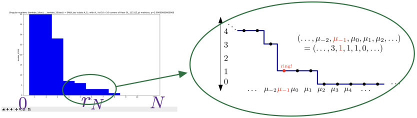
The $\mathbb{Z}_p \leftrightarrow \mathbb{C}$ analogy is actually extremely close



Macdonald processes used for real/complex matrix products (Ahn, Borodin, Gorin, Strahov, Sun 2015+), are also a key tool for us.

Conclusion



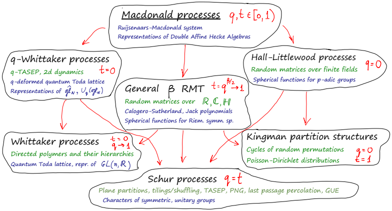


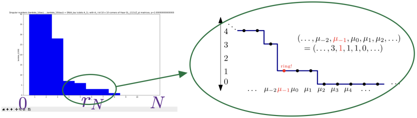
Structural analogies

$$\mathbb{Z}_p \iff \mathbb{R}, \mathbb{C}$$

Probabilistic differences

Conclusion





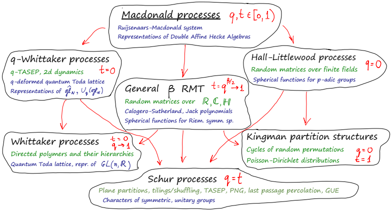
Structural analogies

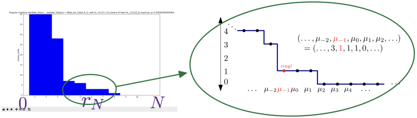
$$\mathbb{Z}_p \iff \mathbb{R}, \mathbb{C}$$

Probabilistic differences

The reflecting Poisson sea is the universal discrete analogue of deformed sine/Airy kernels ([Akemann-Burda-Kieburg 2018], [Liu-Wang-Wang 2018]), but is a (non-determinantal!) local interacting particle system.

Conclusion





Structural analogies

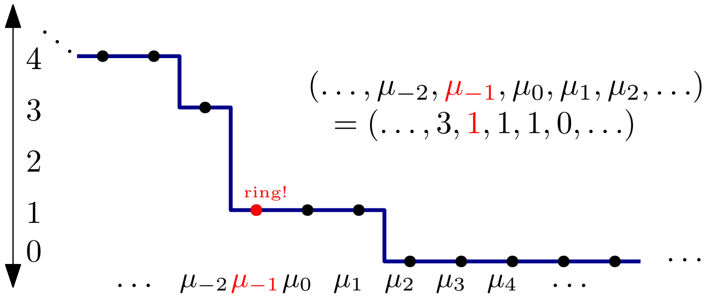
$$\mathbb{Z}_p \iff \mathbb{R}, \mathbb{C}$$

Probabilistic differences

The reflecting Poisson sea is the universal discrete analogue of deformed sine/Airy kernels ([Akemann-Burda-Kieburg 2018], [Liu-Wang-Wang 2018]), but is a (non-determinantal!) local interacting particle system.

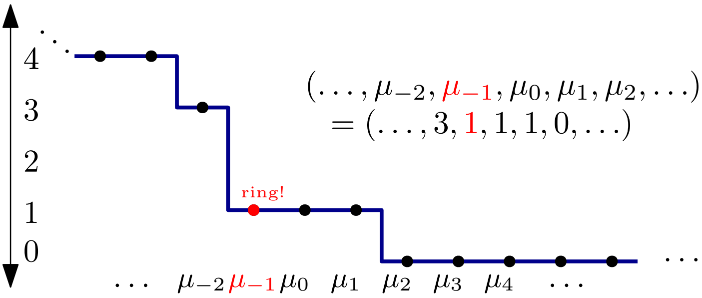
Thanks!

Bonus 1: An infinite amount of ringing



Infinitely many clocks ring on any time interval—nontrivial even to formally define reflecting Poisson sea! [VP 2023]

Bonus 1: An infinite amount of ringing



Infinitely many clocks ring on any time interval—nontrivial even to formally define reflecting Poisson sea! [VP 2023]

However, for 'nice initial conditions' $\mu(0)$ with $\lim_{i \to -\infty} \mu_i(0) = \infty$ as in above picture, projections suffice. In general one must take limit of a version $(\mu_{-D}(T), \mu_{-D+1}(T), \ldots)$.

Bonus 2: more formal statement of bulk limit

Theorem (VP 2023)

Let $r \in (0,1)$, μ a 'nice initial condition', and for each $n \geq 1$, let

- $A_i^{(n)}, i \geq 1$ *iid $n \times n$ matrices with distribution invariant under* $\mathrm{GL}_n(\mathbb{Z}_p)$*,*
- $B^{(n)} \in \mathrm{Mat}_n(\mathbb{Z}_p)$ *fixed 'initial condition matrix' with singular numbers* $\mathrm{SN}(B^{(n)})_{\lfloor rn \rfloor + i} \to \mu_i$ *for all i.*
- ${}^{(n)}(\tau) = \mathrm{SN}(A_\tau^{(n)} \cdots A_1^{(n)} B^{(n)})$*.*

Then $L_i^{(n)}(T) := {}_{\lfloor rn \rfloor + i}(\lfloor c_n^{-1} T \rfloor), i \in \mathbb{Z}, T \geq 0$ converges to reflecting Poisson sea $(\mu_i(T))_{i \in \mathbb{Z}}$ with $\mu(0) = \mu$, for $c_n = c(r, \mathit{Law}(\mathrm{SN}(A_i^{(n)})))$ explicit, provided that

1. $\mathrm{SN}(A_i^{(n)})$ *is not identically $(0, \ldots, 0)$, and*
2. $X_n := \mathrm{corank}(A_i^{(n)} \pmod p) \ll rn$ *w.h.p. (formally,* $\lim_{n \to \infty} \Pr(X_n > rn - j | X_n > 0) = 0$ *for any $j \in \mathbb{N}$).*

Bonus 3: what is $\mathcal{L}_{k,\chi}$ really?

Definition (VP 2023)

For $(L_1, \ldots, L_k) \in \mathrm{Sig}_k$,

$$\Pr(\mathcal{L}_{k,\chi} = (L_1, \ldots, L_k)) := \sum_{d \le L_k} \frac{e^{-\chi t^d} t^{\sum_{i=1}^{k} \binom{L_i - d}{2}}}{(t;t)_{L_k - d} \prod_{i=1}^{k-1} (t;t)_{L_i - L_{i+1}}}$$

$$\times \frac{1}{(t;t)_\infty} \sum_{\substack{\mu \in \mathrm{Sig}_{k-1} \\ L_1 \ge \mu_1 \ge L_2 \ge \mu_2 \ge \ldots}} (-1)^{\sum_{i=1}^{k} L_i - \sum_{i=1}^{k-1} \mu_i - d} \prod_{i=1}^{k-1} \begin{bmatrix} L_i - L_{i+1} \\ L_i - \mu_i \end{bmatrix}_t$$

$$\times Q_{(\mu_1 - d, \ldots, \mu_{k-1} - d)'}(\gamma(\chi(1-t)t^d), \alpha(1); 0, t)$$

where again $t = 1/p$ *and last term is a Hall-Littlewood polynomial specialized with* α *and Plancherel parameters* 1 *and* $\chi(1-t)t^d$.