Unconditional foundations of supersingular isogeny-based cryptography

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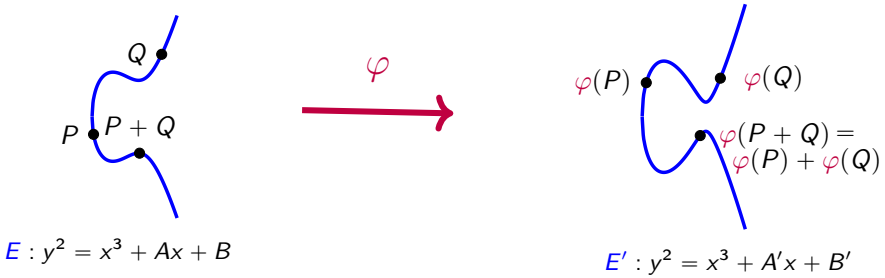
Outline

Hard problems in isogeny-based cryptography

Elliptic curves, isogenies and schemes



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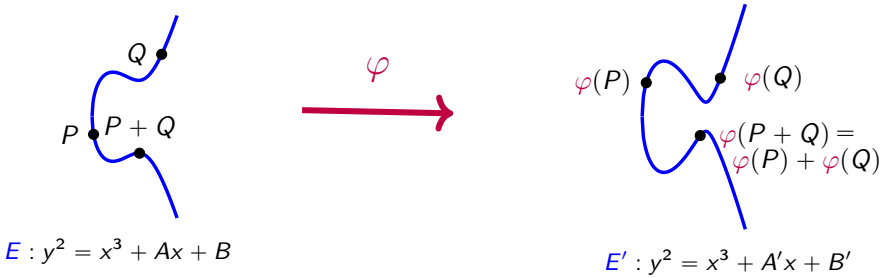
- **Isogeny:** non-constant rational map inducing a group homomorphism.

Elliptic curves, isogenies and schemes



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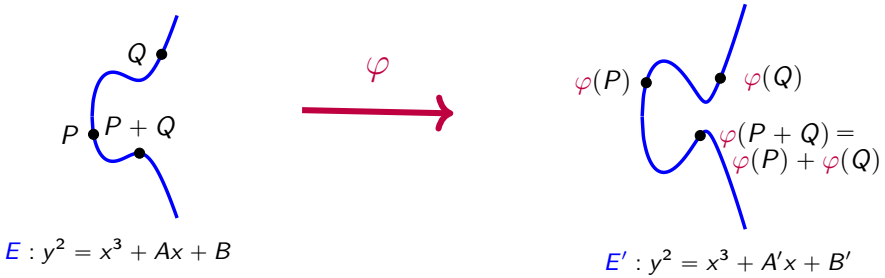

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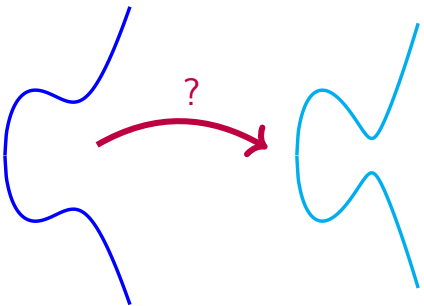
- **Hash function** (CGL), **Key exchange** (CSIDH), **Digital signature** (SQISign) and more.

Hard problems Zoo

Isogeny Problem



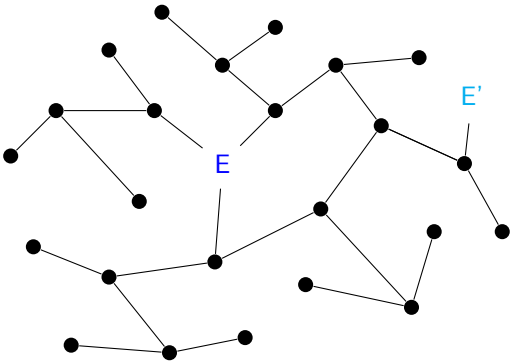
Isogeny

Polynomial reductions between isogeny-based problems

Hard problems Zoo

Let $\ell \neq p$ a prime.

ℓ-IsogenyPath Problem



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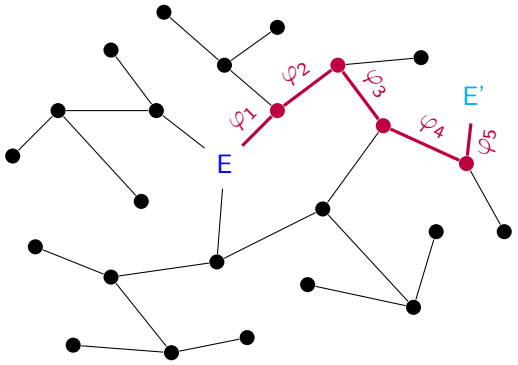

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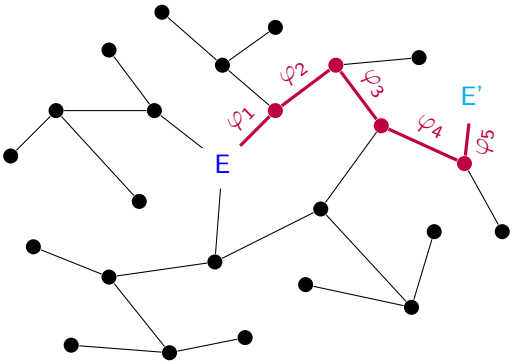

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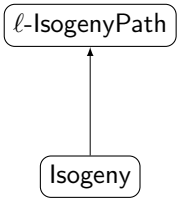

Polynomial reductions between isogeny-based problems

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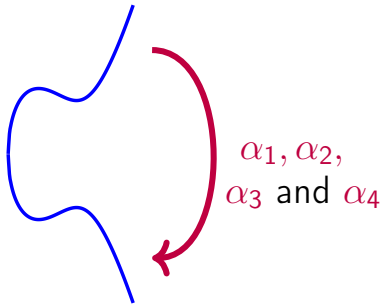
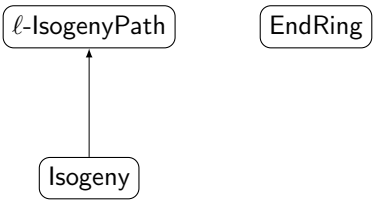
Endomorphism Ring problem



$\mathsf{End}(\curvearrowleft) = \alpha_1\mathbb{Z} + \alpha_2\mathbb{Z} + \alpha_3\mathbb{Z} + \alpha_4\mathbb{Z}$



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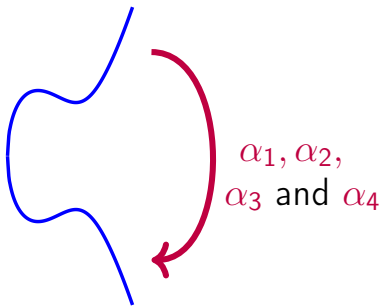
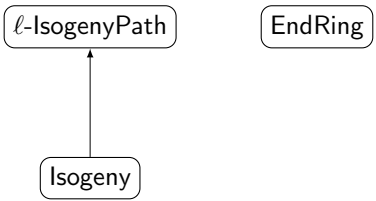


$$\mathsf{End}(\mathcal{E}) = \mathbb{Z} + \alpha_2\mathbb{Z} + \alpha_3\mathbb{Z} + \alpha_4\mathbb{Z}$$

$n \mapsto [n]$ s.t. $[n]P = \underbrace{P + \cdots + P}_{n \text{ times}}$



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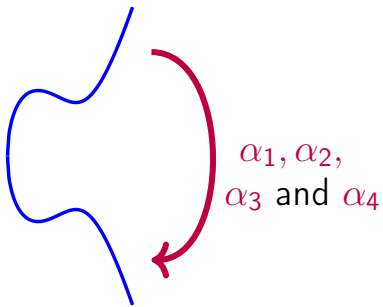

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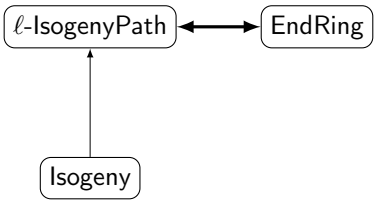

Polynomial reductions between isogeny-based problems

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* : $(\frac{-p,-q_p}{\mathbb{Q}}) = \mathbb{Q}+i\mathbb{Q}+j\mathbb{Q}+ij\mathbb{Q}$ such that $i^2 = -p$, $j^2 = -q_p$ and $ij = -ji$ is a quaternion algebra.

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* `IdealToIsogeny` is easy if:

1. The ideal is smooth enough
2. We know a **special curve** E_0, i.e. such that $\varepsilon : \mathcal{O}_0 \xrightarrow{\sim} \mathrm{End}(E_0)$ is explicit.

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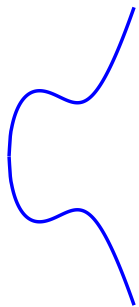
Hard problems Zoo

Maximal Order problem



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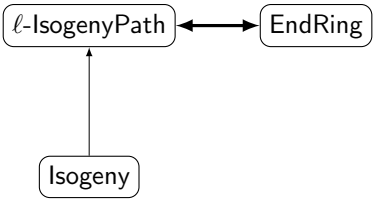

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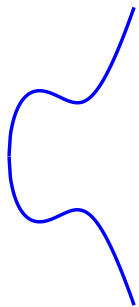

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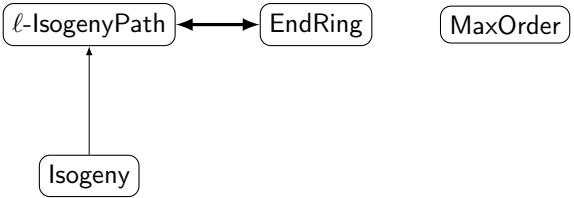

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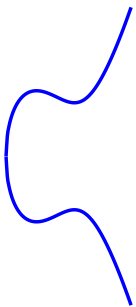

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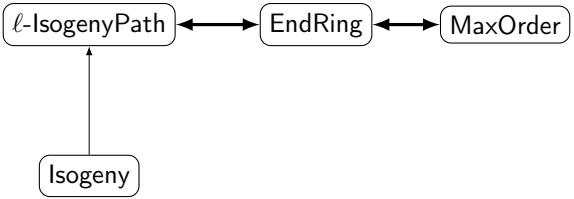

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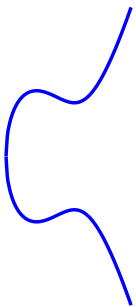

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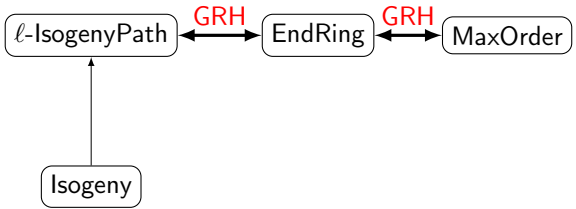
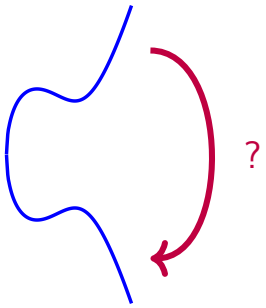
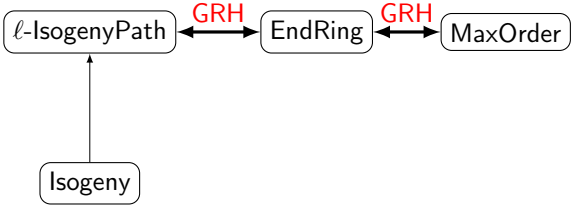


Polynomial reductions between isogeny-based problems
(Non trivial reductions under GRH [Wes21] and
before under heuristics [EHM17; PL17; Eis+18])

Hard problems Zoo

One Endomorphism problem





Polynomial reductions between isogeny-based problems

Hard problems Zoo

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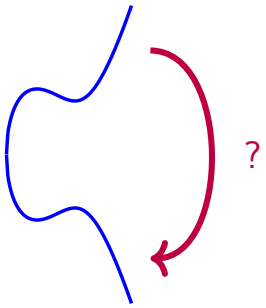

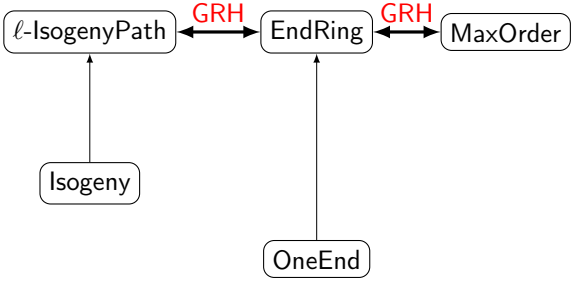

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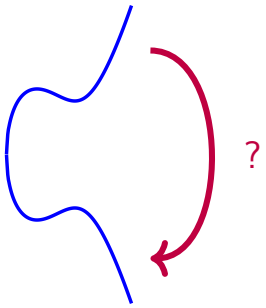

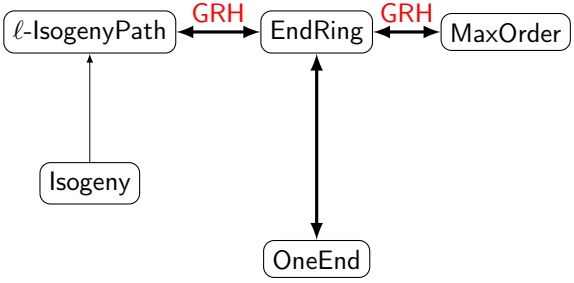

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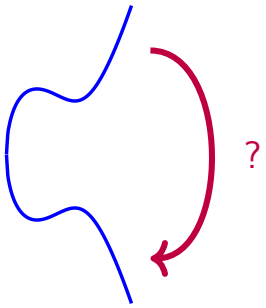

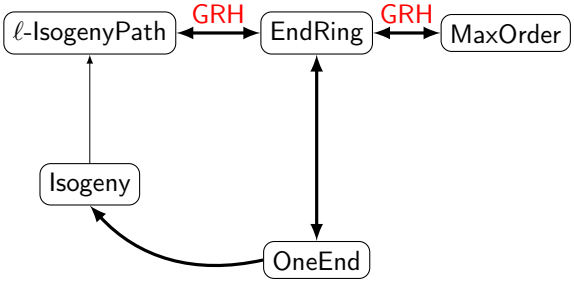

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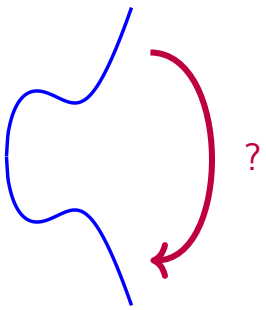

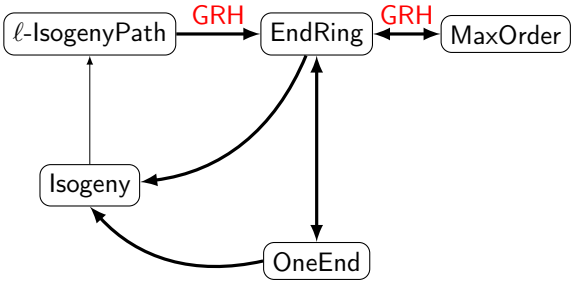

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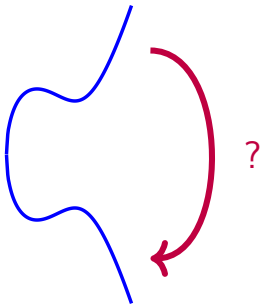

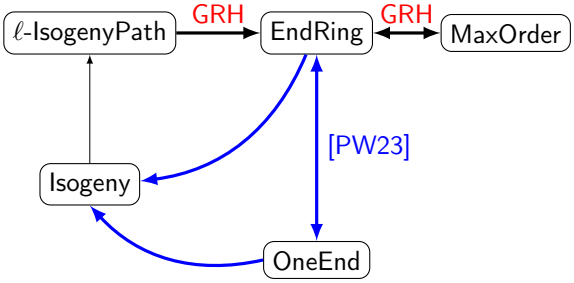

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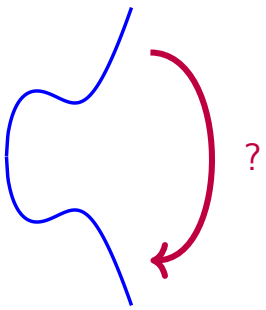
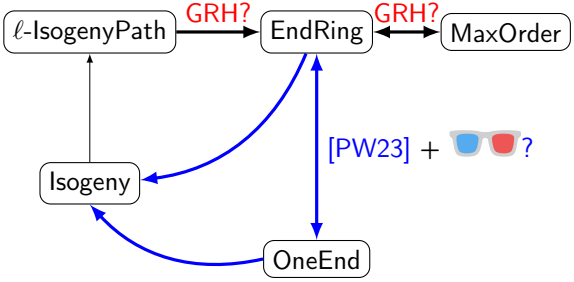


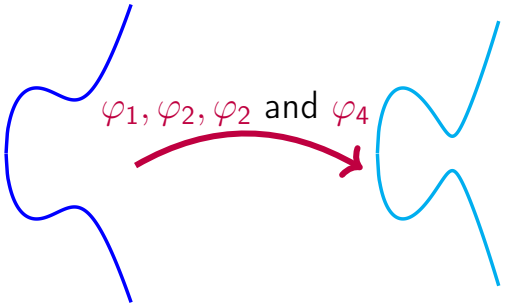


Polynomial reductions between isogeny-based problems

: higher dimensional results published after SIDH's attacks [CSV22; Mai+23; Rob22a]

Hard problems Zoo

Homomorphism Module Problem



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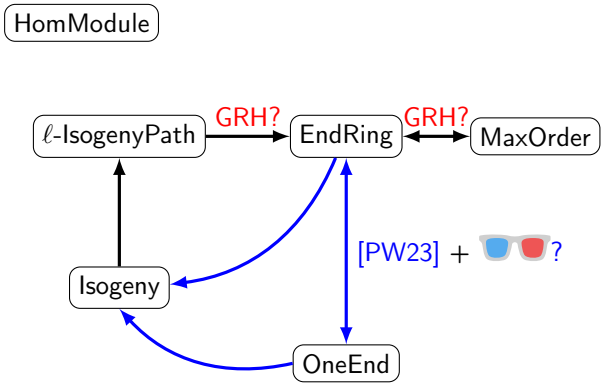

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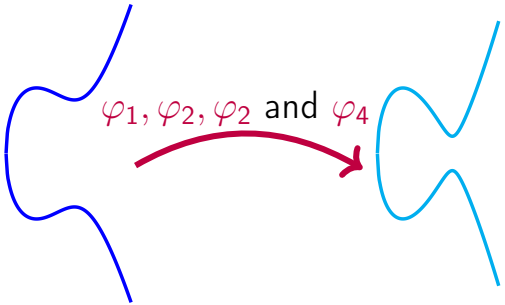

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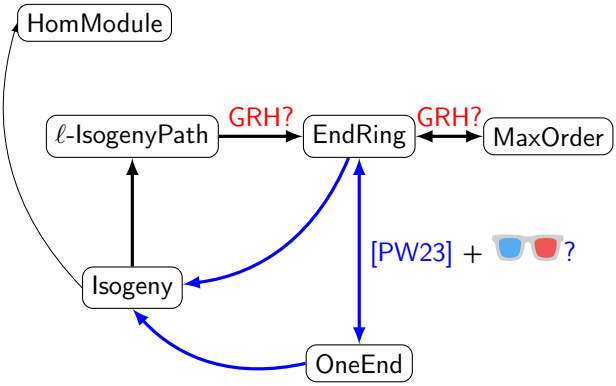

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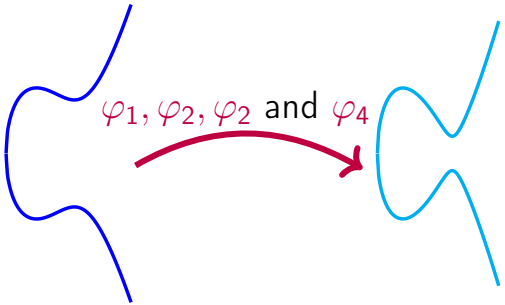

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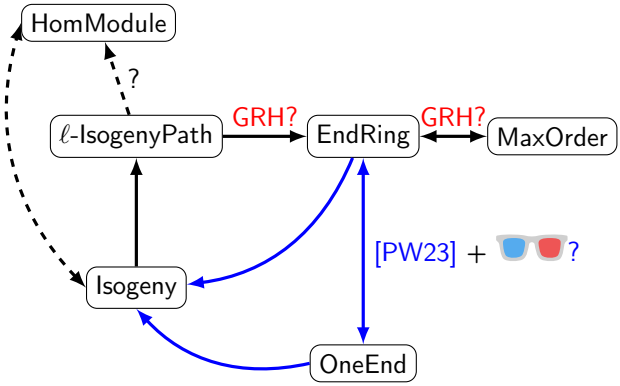

The Generalised Riemann hypothesis

The Riemann hypothesis (RH):

Let ζ be the complex Riemann zeta function (defined from $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \forall \Re(s) > 1$).
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$$\textbf{GRH} \implies \pi(x, a, d) := |\{p \text{ prime s.t } p \leqslant x \text{ and } p \equiv a \mod d\}| = \frac{1}{\varphi(d)} \int_2^x \frac{dt}{lnt} + O(\sqrt{x}\ln x)$$

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Reductions **without** the generalised Riemann hypothesis

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Let $\varphi : E \to E'$ be an isogeny. Given $\varphi(E[\ell_i]^*)$, for enough small primes ℓ_i, one can compute φ in polynomial time.

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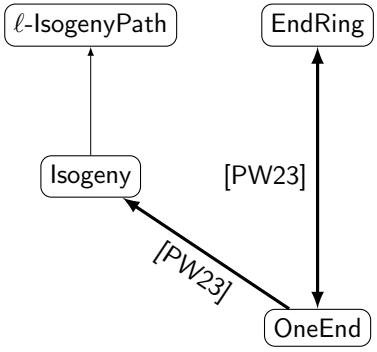

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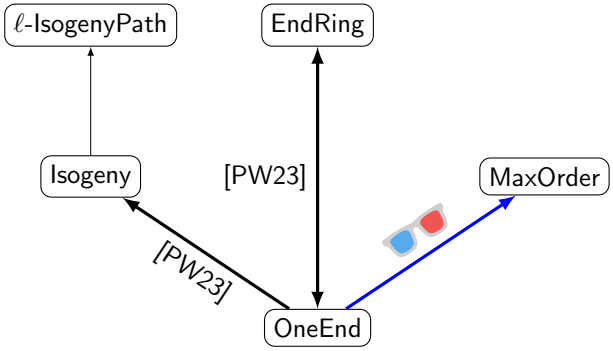

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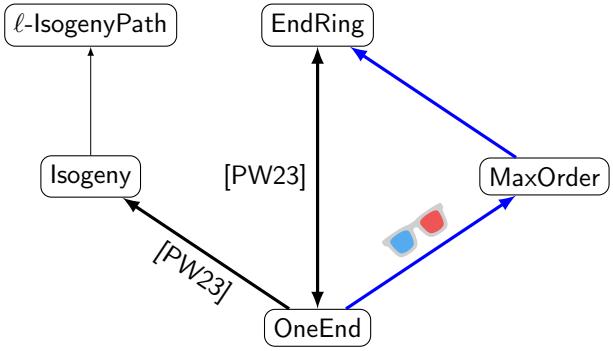

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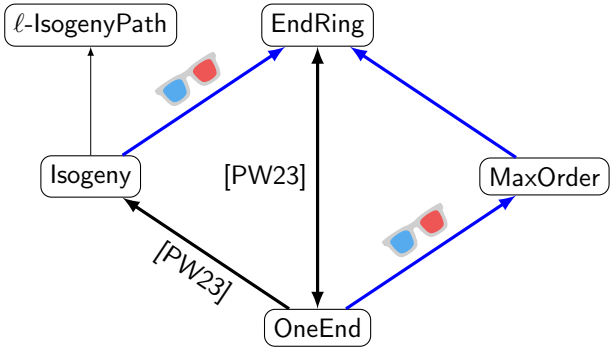

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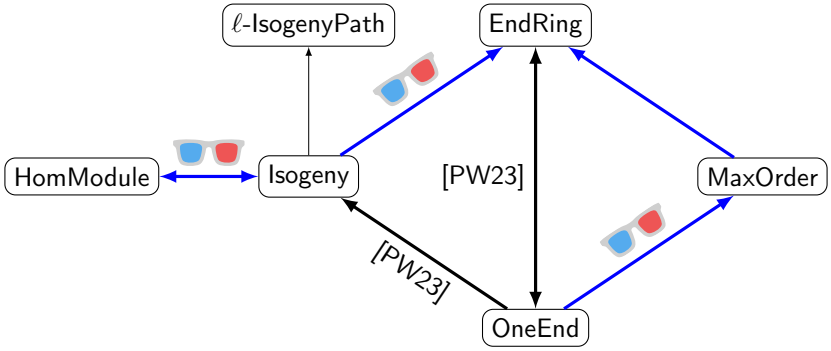

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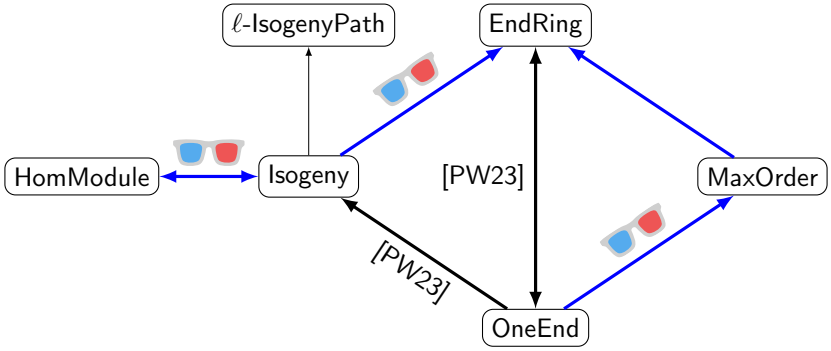

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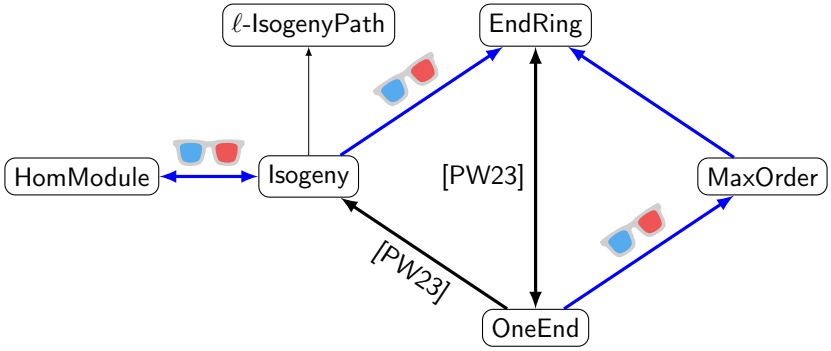

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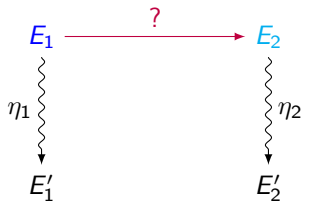

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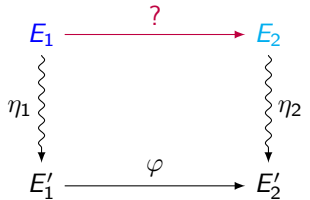

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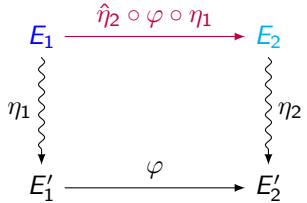

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Thanks for your attention!
https://ia.cr/2025/271

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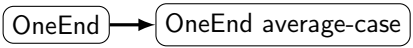

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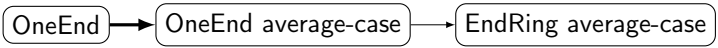

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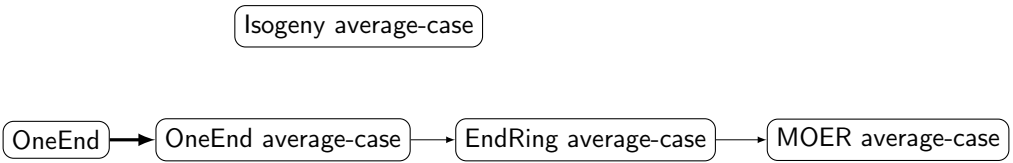

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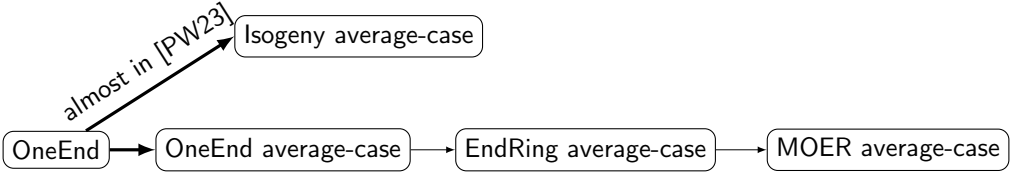

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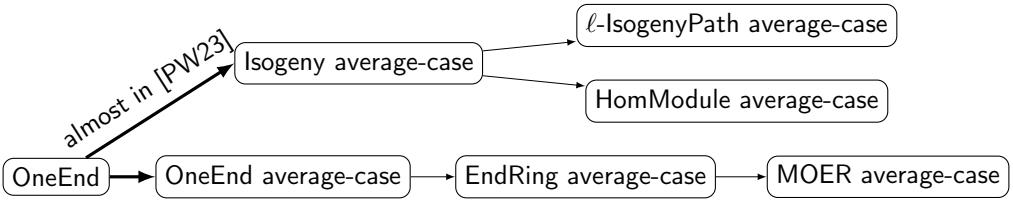

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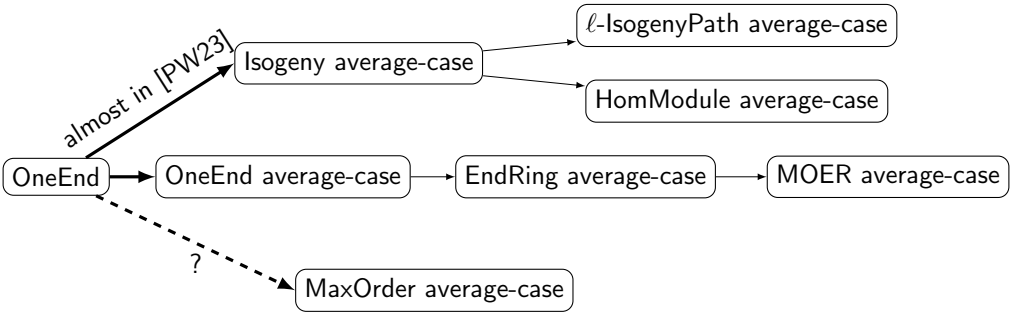

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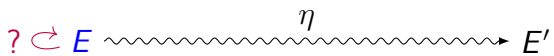

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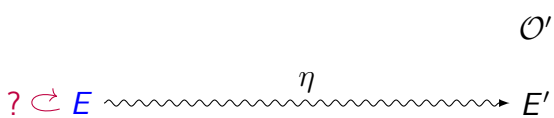

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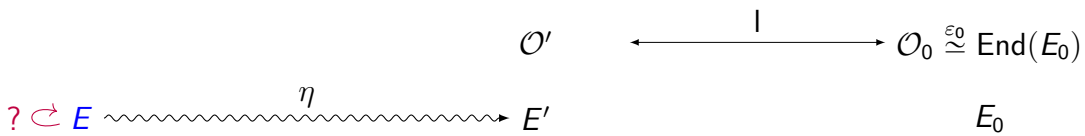

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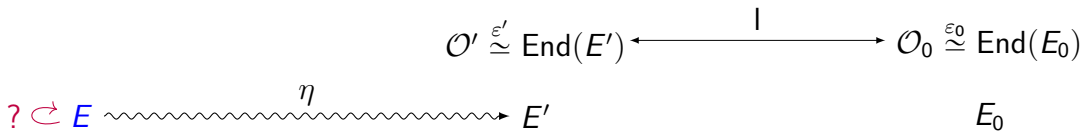

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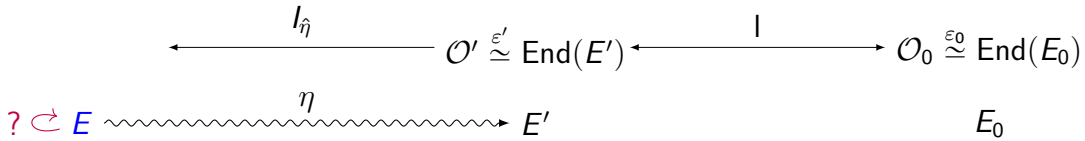

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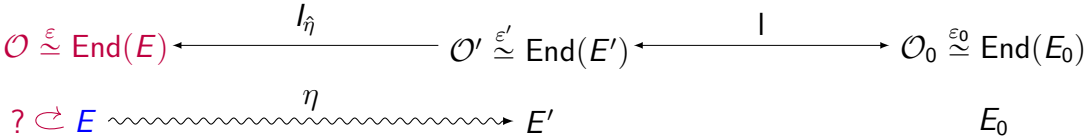

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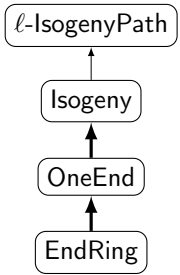

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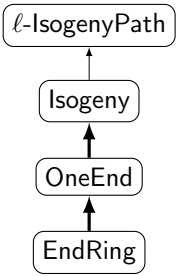

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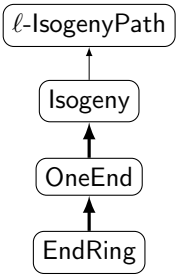

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The reduction

1. Compute $\mathrm{End}(E)$ and $\mathrm{End}(E')$
2. Compute $\varphi : E \to E'$
3. Compute a basis $(\gamma_1, \ldots, \gamma_4)$ of $\mathrm{span}_{\mathbb{Z}}(\mathrm{End}(E')\varphi\,\mathrm{End}(E)) = m\,\mathrm{Hom}(E, E')$
4. Return this basis divided by m [Rob22b; HLMW23]