Asymptotic expansion of the Laplace Functional for the Sine Process

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Overview

Let $\mathrm{Conf}(\mathbb{R})$ be the space of all the configurations of points $\{\xi_j\}_1^n$ in the real line. We have a *"n-point process"* when we introduce a probability measure $\mathrm{d}\mathbb{P}_n$ on $\mathrm{Conf}(\mathbb{R})$.
A n-point process is a Determinantal Point Process (DPP) if the probability measure $\mathrm{d}\mathbb{P}_n$ has the form

$$\mathrm{d}\mathbb{P}_n(\xi_1, \dots, \xi_n) = \frac{1}{n!} \det[K(\xi_j, \xi_k)]_{j,k=1}^n \mathrm{d}\xi_1 \otimes \dots \mathrm{d}\xi_n$$

where $K(x, y)$ is called *"correlation kernel"*.

Sine Process

A important example of DPP is an ∞-point process, the Sine Process, which has a correlation Kernel of the form:

$$K^{sin}(x,y) = \frac{\sin\left[\pi(x-y)\right]}{\pi(x-y)} \quad x,y \in \mathbb{R}. \tag{1}$$

This process is well studied and has a lot of connection with other fields

1 Random Hermitian matrices [Erdös];

2 Orthogonal Polynomials;

3 Integrable systems [Tracy-Widown, Claeys-Tarricone];

4 ...

The Laplace Functional

Let $\mathcal{B}_+(\mathbb{R})$ be the space of bounded non-negative measurable functions $f : \mathbb{R} \to [0, +\infty)$. In general theory of point process, the "Laplace functional" is defined as the map

$$\mathscr{L}_\lambda : \mathcal{B}_+(\mathbb{R}) \to \mathbb{R}^+, \quad f \mapsto \mathscr{L}_\lambda[f], \quad \mathscr{L}_\lambda[f] := \mathbb{E}\left[\mathrm{e}^{-\lambda \int_\mathbb{R} f \mathrm{d}\xi}\right], \tag{2}$$

where $\int_\mathbb{R} f \mathrm{d}\xi = \sum_{x \in \{\xi_j\}_1^n} f(x)$ is a random variable and the average is over the probability measure $\mathrm{d}\mathbb{P}_n$.

For DPP, the Laplace Functional is a Fredholm determinant

$$\mathscr{L}_\lambda[f] = \det\left[\mathrm{id} - \sqrt{\sigma_\lambda}\mathscr{K}\sqrt{\sigma_\lambda}\right], \tag{3}$$

where $\sigma_\lambda(x) := 1 - \mathrm{e}^{-\lambda f(x)}$ and the operator $\mathscr{K} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ has kernel $K(x, y)$ the correlation kernel of the DPP itself.

1 The Laplace functional characterize the point process, i.e. each point process has it's own unique $\mathscr{L}_\lambda$;

2 We can use to prove important results about the distribution of points and the linear statistics of the process, e.g the Chernoff bound

$$\mathbb{P}\left[\int_{\mathbb{R}} f(x) \leq A\right] \leq \mathscr{L}_\lambda[f]\,\mathrm{e}^{\lambda A}.$$

In our work, we study the asymptotic expansion of the Laplace functional $\mathscr{L}_\lambda[f]$ for the Sine Point Process.
We will consider $f \in \mathcal{B}_+(\mathbb{R})$ which satisfies the following conditions:

H.1 f can be extended analytically in a sector $\mathcal{S} \subset \mathbb{C}$ containing $\mathbb{R}$;

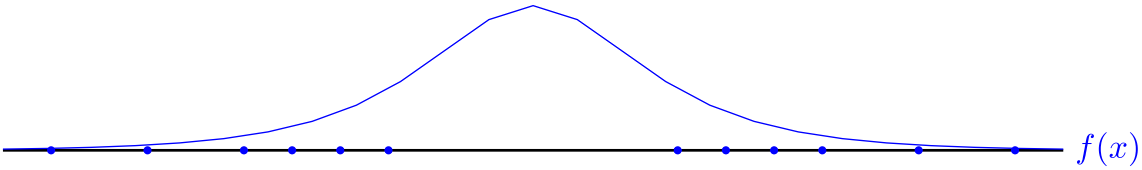
H.2 $f(-x) = f(x)$, with a single maximum at $x = 0$;

H.3 $f(x)$ and $f'(x)$ decays exponentially at $x \sim \pm\infty$:

$$f(x) \sim c_1 \mathrm{e}^{-c_2|x|}, \quad \text{as } x \sim \pm\infty, \tag{4}$$

$$f'(x) \sim \mp c_1 c_2 \mathrm{e}^{-c_2|x|}, \quad \text{as } x \sim \pm\infty, \tag{5}$$

with $c_1, c_2 > 0$.



From the definition of the Lapalce functional, the configurations that contribute more on $\mathscr{L}_\lambda$ are the one with few or no points near the maximum of f.

Main Result

Theorem

*Given $f(z) \in \mathcal{B}_+(\mathbb{R})$ which satisfies the hypothesis **H.1**, **H.2** and **H.3**, then the Laplace functional $\mathscr{L}_\lambda[f]$ for the sine process has the following expansion as $\lambda \to \infty$:*

$$\log \mathscr{L}_\lambda[f] = -\frac{2\pi^2}{c_2^2}(\log\lambda)^2 + \mathcal{O}((\log\lambda)^{\frac{3}{2}}). \tag{6}$$

The Sine Operator

Let $\mathscr{K}^{sin}$ be the integral operator with kernel $K^{sin}(x, y)$, this operator is an *"integrable operator"*, i.e the Kernel has the form

$$K^{sin}(x, y) = \frac{\vec{h}(x)^T\, \vec{t}(y)}{x - y}, \quad \vec{h}(x)^T\, \vec{t}(x) \equiv 0 \tag{7}$$

where $\vec{h}(x), \vec{t}(y)$ are vectors defined as

$$\vec{h}(x) := \frac{1}{2\pi i} \begin{pmatrix} \mathrm{e}^{i\pi x} \\ \mathrm{e}^{-i\pi x} \end{pmatrix} \quad \vec{t}(y) := \begin{pmatrix} \mathrm{e}^{-i\pi y} \\ -\mathrm{e}^{i\pi y} \end{pmatrix}. \tag{8}$$

We can use the results of Its, Izergin, Korepin and Slavnov [IIKS] to study the Fredholm determinant of $\mathscr{K}^{sin}$.

IIKS

If $\mathcal{K} : L^2(\Sigma, |\mathrm{d}z|) \to L^2(\Sigma, |\mathrm{d}z|)$ is an integrable operators, the resolvent operator $\mathcal{R} := -\mathrm{Id} + (\mathrm{Id} - \mathcal{K})^{-1}$ is still an integrable operator with kernel R given by:

$$R(z, w) = \frac{p(z)^T \, \Gamma(z)^T \, (\Gamma^{-1}(w))^T \, q(w)}{z - w}, \tag{9}$$

where $\Gamma(z) \in Mat(r \times r, \mathbb{C})$ is a sectional analytic matrix function which satisfies the Riemann-Hilbert problem:

$$\begin{aligned} &\Gamma_+(z) = \Gamma_-(z) M(z) \text{ for } z \in \Sigma, \qquad M(z) = 1 + 2\pi i p(z)\, q(z)^T, \\ &\Gamma(z) \sim 1 + \mathcal{O}(z^{-1}), \text{ as } z \to \infty. \end{aligned} \tag{10}$$

If the solution of (10) exists then

$$\det(\mathrm{Id} - \mathcal{K}) \neq 0.$$

From the Jacobi Identity we get that

$$\begin{aligned}
\partial_\lambda \log \mathscr{L}_\lambda[f] &= \partial_\lambda \log \det \left[\mathrm{id} - \sqrt{\sigma_\lambda}\mathscr{K}^{sin}\sqrt{\sigma_\lambda}\right] \\
&= -\mathrm{Tr}\left[(\mathrm{id} - \sqrt{\sigma_\lambda}\mathscr{K}^{sin}\sqrt{\sigma_\lambda})^{-1} \circ \partial_\lambda(\sqrt{\sigma_\lambda}\mathscr{K}^{sin}\sqrt{\sigma_\lambda})\right] \\
&= -\mathrm{Tr}[\partial_\lambda \sigma_\lambda \mathscr{K}^{sin}(\mathrm{Id} - \sigma_\lambda \mathscr{K}^{sin})^{-1}] \\
&= -\int_{\mathbb{R}} f(x) L_\lambda(x,x)\mathrm{d}x
\end{aligned}$$

where $L_\lambda(x,y)$ is a smooth kernel of the operator

$$\mathcal{L}_\lambda = \sqrt{1-\sigma_\lambda}\mathscr{K}^{sin}(\mathrm{Id} - \sigma_\lambda \mathscr{K}^{sin})^{-1}\sqrt{1-\sigma_\lambda} = \sqrt{\sigma_\lambda^{-1} - 1}\,\mathscr{R}^{sin}\sqrt{\sigma_\lambda^{-1} - 1} \tag{11}$$

where $\mathscr{R}^{sin}$ is the resolvent of the operator $\mathscr{K}^{sin}$.

We apply the result (9) to the operator $\mathscr{R}^{sin}$ and we get that $\mathcal{L}_\lambda$ has Kernel

$$L_\lambda(x,y) = \sqrt{1-\sigma_\lambda(x)}\frac{\vec{h}(x)^T\, Y^T(x)(Y^{-1}(y))^T\vec{t}(y)}{x-y}\sqrt{1-\sigma_\lambda(y)}; \tag{12}$$

Where $Y(z)$ is a 2×2 matrix, analytic in $z \in \mathbb{C}\setminus\mathbb{R}$, which satisfy the Riemann-Hilbert problem

$Y_+(x) = Y_-(x)J_Y(x)$ as $x \in \mathbb{R}$, where the jump matrix has the form

$$J_Y(x) := \begin{bmatrix} 1-\sigma_\lambda(x) & \sigma_\lambda(x)\mathrm{e}^{2i\pi x} \\ -\sigma_\lambda(x)\mathrm{e}^{-2i\pi x} & 1+\sigma_\lambda(x) \end{bmatrix} \tag{13}$$

$Y(z) \sim 1 + \mathcal{O}(z^{-1}),$ as $z \to \infty$.

We apply the following transformation

$$\Psi(z) = Y(z) \begin{cases} \begin{bmatrix} \mathrm{e}^{2i\pi z} & \mathrm{e}^{2i\pi z} \\ \mathrm{e}^{-2i\pi z} & 0 \end{bmatrix}; & \mathrm{Im}[z] > 0; \\ \begin{bmatrix} \mathrm{e}^{2i\pi z} & 0 \\ \mathrm{e}^{-2i\pi z} & -\mathrm{e}^{-2i\pi z} \end{bmatrix}; & \mathrm{Im}[z] < 0; \end{cases} \tag{14}$$

Then the Riemann-Hilbert problem becomes

$$\Psi_+(x) = \Psi_-(x) \begin{bmatrix} 1 & \mathrm{e}^{-\lambda f(x)} \\ 0 & 1 \end{bmatrix} \text{ as } x \in \mathbb{R} \tag{15}$$

$$\Psi(z) \sim \mathrm{e}^{iz\pi\sigma_3} \begin{cases} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}; & \mathrm{Im}[z] > 0; \\ \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}; & \mathrm{Im}[z] < 0; \end{cases} \text{ as } z \sim \infty \tag{16}$$

Then we have that the diagonal term of the Kernel $L_\lambda(x, y)$ depends on the matrix $\Psi(x)$ as follows:

$$L_\lambda(x,x) = \frac{1-\sigma_\lambda(x)}{2\pi i} \left[\Psi(x)^{-1}\partial_x \Psi(x)\right]_{21} \tag{17}$$

and the Fredholm determinant becomes

$$\partial_\lambda \log \mathscr{L}_\lambda[f] = \int_{\mathbb{R}} \partial_\lambda(\mathrm{e}^{-\lambda f(x)}) \left[\Psi(x)^{-1}\partial_x \Psi(x)\right]_{21} \frac{\mathrm{d}x}{2\pi i} \tag{18}$$

This implies that, by knowing the large asymptotics of the matrix $\Psi(x)$ and $\partial_x \Psi(x)$ we can find the asymptotic of the Laplace functional $\mathscr{L}_\lambda[f]$.

Non-linear steepest descent

We define the transformation:

$$\hat{\Psi}(z) = \Psi(z)\mathrm{e}^{-\lambda(g(z)+\frac{l}{2})\sigma_3} \tag{19}$$

where l is a constant and $g(z)$ is a complex function, analytic in $z \in \mathbb{C} \setminus \mathbb{R}$
Then the RHP (15) becomes

$$\hat{\Psi}_+(x) = \hat{\Psi}_-(x) \begin{bmatrix} \mathrm{e}^{\lambda(g_-(x)-g_+(x))} & e^{\lambda(g_+(x)+g_-(x)-f(x)+l)} \\ 0 & \mathrm{e}^{-\lambda(g_-(x)-g_+(x))} \end{bmatrix} \text{ for } z \in \mathbb{R} \tag{20}$$

$$\hat{\Psi}(z) = (1 + \mathcal{O}(z^{-1})) \begin{cases} \begin{bmatrix} \mathrm{e}^{2i\pi z} & 1 \\ 1 & 0 \end{bmatrix}; & \mathrm{Im}[z] > 0; \\ \begin{bmatrix} 1 & 0 \\ -\mathrm{e}^{-2i\pi z} & -1 \end{bmatrix}; & \mathrm{Im}[z] < 0; \end{cases} \quad \text{for } z \sim \infty \tag{21}$$

the g-function

We define $B_\lambda \subset \mathbb{R}$ the set on which the approximation of $L_\lambda(x,x)$ is supported. According to the properties that the function $f(x)$ has, we suppose the

$$B_\lambda = (-\infty; -a(\lambda)) \cup (a(\lambda); +\infty).$$

We are looking for a g-function $g(z)$ which solves the scalar Riemann-Hilbert problem:

$$g_+(x) + g_-(x) - f(x) + l = 0 \text{ for } x \in B_\lambda \tag{22}$$
$$g_+(x) - g_-(x) = 0 \text{ for } x \in \mathbb{R} \setminus B_\lambda \tag{23}$$

with boundary condition

$$g(z) = \mp \frac{i\pi z}{\lambda} - \frac{l}{2} + \frac{g_1}{z} + \mathcal{O}(z^{-2}) \text{ as } z \sim \infty \quad \pm \mathrm{Im}[z] > 0 \tag{24}$$
$$g(z) \sim \mp i(z \pm a(\lambda))^{3/2} \text{ for } z \sim \pm a(\lambda). \tag{25}$$

By deriving the scalar RHP respect to x and solving the new RHP we get

$$g'(z) = \frac{R(z)}{2\pi i} \int_{B_\lambda} \frac{f'(w)}{R_+(w)(w-z)} \mathrm{d}w \tag{26}$$

where $R(z) := \sqrt{(z+a(\lambda))(a(\lambda)-z)}$ is a multi valued function, analytic in $z \in \mathbb{C} \setminus B_\lambda$. From the boundary condition at infinity we get a equation for the point $a(\lambda)$ we obtain an equation for the point $a(\lambda)$

$$\frac{\pi^2}{\lambda} = \int_a^{+\infty} \frac{|f'(w)|}{|R(w)|} \mathrm{d}w. \tag{27}$$

By analysing the equation (27) as $\lambda \sim +\infty$, we get that $a(\lambda)$ has the following expansion:

$$a(\lambda) = \frac{\log \lambda}{c_2} + C_f + \mathcal{O}(\log \log \lambda). \tag{28}$$

The φ-function

We define $\varphi(z) := 2g(z) - f(z) + l$

Proposition

The function $\varphi(z)$ has the following expansions as $\lambda \sim +\infty$:

$$\varphi(z) \sim \mp \frac{2\pi c_1}{\lambda} R(z) \text{ for } |z| >> a(\lambda) \tag{29}$$

$$\varphi(z) \sim \mp 2\pi i c_1^2 \frac{[R(z)]^3}{\lambda} \text{ for } |z \pm a(\lambda)| << 1 \tag{30}$$

where we have the $-$ sign when $\mathrm{Re}(z) \in (a(\lambda); +\infty)$ and $+$ when $\mathrm{Re}(z) \in (-\infty, a(\lambda))$

Idea of proof

Another way to represent $g'(z)$ is by integrating (26) by parts

$$g'(z) = \frac{R(z)}{2\pi i}\int_{B_\lambda} \frac{f'(w)}{R_+(w)(w-z)}\mathrm{d}w$$
$$= \frac{z}{2\pi i}\int_{a(\lambda)}^{+\infty} \frac{\mathrm{d}}{\mathrm{d}w}\left(\frac{f'(w)}{w}\right)\log\left[\frac{1+\frac{\sqrt{w^2-a^2}}{iR(z)}}{1-\frac{\sqrt{w^2-a^2}}{iR(z)}}\right]\mathrm{d}w \tag{31}$$

Since $R(z) = \pm i\sqrt{z^2-a^2}$ and $R_+(z) = -R_-(z)$ for $z \in B_\lambda$, then we have that

$$g'_+(z) - g'_-(z) = \pm\frac{z}{\pi i}\int_{a(\lambda)}^{+\infty} \frac{\mathrm{d}}{\mathrm{d}w}\left(\frac{f'(w)}{w}\right)\log\left[\frac{1+\sqrt{\frac{w^2-a^2}{z^2-a^2}}}{1-\sqrt{\frac{w^2-a^2}{z^2-a^2}}}\right]\mathrm{d}w \tag{32}$$

where we have the sign $+$ when $z \in (a(\lambda); +\infty)$ and the sign $-$ when $z \in (-\infty; -a(\lambda))$.

From the definition of $\varphi(z)$

$$\varphi'_+(z) = g'_+(z) - g'_-(z) = \pm \frac{z}{\pi i} \int_{a(\lambda)}^{+\infty} \frac{\mathrm{d}}{\mathrm{d}w} \left(\frac{f'(w)}{w} \right) \log \left[\frac{1 + \sqrt{\frac{w^2 - a^2}{z^2 - a^2}}}{1 - \sqrt{\frac{w^2 - a^2}{z^2 - a^2}}} \right] \mathrm{d}w$$

Then we define $\varphi'(z)$ as the analytic continuation of $\varphi'_+(z)$ analyze the asymptotic expansion as $\lambda \sim +\infty$. We have to consider two kinds of limits :

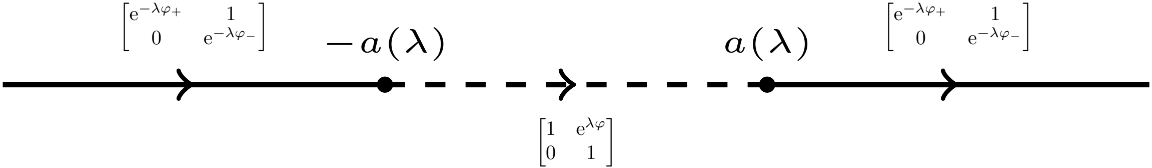
$\lambda \sim +\infty$ and $z >> a(\lambda)$ (or $z << -a(\lambda)$)

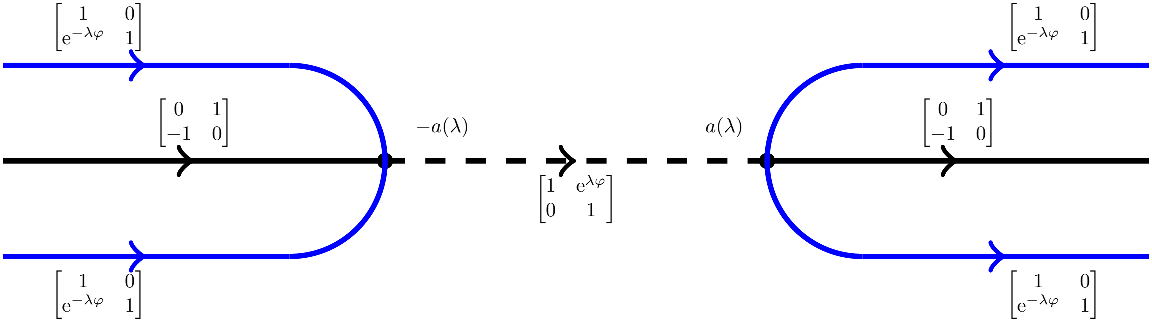
$$\varphi'(z) = \mp \frac{2\pi i c_1 z}{\lambda \sqrt{z^2 - a^2}} + \mathcal{O}((\lambda a(\lambda))^{-1}) \tag{33}$$

$\lambda \sim +\infty$ and $|z \pm a(\lambda)| << 1$

$$\varphi'(z) = \mp 2\pi i c_1^2 \frac{z\sqrt{z^2 - a^2(\lambda)}}{a(\lambda)\lambda} [1 + \mathcal{O}((\lambda a(\lambda))^{-1})] + \mathcal{O}((\lambda \sqrt{a(\lambda)})^{-1}) \tag{34}$$

Opening of the Lenses





$\begin{bmatrix} 1 & 0 \\ e^{-\lambda\varphi} & 1 \end{bmatrix}$
$\begin{bmatrix} 1 & 0 \\ e^{-\lambda\varphi} & 1 \end{bmatrix}$
$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$
$-a(\lambda)$
$a(\lambda)$
$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$
$\begin{bmatrix} 1 & e^{\lambda\varphi} \\ 0 & 1 \end{bmatrix}$
$\begin{bmatrix} 1 & 0 \\ e^{-\lambda\varphi} & 1 \end{bmatrix}$
$\begin{bmatrix} 1 & 0 \\ e^{-\lambda\varphi} & 1 \end{bmatrix}$

Proposition

As $\lambda \sim +\infty$ *function* $\varphi(z)$ *satisfy the following inequalities:*

$$\mathrm{Re}(\lambda\varphi(z)) > 0 \qquad \text{for } z \in \mathcal{U}_r; \tag{35}$$

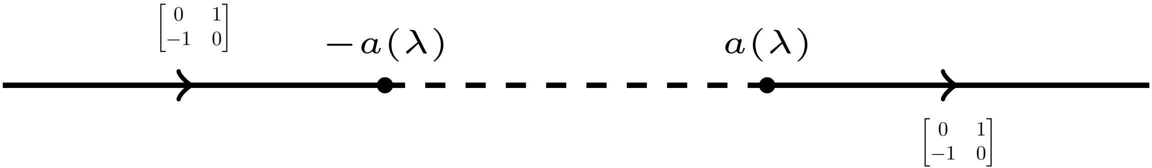
$$\mathrm{Re}(\lambda\varphi(z)) < 0 \qquad \text{for } z \in (-a(\lambda) + r, a(\lambda) - r). \tag{36}$$

where $\mathcal{U}_r := \{z \in \mathbb{C} : |\,\mathrm{Im}(z)| = r, |\,\mathrm{Re}(z)| > r, r > 0\}$.

This means that the Jump matrices along the lenses tends to the identity as $\lambda \sim +\infty$.

Model problem $P^\infty(z)$

We are looking for a matrix $P^\infty(z) \in GL(2,\mathbb{C})$, analytic in $\mathbb{C} \setminus B_\lambda$, that solves the RHP



The solution is the following:

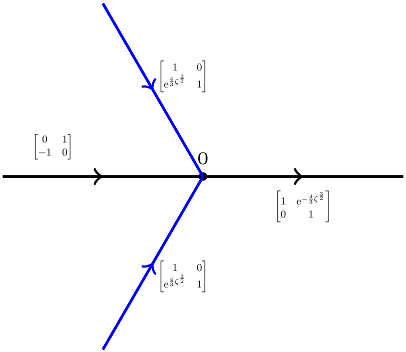
$$P^\infty(z) = U\,\gamma(z)^{\frac{\sigma_3}{4}}\,\mathrm{e}^{-\frac{i\pi}{4}\sigma_3}\,U^\dagger \begin{cases} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \text{ for } \mathrm{Im}(z) > 0; \\ 1 \text{ for } \mathrm{Im}(z) < 0. \end{cases} \tag{37}$$

where $\gamma(z) := \left(\frac{a+z}{a-z}\right)^{\frac{1}{4}}$ and $U := \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$.

In the regions near the points $\pm a(\lambda)$, i.e. the disk $\mathbb{D}_r(\pm a) = \{z \in \mathbb{C} : |z \pm a(\lambda)| < r\}$, we define the conformal map

$$\zeta(z) := \mathrm{e}^{i\frac{2}{3}\pi}\left(\frac{3}{4}\lambda\varphi(z)\right)^{\frac{2}{3}}.$$

then the RHP near the points $\pm a(\lambda)$ is mapped to the following problem



Idea of proof of the Theorem

From our previews results, we can estimate the 21 term of the matrix $\Psi(z)^{-1}\partial_z\Psi(z)$

$$\left[\Psi(z)^{-1}\partial_z\Psi(z)\right]_{21} = -\lambda\varphi_+'(z)\mathrm{e}^{+\lambda f(z)} + \text{ Terms which depends on } P^\infty \text{ and the local parametrix} \tag{38}$$

but this last terms, as $\lambda \sim +\infty$, are smaller respect the the leading order of the first term. Then, by substituting (38) inside the formula (18) and using the fact that the leading order of the Integral is given by the expansion $x \sim a(\lambda)$, we get that

$$\partial_\lambda \log \mathscr{L}_\lambda[f] \sim \int_{a(\lambda)}^{+\infty} \lambda f(x)\varphi_+'(x)\frac{\mathrm{d}x}{\pi i} \sim -\int_a^{+\infty} 2\pi i c_1^2 \frac{x\sqrt{x^2 - a^2(\lambda)}}{a(\lambda)} e^{-c_2 x}\frac{\mathrm{d}x}{\pi i} \sim -\frac{2\pi^2}{c_2^2}\frac{\log\lambda}{\lambda}$$

By integrating this result we have (6). □

Large tail probability

From the Chernoff's bound, for $\lambda = 1/A$ and the result (6), we have that

$$\log \mathbb{P}\left[\int_{\mathbb{R}} f(x) \leq A\right] \leq \log \mathscr{L}_\lambda[f] + \lambda A \sim -\frac{\pi^2}{c_2^2}(1 + o(1)) \log^2 A$$

as $A \to 0$. We would like to find a lower bound such that

$$\log \mathbb{P}\left[\int_{\mathbb{R}} f(x) \leq A\right] \sim -\frac{\pi^2}{c_2^2}(1 + o(1)) \log^2 A$$

as $A \to 0$.

Conjecture

$$\lim_{s \to 0} \frac{\mathbb{P}\left[\int_{\mathbb{R}} f(x) \leq s\right]}{\log^2 s} = -\frac{\pi^2}{c_2^2} \tag{39}$$

Open problems

- Large tail probability;
- More general functions $f(x) \in \mathcal{B}_+(\mathbb{R})$;
- Connection with integrable systems.

Thank you

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