Bipartite spherical spin glass at critical temperature

(with a random matrix detour)

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April 29, 2024

The Journey of this project

Question: What happens to the free energy of spherical spin glasses near the critical temperature threshold?

- **Recent breakthroughs** for Spherical Sherrington-Kirkpatrick model
  - Baik, Lee (2016)
  - Landon (2022)
  - Johnstone, Klochkov, Onatski, Pavlyshyn (2022)
- **Our goal:** Obtain similar results to bipartite spherical spin glasses
- **Missing ingredient:** CLT for a certain statistic of random matrix eigenvalues

Today's talk:

- Spin glass background
- Our result for bipartite spin glasses
- Random matrix project we did along the way

Background on Spin Glass models

The **Sherrington-Kirkpatrick model** has the following set-up:

- Particles are labeled $\{1, 2, 3, ..., N\}$.
- Each particle is assigned a **spin**, either $+1$ or -1.
- A **spin configuration** σ is the vector of spins:

$$\boldsymbol{\sigma} = (\sigma_1, \sigma_2, ..., \sigma_N) \in \{\pm 1\}^N$$

- The **Hamiltonian** is

$$\mathcal{H}(\boldsymbol{\sigma}) = \tfrac{1}{2\sqrt{N}} \sum_{i,j=1}^{N} J_{ij}\sigma_i\sigma_j$$

  where the "interaction coefficients" J_{ij} are Gaussian and independent up to symmetry ($J_{ij} = J_{ji}$).

- **Notice:** $\mathcal{H}$ is maximized when the signs of σ_i, σ_j agree for $J_{ij} > 0$ but disagree for $J_{ij} < 0$.

Background on Spin Glass models

- For ease of notation, we rewrite the Hamiltonian

$$\mathcal{H}(\boldsymbol{\sigma}) = \frac{1}{2\sqrt{N}} \sum_{i,j=1}^{N} J_{ij}\sigma_i\sigma_j = \frac{1}{2}\boldsymbol{\sigma}^T M\boldsymbol{\sigma}, \qquad \text{where } M_{ij} = \frac{1}{\sqrt{N}} J_{ij}$$

- M is a **GOE matrix** (Gaussian Orthogonal Ensemble).
  - It is symmetric.
  - Its entries are Gaussian and independent up to symmetry.
- In the **Spherical Sherrington Kirkpatrick (SSK)** model, $\boldsymbol{\sigma}$ no longer takes discrete values but rather

$$\boldsymbol{\sigma} \in S_{N-1}, \text{ the sphere of radius } \sqrt{N} \text{ in } \mathbb{R}^N.$$

- SSK is similar to the SK model but some analyses are easier due to its continuous nature.
  - Example: $\mathcal{H}(\boldsymbol{\sigma})$ is maximized when $\boldsymbol{\sigma}$ aligns with the leading eigenvector of M. This vector is almost surely not in $\{\pm 1\}^N$.

Free Energy

- The **free energy** of the model is

$$\mathcal{F}_N(\beta) := \frac{1}{N}\log \mathcal{Z}_N, \qquad \mathcal{Z}_N = \int_{S_{N-1}} e^{\frac{\beta}{2}\boldsymbol{\sigma}^T M \boldsymbol{\sigma}} d\omega_N(\boldsymbol{\sigma})$$

  - $\beta > 0$ is the inverse temperature parameter

Note: Integrand in $\mathcal{Z}_N$ is maximized when $\boldsymbol{\sigma}$ aligns with $\mathbf{u}_1$, the eigenvector of λ_1 (largest eigenvalue of M).

- The effect of temperature (heuristic observations):
  - At **high temperature**, β gets closer to 0, the integrand approaches a constant function.
  - At **low temperature**, β becomes large, the integrand has spikes at $\boldsymbol{\sigma} = \pm\mathbf{u}_1$, with height depending on λ_1.
- We might guess that
  - At very high temperatures, $\mathcal{F}_N$ depends on all eigenvalues
  - At very low temperatures, $\mathcal{F}_N$ depends mostly on λ_1

Background on free energy

The **limiting free energy** of SSK as $N \to \infty$ is

$$\mathcal{F}_N(\beta) \to \mathcal{F}(\beta) := \begin{cases} \frac{1}{4}\beta^2 & \beta \leq 1 \text{ (high temp)} \\ \beta - \frac{1}{2}\log\beta - \frac{3}{4} & \beta \geq 1 \text{ (low temp)} \end{cases}$$

- Kosterlitz, Thouless, Jones (1976) proposed the SSK model and computed $\mathcal{F}(\beta)$.
- Parisi (1980) and Crisanti, Sommers (1992) computed $\mathcal{F}(\beta)$ in more generalized settings.
- Talagrand (2006) rigorously proved the formulas of Parisi and Crisanti, Sommers.

What can be said about the **fluctuations** of $\mathcal{F}_N(\beta)$?

Free energy fluctuations

Baik and Lee (2016) obtained the fluctuations of $\mathcal{F}(\beta)$

$$N\Big(\mathcal{F}_N(\beta) - \mathcal{F}(\beta)\Big) \to \mathcal{N}(\mu_\beta, \sigma_\beta^2) \qquad \beta < 1 \text{ (high temp)}$$

$$N^{2/3}\Big(\mathcal{F}_N(\beta) - \mathcal{F}(\beta)\Big) \to \tfrac{\beta-1}{2} TW_1 \qquad \beta > 1 \text{ (low temp)}$$

- **Fluctuations have different magnitudes** (N^{-1} at high temp vs. $N^{-2/3}$ at low temp)
- **High temp: Gaussian**, depending on all eigenvalues
- **Low temp: Tracy-Widom**, depending on largest eigenvalue

Conjecture of Baik and Lee (critical temp window):
When $|\beta - 1| = O(N^{-1/3}\sqrt{\log N})$, fluctuations have order $N^{-1}\sqrt{\log N}$.

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[1] TW_1 denotes the GOE Tracy-Widom distribution (i.e. the rescaled fluctuations of the largest eigenvalue of GOE).

Free Energy - Temperature transition

The transitional regime conjectured by Baik and Lee was analyzed in two recent papers (independently)

- Landon (2022)
- Johnstone, Klochkov, Onatski, Pavlyshyn (2022)

Theorem (Johnstone, Klochkov, Onatski, Pavlyshyn (2022))

Given an SSK model with inverse temperature $\beta = 1 + bN^{-1/3}\sqrt{\log N}$ *for constant* $b \in \mathbb{R}$*, the free energy has the following convergence:*

$$\frac{N}{\sqrt{\frac{1}{6}\log N}}\left(\mathcal{F}_N(\beta) - \mathcal{F}(\beta) + \frac{\log N}{12N}\right) \xrightarrow{d} \mathcal{N}(0,1) + \sqrt{\frac{3}{2}}b_+ TW_1$$

where TW_1 *is a GOE Tracy-Widom distribution, independent from* $\mathcal{N}(0,1)$*, and* $b_+ = \max\{0, b\}$.

- **Fluctuations have order** $N^{-1}\sqrt{\log N}$ **throughout critical window**
- High temp side: Gaussian
- Low temp side: **Gaussian + Tracy-Widom (independent)**

Extending the result to bipartite spin glasses (C-W, Le)

- **SSK is a mean field model** (N particles, all pairs interact)

$$\mathcal{H}(\boldsymbol{\sigma}) = \frac{1}{2\sqrt{N}} \sum_{i,j=1}^{N} J_{ij} \boldsymbol{\sigma}_i \boldsymbol{\sigma}_j, \qquad \boldsymbol{\sigma} \in S_{N-1}$$

- **Bipartite SSK has two "species"** of sizes n, m and particles only interact with those from the other species

$$\mathcal{H}(\boldsymbol{\sigma}, \boldsymbol{\tau}) = \frac{1}{\sqrt{n+m}} \sum_{i=1}^{n} \sum_{j=1}^{m} A_{ij} \sigma_i \tau_j, \qquad \boldsymbol{\sigma} \in S_{n-1}, \ \boldsymbol{\tau} \in S_{m-1}$$

  Matrix A is $n \times m$ with i.i.d. Gaussian entries.

- **Why study this model?**
  - It is a step away from the mean field model
  - Has some applications (e.g. in biology and neural networks)
- **Free energy fluctuations of bipartite model** - Baik, Lee (2018)
  - Gaussian at high temp, Tracy-Widom at low temp
  - Critical inverse temp is not $\beta = 1$ but $\beta = \beta_c := (\frac{m}{n})^{1/4} \sqrt{1 + \frac{n}{m}}$

Temperature transition in bipartite model

Theorem (C-W, Le (2023))

Let $\mathcal{F}_{n,m}(\beta)$ denote the free energy of a **bipartite spherical model** *with species sizes n, m and let $\beta = \beta_c + bn^{-1/3}\sqrt{\log n}$. Then, as $n, m \to \infty$* **with fixed ratio** *$n/m = r + O(n^{-1})$ for $0 < r \leq 1$, the free energy has the convergence:*

$$\frac{n+m}{\sqrt{\frac{1}{6}\log n}}\left(\mathcal{F}_{n,m}(\beta) - \mathcal{F}_r(\beta) + \frac{\log n}{12n}\right) \xrightarrow{d} \mathcal{N}(0,1) + C_r b_+ TW_1$$

where $\mathcal{F}_r(\beta)$ denotes the limiting free energy, $b_+ = \max\{0, b\}$, and $\mathcal{N}(0,1)$, TW_1 denote independent Gaussian and Tracy-Widom terms.

Theorem (Johnstone, Klochkov, Onatski, Pavlyshyn (2022))

Given an **SSK model** *with $\beta = 1 + bN^{-1/3}\sqrt{\log N}$ for constant $b \in \mathbb{R}$,*

$$\frac{N}{\sqrt{\frac{1}{6}\log N}}\left(\mathcal{F}_N(\beta) - \mathcal{F}(\beta) + \frac{\log N}{12N}\right) \xrightarrow{d} \mathcal{N}(0,1) + \sqrt{\frac{3}{2}}b_+ TW_1$$

Key ingredients in proof

Contour integral representation of $\mathcal{Z}_N$

Due to specific properties of these models, one can simplify their partition functions

SSK spherical form: $\mathcal{Z}_N = \int_{S_{N-1}} e^{\beta \mathcal{H}(\boldsymbol{\sigma})} d\omega_N(\boldsymbol{\sigma})$

contour form: $\mathcal{Z}_N = C_N \int_{\gamma - i\infty}^{\gamma + i\infty} e^{N G_\beta(z)} dz$

Bipartite spherical form: $\mathcal{Z}_{n,m} = \int_{S_{m-1}} \int_{S_{n-1}} e^{\beta \mathcal{H}(\boldsymbol{\sigma}, \boldsymbol{\tau})} d\omega_n(\boldsymbol{\sigma}) d\omega_m(\boldsymbol{\tau})$

contour form: $\mathcal{Z}_{n,m} = C_{n,m} \int_{\gamma_1 - i\infty}^{\gamma_1 + i\infty} \int_{\gamma_2 - i\infty}^{\gamma_2 + i\infty} e^{n G_\beta(z_1, z_2)} dz_2 dz_1$

C_N, $C_{n,m}$ are constants, $G_\beta(z)$, $G_\beta(z_1, z_2)$ depend on eigenvalues of M:

- M is GOE (rescaled interaction matrix) for SSK
- M is Wishart ($M := \frac{1}{m} A A^T$ where A is interaction) for bipartite

Steepest descent analysis

- Note: G is a random function so it's saddle point is also random.
- Properties of random matrices (eigenvalue rigidity) enable analysis.

Key ingredients in proof

Asymptotic expression for free energy (obtained using contour integral and steepest descent analysis)

$$\mathcal{F}_{n,m}(\beta) = c_{\beta,n,m} - \frac{1}{n+m}\sum_{j=1}^{n} \log|d_+ - \lambda_j| + (\beta - \beta_c)_+(\lambda_1 - d_+)$$

where d_+ is upper edge of limiting spectral measure.

- $\frac{1}{n+m}\sum_{j=1}^{n} \log|d_+ - \lambda_j|$ has **Gaussian** fluctuations, order $n^{-1}\sqrt{\log n}$.
- $(\beta - \beta_c)_+(\lambda_1 - d_+)$ has **Tracy-Widom** fluctuations, order $n^{-1}\sqrt{\log n}$

CLT for the sum of logs term:

- This is delicate because d_+ is right at the spectral edge.
- SSK requires this CLT for <u>GOE matrix</u>
  (Lambert, Paquette 2020 and Johnstone et al 2020)
- Bipartite requires this CLT for <u>Wishart matrix</u>
  (C-W, Le 2022 - **second part of this talk**)

Key ingredients in proof

$$\mathcal{F}_{n,m}(\beta) = c_{\beta,n,m} - \underbrace{\frac{1}{n+m}\sum_{j=1}^{n}\log|d_+ - \lambda_j|}_{\text{asymptotically Gaussian}} + \underbrace{(\beta-\beta_c)_+(\lambda_1 - d_+)}_{\text{asymptotically Tracy-Widom}}$$

Asymptotic independence of sum and λ_1

- It has been demonstrated numerically (eg Edelman, Wang 2013) that λ_1 depends (asymptotically) only on a matrix minor of size $O(n^{1/3})$ in the tridiagonal form of the matrix.
- Johnstone et al and C-W, Le verify this (for GOE and Wishart matrices respectively) using recursive formulas on matrix minors.
- Asymptotically, λ_1 depends on a minor of size $n^{1/3}(\log\log n)^3$, while $\sum_{i=1}^{n}\log|d_+ - \lambda_i|$ is determined by the rest of the matrix.

SECTION 2: Edge CLT result for Wishart random matrices

GOAL: Given eigenvalues $\{\lambda_i\}$ of a Wishart matrix, derive a CLT for

$$\sum_{i=1}^{n} \log |d_+ - \lambda_i|$$

RESULT: We prove this CLT with the following generalizations:

- Wishart matrices $\longrightarrow$ Laguerre beta ensembles (LβE)
- $\sum \log |d_+ - \lambda_i| \longrightarrow \sum \log |\gamma_n - \lambda_i|$ where $\gamma_n = d_+$ or $\gamma_n \to d_+$ sufficiently fast

NOTE: This in not the same β from the spin glass theorems.

Laguerre random matrices

- **Laguerre Orthogonal Ensemble (LOE)** aka real Wishart matrix is an $n \times n$ random matrix $M_{n,m}$ constructed as

$$M_{n,m} := \frac{1}{m} A A^T$$

  where A is an $n \times m$ matrix (for $n \leq m$) with i.i.d $\mathcal{N}(0,1)$ entries.

- **Laguerre Beta Ensembles (LβE)** are matrices with joint eigenvalue distribution given by

$$p(\lambda_1, \lambda_2, \ldots, \lambda_n) = C_{n,m,\beta} \prod_{i<j} |\lambda_i - \lambda_j|^\beta \prod_{i=1}^{n} \left(\lambda_i^{\frac{\beta}{2}(m-n+1)-1} e^{\lambda_i/2} \right)$$

- Remarks:
  - LOE is a special case of LβE with $\beta = 1$.
  - We focus on the case where $\frac{n}{m} \to r$ as $n, m \to \infty$ for $0 < r \leq 1$.
  - Eigenvalues converge to the Marcenko-Pastur distribution which is supported on the interval $[d_-, d_+]$ where $d_\pm = (1 \pm \sqrt{r})^2$.

Log determinants

- We consider the quantity

$$\sum_{i=1}^{n} \log|\lambda_i - \gamma_n| = \log|\det(M_{n,m} - \gamma_n I_n)|$$

where γ_n approaches the upper edge of the spectrum of $M_{n,m}$.

- **CLT for linear eigenvalue statistics**: In the case of fixed $\gamma_n = \gamma$ outside the spectral support (see Bai, Silverstein 2004):

$$\sum_{i=1}^{n} \log|\lambda_i - \gamma| - n \int \log|x - \gamma| d\rho_{MP}(x) \ \to \ \mathcal{N}(\mu, \sigma^2)$$

for μ, σ^2 not n-dependent and ρ_{MP} the Marchenko-Pastur measure.
**When γ_n approaches the spectral edge, this theorem does not apply.*

- **Edge CLTs for the log determinant**
  - Johnstone, Klochkov, Onatski, Pavlyshyn (2020) - Obtain such a CLT for GβE and then extend to Wigner matrices.
  - Lambert, Paquette (2020) - Obtain such a CLT for GβE as a corrollary of a more detailed result on the characteristic polynomial.
  - Collins-Woodfin, Le (2022) - Obtain such a CLT for LβE.

Theorem (C-W, Le 2022)

Let $M_{n,m}$ be $L\beta E$ with $\frac{n}{m} = r + O(n^{-1})$ for $0 < r \leq 1$. Let $\gamma_n = d_+ + \sigma_n n^{-2/3}$ where d_+ is upper edge of Marchenko-Pastur measure. We take $(\log\log n)^2 \ll \sigma_n \ll (\log n)^2$. For $\beta = 1, 2$, allow $-C < \sigma_n \ll (\log n)^2$ Then,

$$\frac{\log\left|\det\left(M_{n,m} - \gamma_n\right)\right| - \mu_{n,m}}{\sqrt{\frac{2}{3\beta}\log n}} \to \mathcal{N}(0,1),$$

$$\mu_{n,m} = \left((1 - r^{-1})\log(1 + r^{\frac{1}{2}}) + \log(r^{\frac{1}{2}}) + r^{-\frac{1}{2}}\right) n$$
$$+ \frac{1}{r^{1/2}(1+r^{1/2})}\sigma_n n^{1/3} - \frac{2}{3r^{3/4}(1+r^{1/2})^2}\sigma_n^{3/2} - \frac{1}{6}\left(\frac{2}{\beta} - \left(\frac{1}{4} + \frac{3r^{1/2}}{2(r^{1/2}+1)^2}\right)\right)\log n$$

Key take-away: This log determinant (which is order n) has Gaussian fluctuations of order $\sqrt{\log n}$

Note comparison to:

- Sum of i.i.d variables – order $\sqrt{n}$ fluctuations
- Log-determinant with γ_n away from spectrum – order 1 fluctuations

This result is very similar to the one for $G\beta E$ and our proof methods are inspired by those of Johnstone et al.

Proof Sketch (and cool RMT results we used en route)

1. **Tridiagonal representation of LβE** (Dumitriu, Edelman 2002) An LβE matrix $M_{n,m}$ has the same joint eigenvalue distribution as $\frac{1}{m}BB^T$ for a bidiagonal matrix B satisfying

$$B = \begin{bmatrix} a_1 & & & \\ b_1 & a_2 & & \\ & \ddots & \ddots & \\ & & b_{n-1} & a_n \end{bmatrix}, \quad BB^T = \begin{bmatrix} a_1^2 & a_1 b_1 & & \\ a_1 b_1 & a_2^2 + b_1^2 & & \\ & & \ddots & a_{n-1} b_{n-1} \\ & & a_{n-1} b_{n-1} & a_n^2 + b_{n-1}^2 \end{bmatrix}$$

   where $\{a_i\}$, $\{b_i\}$ are independent, χ-distributed, with

$$a_i^2 \sim \frac{1}{\beta}\chi^2_{\beta(m-n+i)}, \qquad b_i^2 \sim \frac{1}{\beta}\chi^2_{\beta i}.$$

2. **Recurrence on determinants of matrix minors**:
   - We want to study $\det(BB^T - \gamma_n m I_n)$.
   - Definition: $D_i = \det\left(i \times i \text{ principal minor of } (BB^T - \gamma_n m I_n)\right)$
   - Recurrence: $D_i = (a_i^2 + b_{i-1}^2 - \gamma_n m)D_{i-1} - a_{i-1}^2 b_{i-1}^2 D_{i-2}$

Proof Sketch

1. Tridiagonal representation of LβE (Dumitriu, Edelman 2002)
2. Recurrence on determinants of matrix minors:

$$D_i = (a_i^2 + b_{i-1}^2 - \gamma_n m)D_{i-1} - a_{i-1}^2 b_{i-1}^2 D_{i-2}$$

3. **Transform to an approximately linear recurrence**
For R_i, a suitable rescaled and shifted version of the ratio D_i/D_{i-1},

$$R_i = \xi_i + \omega_i R_{i-1} + \varepsilon_i, \text{ where}$$

   - ξ_i depends on a_i, b_i
   - ω_i is deterministic with $0 < \omega_i < 1$,
   - ε_i is small.
4. **Analyze the recurrence** on R_i, obtain CLT for $\log D_n$ in the case where $(\log \log n)^2 \ll \sigma_n \ll \log^2 n$.
   - Involves concentration bounds on sub-gamma random variables, Hanson-Wright tail bounds on quadratic forms, etc.

Proof Sketch

1. Tridiagonal representation of LβE
2. Recurrence on determinants of matrix minors
3. Transform to an approximately linear recurrence
4. Analyze recurrence, obtain CLT for $(\log\log n)^2 \ll \sigma_n \ll \log^2 n$.
5. **Extend result to $-C < \sigma_n \ll \log^2 n$ in the case of $\beta = 2$**
   - Relies on result specific to $\beta = 2$ (Götze, Tikhomorov 2005).
6. **Obtain extension for $\beta = 1$ by relating LUE, LOE eigenvalues**
   - **Theorem** *(Forrester, Rains 2001): Let $LOE_{n,m}$, $LOE_{n+1,m+1}$, $LUE_{n,m}$ denote the eigenvalue sets of independent matrices with the given parameters. Then*

     $$even(LOE_{n,m} \cup LOE_{n+1,m+1}) = LUE_{n,m}$$

     where the equality is in distribution.

How does this compare to the proof of Johnstone et al for GβE?

- General approach is similar
- LβE analysis is more complicated due to the more intricate tridiagonal structure (dependence between adjacent entries, non-identical distributions on the diagonal)

Statistical application - critically spiked matrix models

Spiked matrix models

$$M = H + c\mathbf{x}\mathbf{x}^T$$

- H is a Gaussian or Wishart random matrix ("noise").
- $\mathbf{x}$ is a deterministic vector ($\mathbf{x}\mathbf{x}^T$ is "spike" or "signal").
- c is the spike magnitude.

BBP transition (Baik, Ben Arous, Péché)

- For fixed $c > d_+$, largest eigenvalue of M separates from the bulk.
- For fixed $c \leq d_+$, largest eigenvalue does not separate (spectrum resembles that of H)
- In the case $c \downarrow d_+$, other methods are needed.

Role of edge CLTs

- These results (Johnstone et al for GOE; C-W and Le for LOE) are relevant to analyzing log-likelihood ratios for critically spiked models (Gaussian, Wishart respectively).
- The result for LOE is particularly of interest due to connection with sample covariance matrices.

Thank you for listening!