Resurgence and Irregular singular ODEs

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Equations différentielles motiviques et au-delà
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Irregular singular ODEs

DEFINITION: A homogeneous differential equation

$$Y^{(d)} + \sum_{j=0}^{d-1} p_j(z) Y^{(j)} = 0$$

has a regular singularity at $z = a$ if $p_{n-j}(z)$ has a pole of order at most j at a.

Otherwise $z = a$ is an irregular singularity

If the equation is non-homogeneous we refer to its singular behaviour as to the one of its homogeneous part.

Examples

1. $Y' = Y + \frac{1}{z}$ irregular singular at $z = \infty$

2. $Y'' - Y + \frac{1}{z}Y' + \left(\frac{m}{n}\right)^2 \frac{1}{z^2}Y = 0$ irregular singular at $z = \infty$, $m, n \in \mathbb{Z}$

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3. $\zeta(1-\zeta)y'' + 3\left(\frac{1}{2} - \zeta\right)y' + \left[1 - \left(\frac{m}{n}\right)^2\right]y = 0$ regular singular at $\zeta = 0,1,\infty$

Some features of irregular ODEs

Irregular singular ODEs	Regular singular ODEs
Formal divergent solutions	Formal convergent solutions
Monodromy + Stokes data	Monodromy data

Formal divergent solutions

Ansatz: $\sum_{n=0}^{\infty} a_n z^{-n+\alpha}$, $\alpha \in \mathbb{Q}$ and determine the coefficients a_n solving order by order

1. $Y' = Y + \frac{1}{z}$ $\longrightarrow$ $Y = \sum_{n=0}^{\infty} n!\, z^{-n-1}$

2. $Y'' - Y + \frac{1}{z}Y' + \left(\frac{m}{n}\right)^2 \frac{1}{z^2} Y = 0$ $\longrightarrow$ $Y = 0$

Ansatz: $\sum_{n=0}^{\infty} a_n \zeta^{n+\alpha}$, $\alpha \in \mathbb{Q}$

3. $\zeta(1-\zeta)y'' + 3\left(\frac{1}{2} - \zeta\right)y' + \left[1 - \left(\frac{m}{n}\right)^2\right]y = 0$ $\longrightarrow$ $y = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}-\frac{m}{n}\right)_k \left(\frac{1}{2}+\frac{m}{n}\right)_k}{k!\,\left(\frac{1}{2}\right)_k} \zeta^{n-\frac{1}{2}}$

$(a)_s = \frac{\Gamma(a+s)}{\Gamma(a)}$

Formal divergent solutions

Ansatz: $\sum_{n=0}^{\infty} a_n z^{-n+\alpha}$, $\alpha \in \mathbb{Q}$ and determine the coefficients a_n solving order by order

In Example 1. gives a non trivial solution, but in Example 2. gives zero!

Poincaré Ansatz: $e^{\lambda z} \sum_{n=0}^{\infty} a_n z^{-n+\alpha}$, $\alpha \in \mathbb{Q}$, $\lambda \in \mathbb{C}$

Example 2.

$$Y = e^{-z} z^{-1/2} \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2} - \frac{m}{n}\right)_k \left(\frac{1}{2} + \frac{m}{n}\right)_k}{2^k \, k!} z^{-k}$$

Poincaré formal solutions

If the equation is in the form

$$\left[P(\partial_z) + \sum_{j=1}^{d-1} \frac{1}{z^j} Q_j(\partial_z) + \frac{1}{z^2} R(z^{-1})\right] Y = 0$$

with P a degree d polynomial, Q_j a degree $d-j$ polynomial and R holomorphic, then the ansatz will be

$$Y_\lambda(z) = e^{\lambda z} \sum_{n=0}^{\infty} a_n z^{-n+\alpha}$$

with $P(\lambda) = 0$ and $\alpha = -\dfrac{Q_1(-\lambda)}{P'(-\lambda)}$.

Frame of solutions

A frame of (formal) solutions of an irregular singular ODE would be given by trans-monomial of the form $\tilde{\Phi}_\lambda(z) = e^{\lambda z} \sum_{n=0}^{\infty} a_n z^{-n+\alpha}$, $\alpha \in \mathbb{Q}$, $\lambda \in \mathbb{C}$ for different values λ.

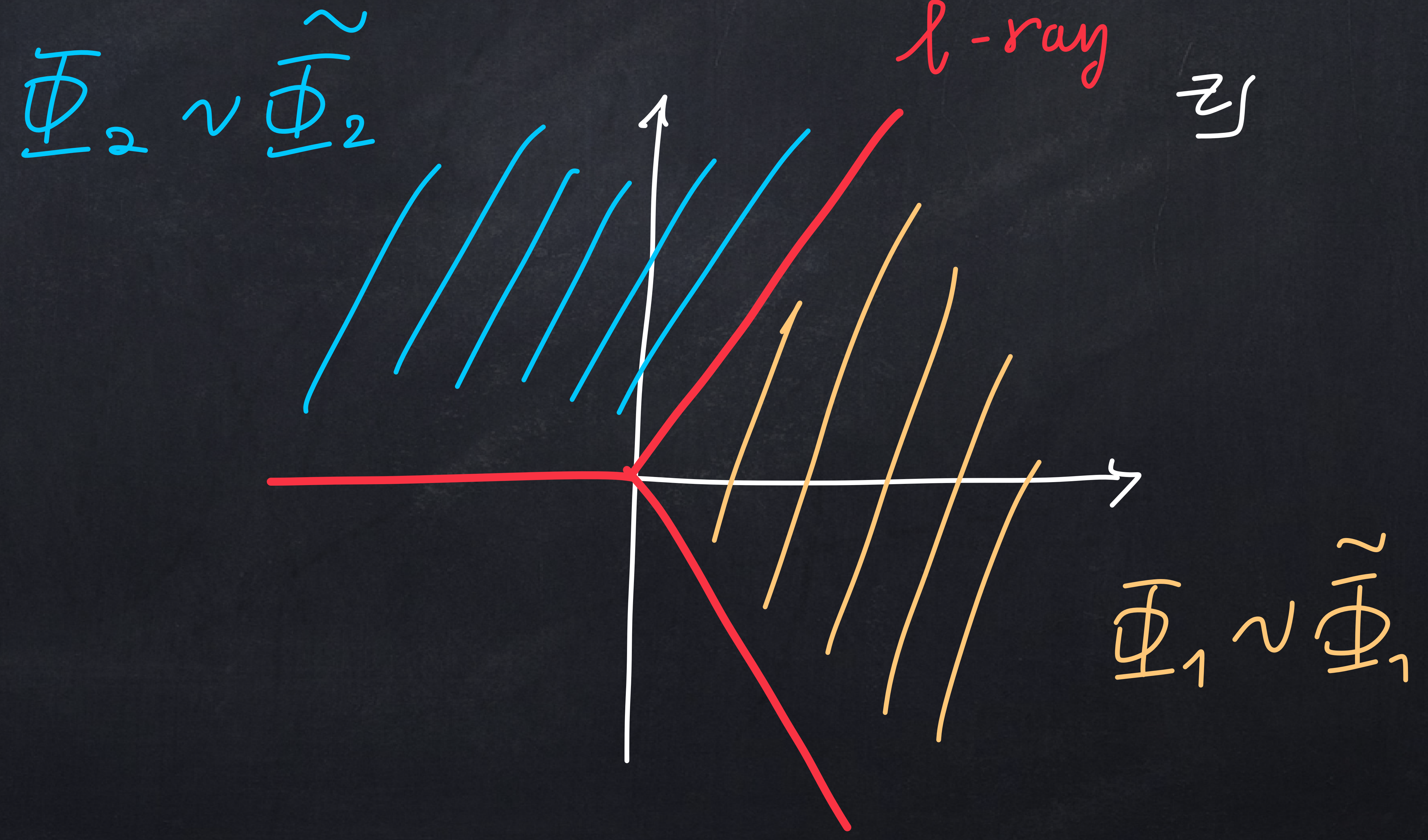
Different $\tilde{\Phi}_\lambda$ will be dominant in different sectors at infinity.

Then a frame of analytic solutions $\Phi_\lambda(z)$ would be characterised by being asymptotic to $\tilde{\Phi}_\lambda$ as $z \to \infty$ in a given sector at infinity.

Stokes data

Stokes data consists of Stokes rays, i.e. the rays that separate two sectors where the the asymptotic behaviour of a frame of solutions changes, and Stokes matrices which encode the jump of the solutions

$$\begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix} = \begin{bmatrix} 1 & S_\ell \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix}$$



Some features of irregular ODEs

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Formal divergent solutions — Resurgence

Having formal divergent solutions makes irregular singular ODEs good candidates to test predictions of resurgence.

The theory of resurgence introduced by Écalle in ‘80 is the theory of divergent power series.

Resurgence — Stokes data

In a nutshell, resurgence take as input *resurgent* formal power series and gives as output the so called *resurgent structure:*

$$\tilde{\Phi} \in \mathbb{C}[[z^{-1}]] \;\; \textit{resurgent} \;\longrightarrow\; \{\omega, S_\omega, \tilde{\phi}_\omega\} \;\textit{resurgent structure}$$

where *resurgent structure* consists of the following data:

- The position of singularities $\omega \in \mathbb{C}_\zeta$
- The Stokes constants $S_\omega \in \mathbb{C}$
- New convergent power series $\tilde{\phi}_\omega \in \mathbb{C}\{\zeta\}$

How does it work?

Borel transform and analytic continuation

DEFINITION: the Borel transform is a formal map $\mathscr{B}: \mathbb{C}[[z^{-1}]] \to \mathbb{C}\delta + \mathbb{C}[[\zeta]]$

$$\mathscr{B}\left[z^{-n-1}\right] := \frac{\zeta^n}{n!}$$

and then extend by *countably* linearity

$$\mathscr{B}\left[\sum_{n=0}^{\infty} a_n z^{-n-1}\right] := \sum_{n=0}^{\infty} a_n \frac{\zeta^n}{n!}$$

Furthermore, one can define $\mathscr{B}[1] := \delta$ where δ is the convolution unit.

How does it work?

Borel transform and analytic continuation

DEFINITION: the Borel transform is a formal map $\mathscr{B}: z^{\alpha}\mathbb{C}[[z^{-1}]] \to \zeta^{-\alpha}\mathbb{C}[[\zeta]]$,

$$\mathscr{B}\left[z^{-n-1+\alpha}\right] := \frac{\zeta^{n-\alpha}}{\Gamma(n+1-\alpha)} \qquad \alpha \in \mathbb{Q}\backslash\mathbb{Z}$$

And then extend by *countably* linearity

$$\mathscr{B}\left[\sum_{n=0}^{\infty} a_n z^{-n-1+\alpha}\right] := \sum_{n=0}^{\infty} a_n \frac{\zeta^{n-\alpha}}{\Gamma(n+1-\alpha)}$$

How does it work?

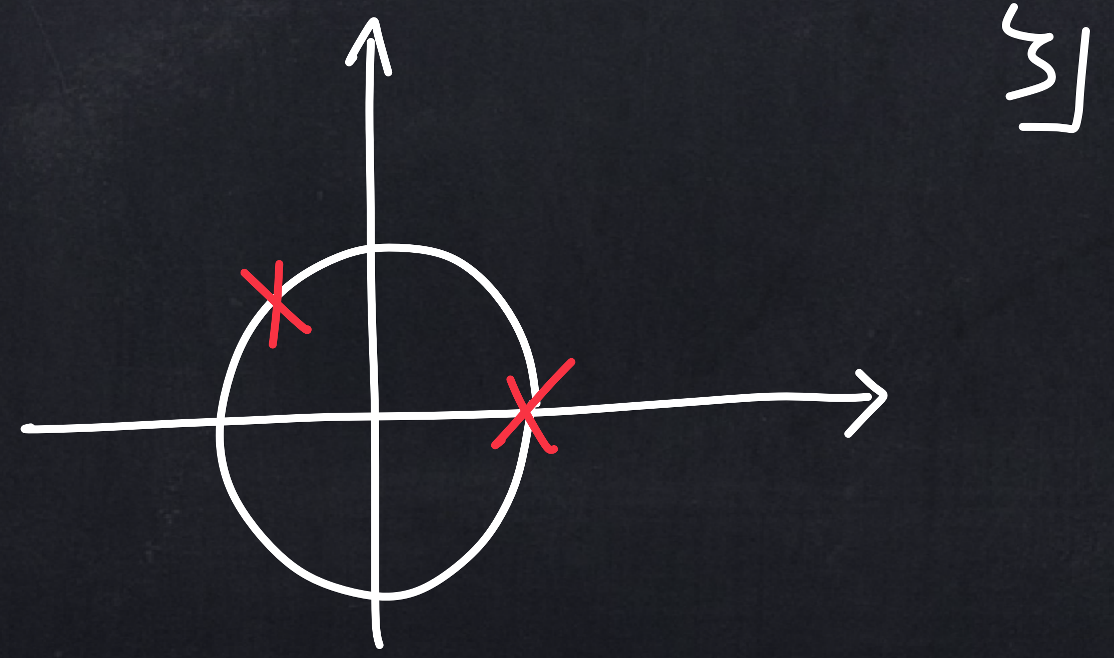
Borel transform and analytic continuation

DEFINITION: A formal series $\tilde{\Phi} \in \mathbb{C}[[z^{-1}]]$ is 1-Gevrey if its coefficients a_n grow as

$$|a_n| \leq C A^n n!\,, \quad C, A > 0.$$

The space of 1-Gevrey series is denoted by $\mathbb{C}[[z^{-1}]]_1$.

REMARK: if $\tilde{\Phi} \in \mathbb{C}[[z^{-1}]]_1$ i.e., then $\tilde{\phi} := \mathscr{B}\tilde{\Phi} \in \mathbb{C}\{\zeta\}$. Thus it is possible to study the analytic continuation of $\tilde{\phi}$



Resurgent functions/series

DEFINITION: An analytic function $\tilde{\phi}(\zeta) \in \mathbb{C}\{\zeta\}$ is resurgent if it can be endlessly analytically continued, i.e. for every L>0 there exists a finite subset $\Omega_L \subset \mathbb{C}$ such that $\tilde{\phi}(\zeta)$ can be analytically continued along every path of length less than L which starts from $\zeta = 0$ and avoids Ω_L.

DEFINITION: A 1-Gevrey series $\tilde{\Phi} \in \mathbb{C}[[z^{-1}]]_1$ is resurgent if its Borel transform $\tilde{\phi}$ is a resurgent function.

Examples

1. $\displaystyle\sum_{n=0}^{\infty} n!\, z^{-n-1} \in \mathbb{C}[\![z^{-1}]\!]_1$ is resurgent

2. $\displaystyle\sum_{n=0}^{\infty} a_n n!\, z^{-n-1} \in \mathbb{C}[\![z^{-1}]\!]_1$, where $a_n = \begin{cases} 1 & \text{if } n = 2^k \\ 0 & \text{otherwise} \end{cases}$ is NOT resurgent

Resurgent 1-Gevrey series $\subset$ 1-Gevrey series

Example 1.

$$\tilde{\Phi} = \sum_{n=0}^{\infty} n!\, z^{-n-1}$$

$$\mathcal{B} \downarrow$$

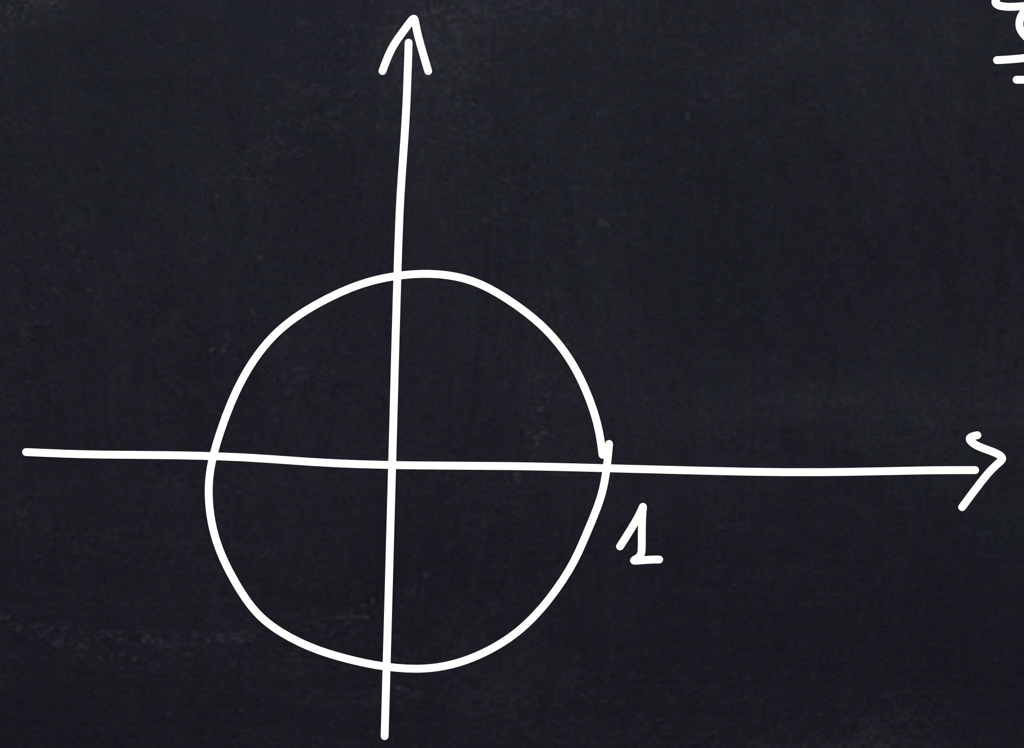
$$\tilde{\phi} = \sum_{n=0}^{\infty} n!\, \frac{\zeta^n}{n!}$$

Example 1.

$$\tilde{\Phi} = \sum_{n=0}^{\infty} n! \, z^{-n-1}$$

$$\Big\downarrow \mathcal{B}$$

$$\tilde{\phi} = \sum_{n=0}^{\infty} \cancel{n!} \frac{\zeta^n}{\cancel{n!}} \in \mathbb{C}\{\zeta\}$$



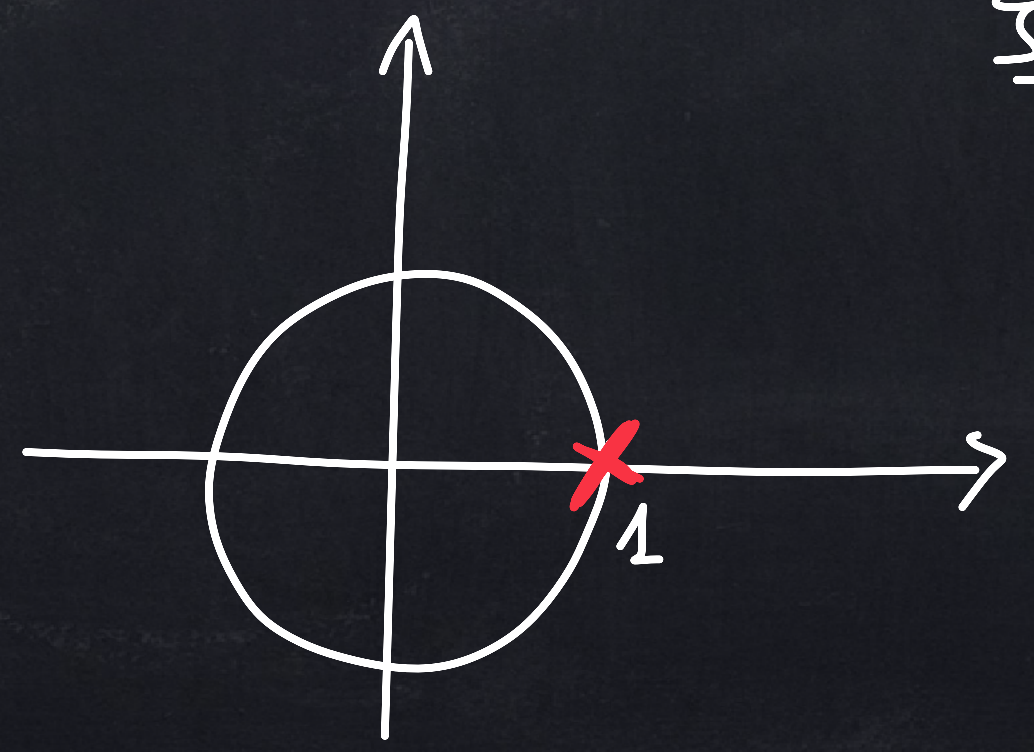
ζ

Example 1.

$$\tilde{\Phi} = \sum_{n=0}^{\infty} n! \, z^{-n-1}$$

$$\Big\downarrow \mathcal{B}$$

$$\tilde{\phi} = \sum_{n=0}^{\infty} \cancel{n!} \, \frac{\zeta^n}{\cancel{n!}} \in \mathbb{C}\{\zeta\} \xrightarrow[\zeta]{\text{sum}} \hat{\phi} = \frac{1}{1-\zeta}$$



simple pole at $\zeta = 1$

Example 2.

$$\tilde{\Phi} = \sum_{n=0}^{\infty} a_n \, n! \, z^{-n-1} \qquad a_n = \begin{cases} 1 & n = 2^k \\ 0 & \text{otherwise} \end{cases}$$

$$\Big\downarrow \mathcal{B}$$

$$\tilde{\phi} = \sum_{n=0}^{\infty} a_n \, n! \, \frac{\zeta^n}{n!}$$

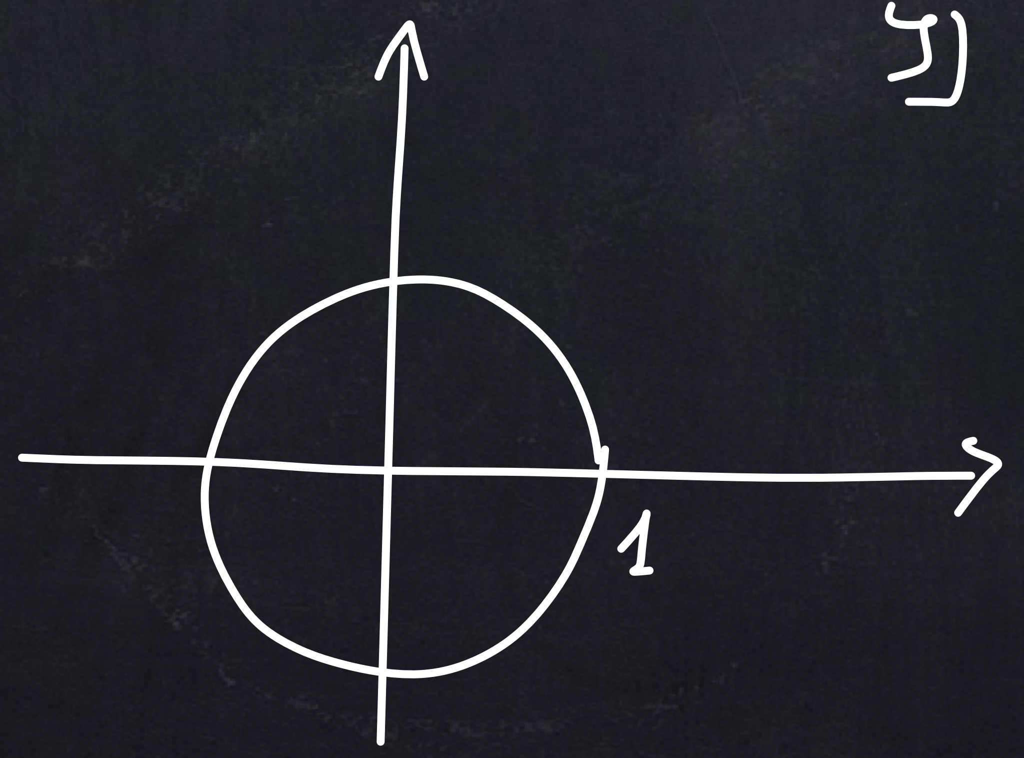
Example 2.

$$\tilde{\Phi} = \sum_{n=0}^{\infty} a_n \, n! \, z^{-n-1}$$

$$a_n = \begin{cases} 1 & n = 2^k \\ 0 & \text{otherwise} \end{cases}$$

$\mathcal{B} \downarrow$

$$\tilde{\phi} = \sum_{n=0}^{\infty} a_n \, n! \, \frac{\zeta^n}{n!}$$

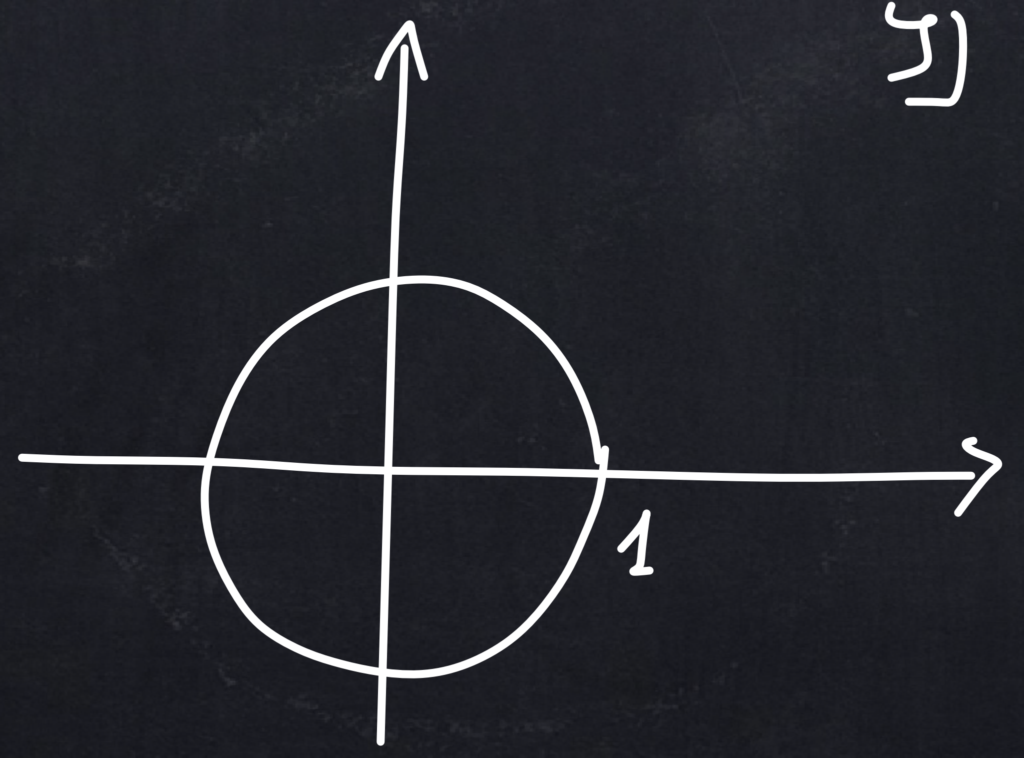


Example 2.

$$\tilde{\Phi} = \sum_{n=0}^{\infty} a_n \, n! \, z^{-n-1} \qquad a_n = \begin{cases} 1 & n = 2^k \\ 0 & \text{otherwise} \end{cases}$$

$$\Big\downarrow \mathcal{B}$$

$$\tilde{\phi} = \sum_{n=0}^{\infty} a_n \, n! \, \frac{\zeta^n}{n!}$$



$\zeta_k^{\ell} = e^{2\pi i \, \ell/2^k}$ is a singular point $\forall \ell \in \mathbb{Z},\ k \in \mathbb{Z}_{>0}$.

$\Longrightarrow$ can't analytically continue $\tilde{\phi}$!

Resurgent structure

DEFINITION: A resurgent function $\tilde{\phi} \in \mathbb{C}\{\zeta\}$ is simple if it has simple poles or logarithmic singularities at $\omega \in \mathbb{C}_\zeta$

$$\hat{\phi}(\zeta + \omega) = \frac{S_\omega}{2\pi i \zeta} + \text{reg.}$$

$$\hat{\phi}(\zeta + \omega) = \frac{S_\omega}{2\pi i} \log(\zeta)\ \tilde{\phi}_\omega(\zeta) + \text{reg.}$$

The set of singularities ω, the Stokes constants S_ω and $\tilde{\phi}_\omega$ constitutes the resurgent structure of $\tilde{\phi}$.

Resurgent structure — Stokes data

If $\tilde{\Phi} \in \mathbb{C}[[z^{-1}]]_1$ is a solution of an irregular singular ODE, the resurgent structure of its Borel transform determines the Stokes data.

WHY?

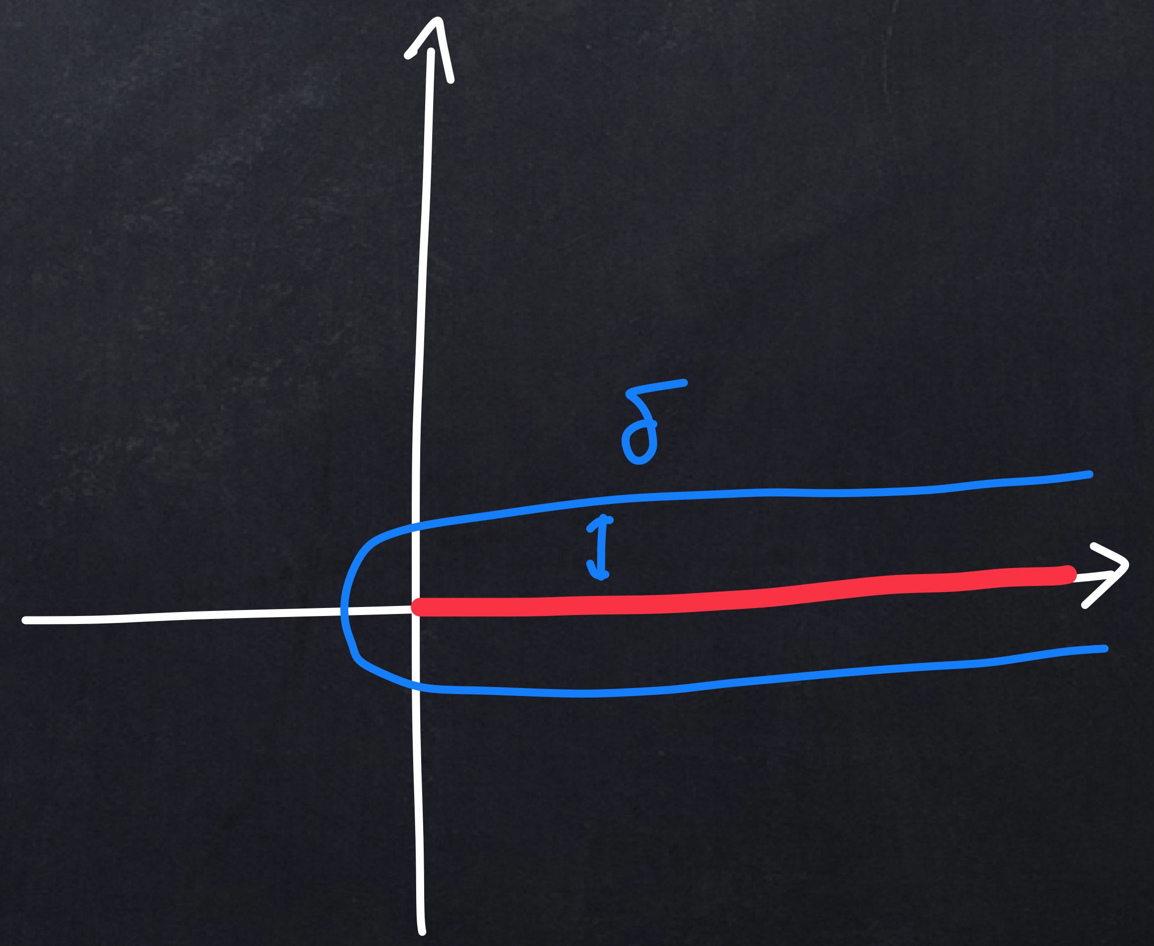
Thinking in terms of Borel-Laplace summabilty…

Laplace transform

DEFINITION: let ϕ be an holomorphic function such that $|\phi| \leq Ce^{A|\zeta|}, A, C > 0$ in a tubular neighbourhood of the ray $[0, +\infty)$,

$$\mathscr{L}\phi(z) := \int_0^\infty e^{-z\zeta}\phi d\zeta$$

and it is an holomorphic function in a half-plane $\Re z > A$.



Laplace transform along a ray in the direction θ

DEFINITION: let ϕ be an holomorphic function such that $|\phi| \leq Ce^{A|\zeta|}, A, C > 0$ in a tubular neighbourhood of the ray $[0, e^{i\theta}\infty)$

$$\mathscr{L}^\theta\phi(z) := \int_0^{\infty e^{i\theta}} e^{-z\zeta}\phi d\zeta$$

and it is an holomorphic function in a half-plane $\Re(e^{-i\theta}z) > A$.

Example 1.

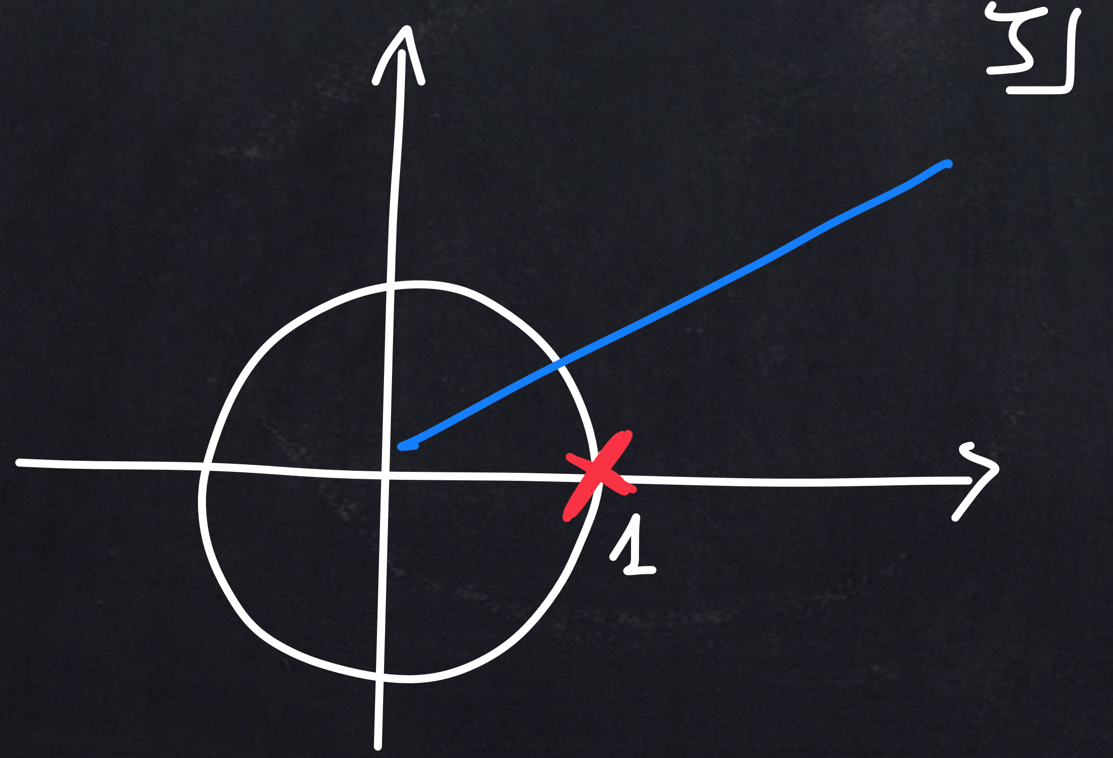
$$\tilde{\Phi} = \sum_{n=0}^{\infty} n!\, z^{-n-1}$$

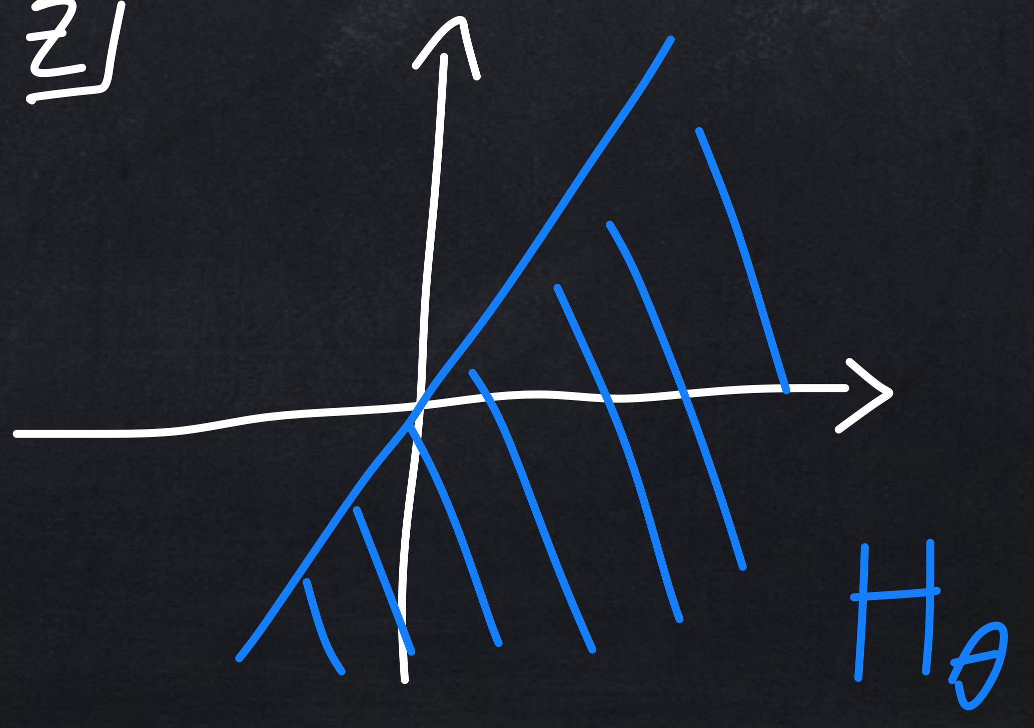
$\downarrow \mathcal{B}$

$$\tilde{\phi} = \sum_{n=0}^{\infty} \cancel{n!}\, \frac{\zeta^n}{\cancel{n!}} \in \mathbb{C}\{\zeta\} \xrightarrow{\text{sum}} \hat{\phi} = \frac{1}{1-\zeta}$$

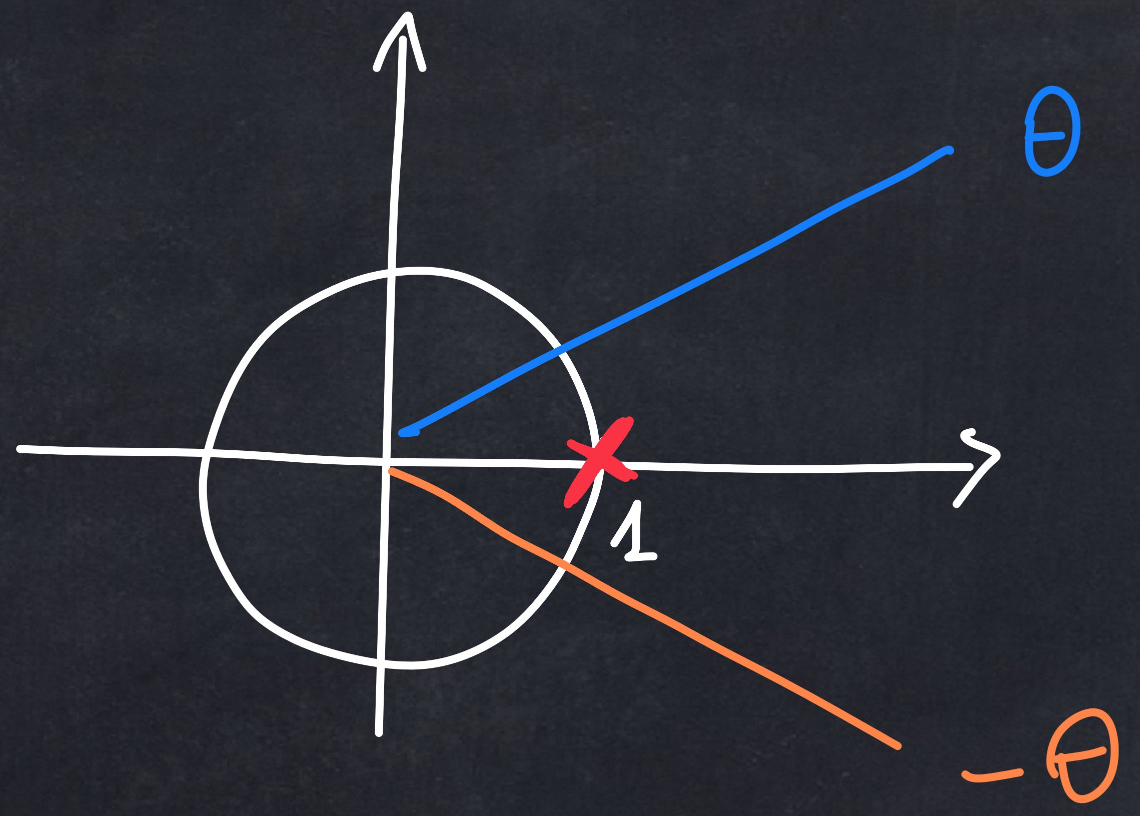
$\uparrow \mathcal{L}^{\theta}$

$$\Phi \in \mathcal{O}(H_\theta)$$





Example 1.

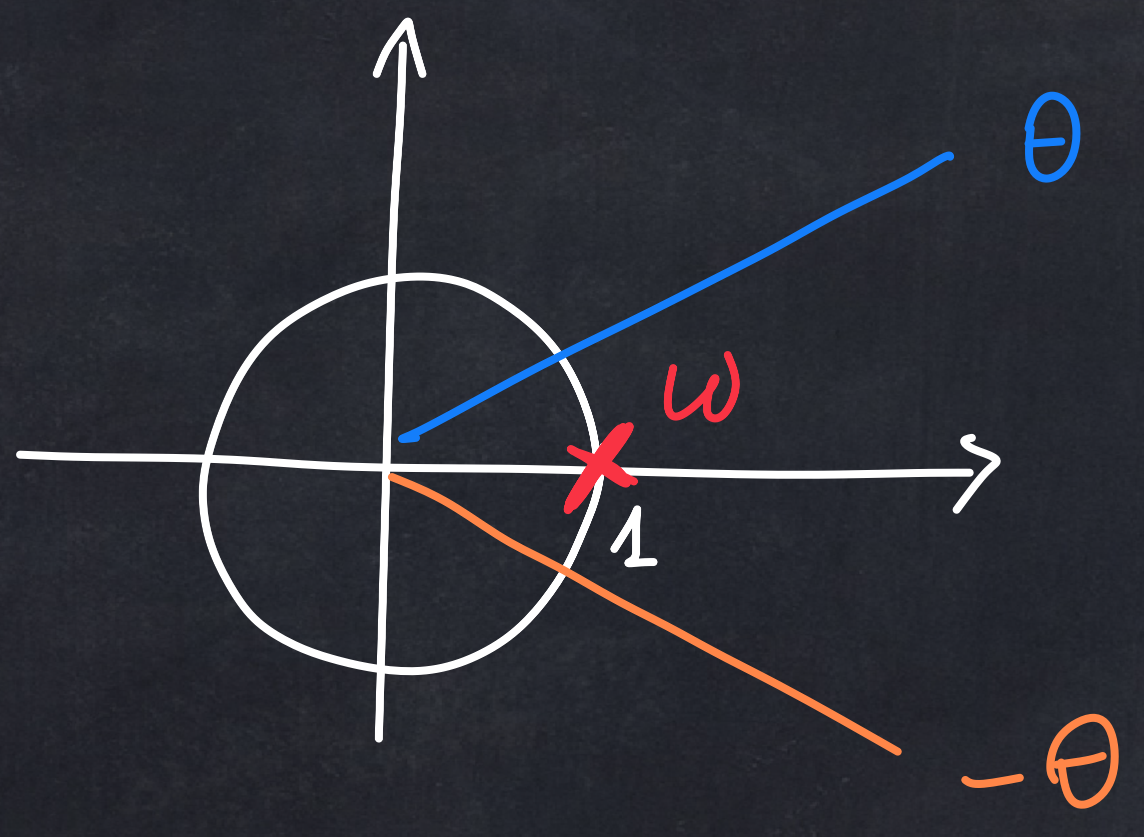


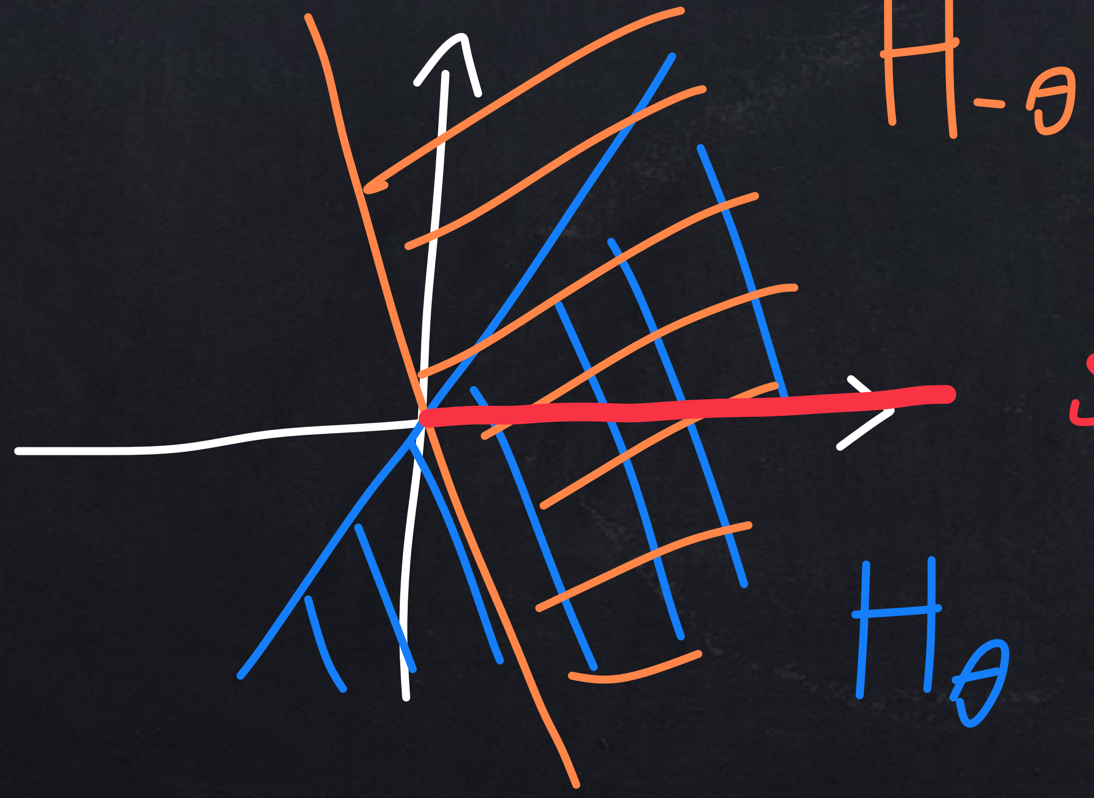
$$\mathcal{L}^{\theta}\hat{\phi} - \mathcal{L}^{-\theta}\hat{\phi} = 2\pi i\, e^{-z}$$

analytic in H_θ

analytic in $H_{-\theta}$

Example 1.





stokes ray

$$\mathcal{L}^{\theta}\hat{\phi} - \mathcal{L}^{-\theta}\hat{\phi} = 2\pi i\, e^{-1\cdot z}$$

analytic in H_θ

analytic in $H_{-\theta}$

$w = 1$

$S_w = 2\pi i$

Example 2.

$$Y'' - Y + \frac{1}{z}Y' + \left(\frac{m}{n}\right)^2 \frac{1}{z^2}Y = 0$$

$$Y_1 = e^{-z} z^{-1/2} \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{1}{2}-\frac{m}{n}\right)_k \left(\frac{1}{2}+\frac{m}{n}\right)_k}{2^k\, k!} z^{-k}$$

$$\mathcal{B} \downarrow$$

$$y_1(\zeta+1) = \zeta^{-1/2} \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}-\frac{m}{n}\right)_k \left(\frac{1}{2}+\frac{m}{n}\right)_k}{2^k\, k!} \frac{(-\zeta)^k}{\Gamma(k+\frac{1}{2})}$$

$$= \zeta^{-1/2}\, {}_2F_1\left(\frac{1}{2}-\frac{m}{n}, \frac{1}{2}+\frac{m}{n}; \frac{1}{2}; -\frac{\zeta}{2}\right)$$

$\omega = 1$

$\omega = -1$

Example 2.

$$Y'' - Y + \frac{1}{z}Y' + \left(\frac{m}{n}\right)^2 \frac{1}{z^2}Y = 0$$

$$y_1(\zeta+1) = \zeta^{-1/2} \; {}_2F_1\left(\frac{1}{2}-\frac{m}{n}, \frac{1}{2}+\frac{m}{n}; \frac{1}{2}; -\frac{\zeta}{2}\right)$$

$w=-1$ is a log singularity

From the analytic continuation you get a new function

$$y_{-1}(\zeta) \propto (1+\zeta)^{-1/2} \; {}_2F_1\left(\frac{1}{2}-\frac{m}{n}, \frac{1}{2}+\frac{m}{n}; \frac{1}{2}; \frac{1+\zeta}{2}\right)$$

Example 2.

$$Y'' - Y + \frac{1}{z}Y' + \left(\frac{m}{n}\right)^2 \frac{1}{z^2}Y = 0$$

$$y_1(\zeta+1) = \zeta^{1/2} \; {}_2F_1\left(\tfrac{1}{2}-\tfrac{m}{n}, \tfrac{1}{2}+\tfrac{m}{n}; \tfrac{1}{2}; -\tfrac{\zeta}{2}\right)$$

$w=-1$ is a log singularity

$$y_{-1}(\zeta) \propto (1+\zeta)^{1/2} \; {}_2F_1\left(\tfrac{1}{2}-\tfrac{m}{n}, \tfrac{1}{2}+\tfrac{m}{n}; \tfrac{1}{2}; \tfrac{1+\zeta}{2}\right) = B\,\tilde{Y}_2$$

and $\tilde{Y}_2$ is another solution!

Example 2.

$$Y'' - Y + \frac{1}{z}Y' + \left(\frac{m}{n}\right)^2 \frac{1}{z^2}Y = 0$$

Formal frame using Poincaré ansatz

$$\tilde{Y}_1(z) = e^{-z}z^{-1/2}\sum_{k=0}^{\infty} a_k z^{-k}$$

$$\tilde{Y}_2(z) = e^{z}z^{-1/2}\sum_{k=0}^{\infty} (-1)^k a_k z^{-k}$$

Then Borel-Laplace summation Y_1, Y_2 analytic solutions

Formal solution using Poincaré ansatz

$$\tilde{Y}_1(z) = e^{-z}z^{-1/2}\sum_{k=0}^{\infty} a_k z^{-k}$$

Then resurgence $\hat{y}_1(\zeta)$ analytic away from the singularity ω,

At ω, we see a new function $\hat{y}_2(\zeta)$

Resurgence vs Borel-Laplace summation

More generally, Écale introduced the Alien calculus and the formalism of singularities to compute the resurgent structure from the study of the *simple* resurgent germ $\tilde{\phi} \in \mathbb{C}\{\zeta\}$ in the Borel plane $\mathbb{C}_\zeta$

- no need to check growth conditions to compute the Laplace transform.
- Borel transform turns irregular singular ODEs into regular singular differential equations. Thus, one could use the theory of regular singular ODEs to extract informations about the analytic continuation of $\tilde{\phi}$.

Geometric interpretation of the Stokes constants

Thimble integrals are solutions of irregular singular ODEs

$$I_\lambda(z) = \int_{\mathscr{C}_\lambda} e^{-zf}\nu$$

When f is regular enough, the Stokes constants can be computed by intersection numbers of thimbles $\mathscr{C}_\lambda$.

In fact, I_λ can be turned into Laplace transform integrals, thus it is equivalently the Borel-Laplace sum of its asymptotics.

Summary

Borel →

← Laplace

Irregular singular ODEs	Regular singular ODEs
Formal divergent solutions	Formal convergent solutions
Monodromy + Stokes data	Monodromy data
$Y'' - Y + \frac{1}{z}Y' + \left(\frac{m}{n}\right)^2 \frac{1}{z^2}Y = 0$	$\zeta(1-\zeta)y'' + 3\left(\frac{1}{2} - \zeta\right)y' + \left[1 - \left(\frac{m}{n}\right)^2\right]y = 0$

Example 2. and 3. : irregular vs regular

$$Y'' - Y + \frac{1}{z}Y' + \left(\frac{m}{n}\right)^2 \frac{1}{z^2}Y = 0 \quad \xrightarrow{\mathcal{B}}$$

$$\zeta_1(1-\zeta_1)y_1'' + 3\left(\tfrac{1}{2} - \zeta_1\right)y_1' + \left[1 - \left(\tfrac{m}{n}\right)^2\right]y_1 = 0$$

$$\zeta_1 = \frac{\zeta - 1}{2}$$

$$\zeta_2(1-\zeta_2)y_2'' + 3\left(\tfrac{1}{2} - \zeta_2\right)y_2' + \left[1 - \left(\tfrac{m}{n}\right)^2\right]y_2 = 0$$

$$\zeta_2 = \frac{\zeta + 1}{2}$$

Thank you for your attention