A Framework for Structured Linearizations of Matrix Polynomials in Various Bases

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- Linearizations of (matrix) polynomials.
- Dual (minimal) bases, related to linear system theory.
- How to take structures from polynomials and preserve them in linearizations (spoiler alert: using dual bases!).
- How to deal with different bases in the definition of a polynomial.

Vector spaces of rational functions

One can consider the vector space $\mathbb{F}^n(\lambda)$ of rational functions in the variable λ:

$$v(\lambda) \in \mathbb{F}^n(\lambda) \iff v(\lambda) = \begin{bmatrix} \frac{p_1(\lambda)}{q_1(\lambda)} \\ \vdots \\ \frac{p_n(\lambda)}{q_n(\lambda)} \end{bmatrix}$$

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$\mathbb{F}(\lambda)$ is a field, so everything is nice. However, every vector can be written as $v(\lambda) = q(\lambda)^{-1}\tilde{v}(\lambda)$ with $v(\lambda) \in \mathbb{F}[\lambda]$.

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Switching between polynomials and rational functions is useful for theoretical reasons.

Bases for rational vector spaces

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- $\mathcal{V}$ admits a basis composed only of *polynomials*.
- Among these bases, we can look for the ones whose sum of column-degrees is *minimal*. The basis is not unique, but its column-degrees are.

We call these bases *minimal*, and their column degrees *minimal indices*.

Dual spaces

We can look for the dual space $\mathcal{V}^{\perp}$, which has dimension $n - r$.

$$\mathcal{V}^{\perp} = \{g(\lambda) \in \mathbb{F}^n(\lambda) \mid g(\lambda)^T f(\lambda) = 0 \ \forall f(\lambda) \in \mathcal{V}\}.$$

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If $B(\lambda)$ is a dual basis to $A(\lambda)$ if $A(\lambda)^T B(\lambda) = 0$. This is equivalent to saying that $B(\lambda)$ is a basis of $\mathcal{V}^\perp$. If both $A(\lambda)$ and $B(\lambda)$ are minimal we say that they form a pair of *dual minimal bases*.

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$$\begin{bmatrix} 1 & -\lambda & & \\ & \ddots & \ddots & \\ & & 1 & -\lambda \end{bmatrix} \begin{bmatrix} \lambda^k \\ \vdots \\ 1 \end{bmatrix} = 0 \qquad \text{are dual minimal bases.}$$

Properties of dual minimal bases

Let $A(\lambda)^T B(\lambda) = 0$ be a pair of dual minimal bases.

- The sum of degrees of a minimal basis and of a minimal basis of the dual space *always coincide*:

$$\sum_{i=1}^{r} \deg A(\lambda)e_i = \sum_{i=1}^{n-r} B(\lambda)e_i.$$

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These facts are crucial in proving backward stability results.

Building linearizations

We are interested in computing *eigenvalues* and *eigenstructure* of a matrix polynomial $P(\lambda)$:

$$P(\lambda) = \sum_{i=0}^{n} P_i \lambda^i, \quad \text{or, more generally,} \quad P(\lambda) = \sum_{i=0}^{n} P_i \phi_i(\lambda).$$

where $\Phi = \{\phi_0(\lambda), \ldots, \phi_n(\lambda)\}$ is some polynomial basis (monomials, Chebyshev, Lagrange, Newton, ...).

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We look for a pencil $\mathcal{L}(\lambda)$ with the same spectral properties. We say that $\mathcal{L}(\lambda)$ is a linearization for $P(\lambda)$ if and only if:

$$P(\lambda) \oplus I_{m(n-1)} = E(\lambda)\mathcal{L}(\lambda)F(\lambda), \quad \det E(\lambda), \det F(\lambda) \text{ invertible in } \mathbb{F}[\lambda].$$

From dual bases to linearizations

Given a basis $\Phi := \{\phi_0(\lambda), \ldots, \phi_n(\lambda)\}$ we say that $L_\phi(\lambda)$ and $\pi_\phi(\lambda)$ are *dual bases associated to* Φ if:

$$L_\phi(\lambda)^T \pi_\phi(\lambda) = 0, \qquad \pi_\phi(\lambda) = \begin{bmatrix} \phi_n(\lambda) \\ \vdots \\ \phi_0(\lambda) \end{bmatrix}$$

Theorem

If $L_\phi(\lambda)$ and $\pi_\phi(\lambda)$ are dual bases associated to Φ then

$$\mathcal{L}(\lambda) := \begin{bmatrix} W^T(\lambda) \\ L_\phi^T(\lambda) \otimes I_m \end{bmatrix}, \qquad W(\lambda) = \begin{bmatrix} W_n(\lambda)^T \\ \vdots \\ W_0(\lambda)^T \end{bmatrix}$$

linearizes $P(\lambda) := W^T(\lambda)(\pi_\phi(\lambda) \otimes I_m) = \sum_{i=0}^n W_i(\lambda)\phi_i(\lambda)$.

Almost a proof

We check what the eigenvalues of such a pencil are:

$$\mathcal{L}(\lambda)v = 0 \iff \begin{cases} W^T(\lambda)v = 0 \\ (L^T(\lambda) \otimes I_m)v = 0 \end{cases}$$

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Since v is in the dual of $L(\lambda)$ we can write it as $v = (\pi(\lambda) \otimes I_m)w_v$ where $w_v \neq 0$ so that:

$$W^T(\lambda)v = 0 \iff W^T(\lambda)(\pi(\lambda) \otimes I_m)w_v = 0 \iff P(\lambda)w_v = 0.$$

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Basic idea: The eigenvector must lie in the dual of $L(\lambda)$, thus has the correct structure.

A well-known example

One can consider the pair of dual minimal bases

$$L(\lambda) = \begin{bmatrix} 1 & & & \\ -\lambda & \ddots & & \\ & \ddots & & 1 \\ & & & -\lambda \end{bmatrix}, \qquad \pi(\lambda) = \begin{bmatrix} \lambda^{n-1} \\ \vdots \\ \lambda \\ 1 \end{bmatrix}$$

so that

$$\mathcal{L}(\lambda) = \begin{bmatrix} \lambda p_n + p_{n-1} & \dots & \dots & p_0 \\ 1 & -\lambda & & \\ & \ddots & \ddots & \\ & & 1 & -\lambda \end{bmatrix} \quad \text{linearizes } p(\lambda) := \sum_{i=0}^{n} p_i \lambda^i.$$

since $p(\lambda) = \begin{bmatrix} \lambda p_n + p_{n-1} & \dots & p_0 \end{bmatrix} \pi(\lambda)$.

Other well-known examples

…the previous approach allows for linearization in arbitrary bases!

$$\mathcal{L}(\lambda) = \begin{bmatrix} \lambda p_n + p_{n-1} & p_{n-2} - p_n & \dots & p_1 & p_0 \\ 1 & -2\lambda & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2\lambda & 1 \\ & & & 1 & -\lambda \end{bmatrix}$$

is a linearization for the polynomial expressed in the Chebyshev basis of the first kind:

$$p(\lambda) = \sum_{i=0}^{n} p_i T_i(\lambda).$$

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Already in [Amirlaslani, Corless, Lancaster, 2009]: all the orthogonal bases and also Newton, Lagrange, Hermite are possible.

Just to get the idea

Assume we have an orthogonal basis $\phi_i(\lambda)$ satisfying the relation:

$$\alpha\phi_{j+1}(\lambda) = (\lambda - \beta)\phi_j(\lambda) - \gamma\phi_{j-1}(\lambda), \qquad \alpha \neq 0, \quad j > 0, \quad (1)$$

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Then

$$L_\phi(\lambda)^T := \begin{bmatrix} \alpha & (\beta-\lambda) & \gamma & & \\ & \ddots & \ddots & \ddots & \\ & & \alpha & (\beta-\lambda) & \gamma \\ & & & \phi_0(\lambda) & -\phi_1(\lambda) \end{bmatrix}$$

is a basis dual to $\pi_\phi(\lambda)$.

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Let Φ, Ψ be two polynomial bases, and $L_\phi(\lambda), \pi_\phi(\lambda)$ and $L_\psi(\lambda), \pi_\psi(\lambda)$ the corresponding dual minimal bases.

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Theorem

The matrix pencil

$$\mathcal{L}(\lambda) := \begin{bmatrix} \lambda M_1 + M_0 & L_\phi(\lambda) \\ L_\psi(\lambda)^T & 0 \end{bmatrix}$$

is a linearization for the polynomial

$$p(\lambda) := \pi_\psi(\lambda)^T (\lambda M_1 + M_0) \pi_\phi(\lambda).$$

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- The Frobenius linearizations (the classical companions) are a particular case.
- Every Fiedler pencil/companion is just a permutation of a pencil in the previous form.
- However, many more linearizations fits in this class!

Equivalence to Fiedler linearizations

Consider $p(\lambda) = p_4\lambda^4 + p_3\lambda^3 + p_2\lambda^2 + p_1\lambda + p_0$. A Fiedler linearization looks like:

$$\lambda B - A = \begin{bmatrix} \lambda p_4 + p_3 & -1 & 0 & 0 \\ p_2 & \lambda & p_1 & p_0 \\ -1 & 0 & \lambda & 0 \\ 0 & 0 & -1 & \lambda \end{bmatrix}.$$

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Permuting the above pencil in a "good way" yields:

$$\left[\begin{array}{ccc|c} \lambda p_4 + p_3 & 0 & 0 & -1 \\ p_2 & p_1 & p_0 & \lambda \\ \hline -1 & \lambda & 0 & 0 \\ 0 & -1 & \lambda & 0 \end{array}\right] = \begin{bmatrix} \lambda M_1 + M_0 & L_\phi(\lambda) \\ L_\psi(\lambda)^T & 0 \end{bmatrix}$$

Exploiting the extra freedom

According to the previous Theorem, we can write the linearized polynomial:

$$\left[\begin{array}{ccc|c} \lambda p_4 + p_3 & 0 & 0 & -1 \\ p_2 & p_1 & p_0 & \lambda \\ \hline -1 & \lambda & 0 & 0 \\ 0 & -1 & \lambda & 0 \end{array}\right] \sim \begin{bmatrix} \lambda & 1 \end{bmatrix} \begin{bmatrix} \lambda p_4 + p_3 & 0 & \\ p_2 & p_1 & p_0 \end{bmatrix} \begin{bmatrix} \lambda^2 \\ \lambda \\ 1 \end{bmatrix}.$$

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The latter product corresponds to the sum of all the elements in the Hadamard product:

$$\begin{bmatrix} \lambda p_4 + p_3 & & \\ p_2 & p_1 & p_0 \end{bmatrix} \circ \begin{bmatrix} \lambda^3 & \lambda^2 & \lambda \\ \lambda^2 & \lambda & 1 \end{bmatrix}$$

which clearly is $p(\lambda)$. However, reshuffling is allowed!

Exploiting the extra freedom

- We can get *symmetric linearizations* for odd degree polynomials:

$$\left[\begin{array}{ccc|cc} \lambda p_5 + p_4 & & & -1 & \\ & \lambda p_3 + p_2 & & \lambda & -1 \\ & & \lambda p_1 + p_0 & & \lambda \\ \hline -1 & \lambda & & 0 & 0 \\ & -1 & \lambda & 0 & 0 \end{array}\right]$$

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Main underlying idea: Construct the two bases $L_\phi(\lambda)$ and $L_\psi(\lambda)$ so that they have the same symmetry of the matrix polynomial, and you will get a *structured linearization*.

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"Famous" example: simulation of vibrations for high speed trains: palindromic matrix polynomials.

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Just take some $\otimes I_m$ and attach them on the right of (almost) every matrix you can see, that is

$$\begin{bmatrix} \lambda M_1 + M_0 & L_\phi(\lambda) \\ L_\psi^T(\lambda) & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} \lambda M_1 + M_0 & L_\phi(\lambda) \otimes I_m \\ L_\psi^T(\lambda) \otimes I_m & 0 \end{bmatrix}$$

are linearizations for:

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I will switch back and forth between the two notations.

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Given such a polynomial, how to construct a $\star$-even/odd linearization? Let's construct two "adapted" $L(\lambda)$ ...

$\star$-even/odd linearizations!

$$L_\phi(\lambda)^T = \begin{bmatrix} 1 & -\lambda & & \\ & \ddots & \ddots & \\ & & 1 & -\lambda \end{bmatrix}, \qquad L_\psi(\lambda)^T = \begin{bmatrix} 1 & \lambda & & \\ & \ddots & \ddots & \\ & & 1 & \lambda \end{bmatrix}$$

Notice that $L_\phi(\lambda) = L_\psi(-\lambda)$, so ...

$$\mathcal{L}(\lambda) = \begin{bmatrix} \lambda M_1 + M_0 & L_\phi(\lambda) \otimes I_m \\ L_\psi^T(\lambda) \otimes I_m & 0 \end{bmatrix}, \qquad M_1 = -M_1^T, \; M_0 = M_0^T$$

is a T-even linearization for the matrix polynomial:

$$P(\lambda) = (\pi_\phi^T(\lambda) \otimes I_m)(\lambda M_1 + M_0)(\pi_\psi(\lambda) \otimes I_m).$$

The final step

We can easily check that:

$$\pi_\phi(\lambda) = \begin{bmatrix} \lambda^{n-1} \\ \vdots \\ \lambda \\ 1 \end{bmatrix}, \qquad \pi_\psi(\lambda) = \begin{bmatrix} (-1)^{n-1}\lambda^{n-1} \\ \vdots \\ -\lambda \\ 1 \end{bmatrix},$$

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and that a reasonable choice for $\lambda M_1 + M_0$ is given by:

$$\lambda M_1 + M_0 = \begin{bmatrix} (-1)^{n-1}(\lambda P_{2n-1} + P_{2n-2}) & & \\ & \ddots & \\ & & P_0 + \lambda P_1 \end{bmatrix}.$$

If $P(\lambda)$ is $\star$-even the $\lambda M_1 + M_0$ is also $\star$-even and so we have built a *structured linearization*.

Palindromic matrix polynomials

Another interesting case:

$$P(\lambda) = \operatorname{rev} P(\lambda)^\star, \qquad P(\lambda) = -\operatorname{rev} P(\lambda)^\star.$$

Here $\operatorname{rev} P(\lambda) := \lambda^{\deg P} P(\lambda^{-1})$. This induces the spectral symmetry

$$\begin{cases} \lambda \text{ eigenvalue} \iff \lambda^{-1} \text{ eigenvalue} & \star \in \{T, 1\} \\ \lambda \text{ eigenvalue} \iff \overline{\lambda}^{-1} \text{ eigenvalue} & \star = * \end{cases}$$

The $\star = 1$ case is not interesting, since it is impossible to build *structured* linearizations. In the other cases, however, we can use more or less the same trick of before.

Building palindromic dual bases

$$L_\phi(\lambda)^T = \begin{bmatrix} 1 & -\lambda & & \\ & \ddots & \ddots & \\ & & 1 & -\lambda \end{bmatrix}, \qquad L_\psi(\lambda)^T = \begin{bmatrix} \lambda & -1 & & \\ & \ddots & \ddots & \\ & & \lambda & -1 \end{bmatrix}$$

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We follow the now usual approach and we have:

$$\pi_\phi(\lambda) = \begin{bmatrix} \lambda^{n-1} \\ \vdots \\ \lambda \\ 1 \end{bmatrix}, \qquad \pi_\psi(\lambda) = \begin{bmatrix} 1 \\ \lambda \\ \vdots \\ \lambda^{n-1} \end{bmatrix},$$

...and then a palindromic linearization

When building the linearization we have the correct symmetries, since $\operatorname{rev} L_\phi(\lambda) = L_\psi(\lambda)$, and so

$$\mathcal{L}(\lambda) = \begin{bmatrix} \lambda M + M^\star & L_\phi(\lambda) \\ L_\psi(\lambda)^T & 0 \end{bmatrix}$$

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is a *palindromic linearization*. How to choose M? Some computations show that

$$M = \begin{bmatrix} 0_m & \cdots & 0_m & P_0^\star \\ \vdots & & \vdots & \vdots \\ 0_m & \cdots & 0_m & P_{n-1}^\star \end{bmatrix}$$

is the right choice for a palindromic polynomial of degree $2n-1$.

Midway conclusions

- We are considering a family of matrices that includes Fiedler linearizations, and we have found structured linearizations which are also possible in the Fielder framework.

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Can we do other interesting things with this framework? Yes, let's see something completely different!

Intersection of polynomials

Consider the problem: find the values of λ such that

$$p_1(\lambda) = p_2(\lambda)$$

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However, what to do if $p_1(\lambda)$ and $p_2(\lambda)$ are represented using *different bases*? Can we still easily solve the problem?

- Convert them to the same basis, and use a companion matrix for that basis.
- Use our framework in a "creative" way.

Let's recap

$$\mathcal{L}(\lambda) = \begin{bmatrix} \lambda M_1 + M_0 & L_\phi(\lambda) \\ L_\psi(\lambda)^T & 0 \end{bmatrix}$$

is a linearization for $p(\lambda) := \pi_\phi^T(\lambda)(\lambda M_1 + M_0)\pi_\psi(\lambda)$.

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Let w_ϕ, w_ψ constant vectors such that $w_\phi^T \pi_\phi(\lambda) = 1 = w_\psi^T \pi_\psi(\lambda)$ (the coordinates of the constant 1 in our basis).

If we set $\lambda M_1 + M_0 = w_\phi p_1^T - p_2 w_\psi^T$ we have:

$$p(\lambda) = p_1^T \pi_\psi(\lambda) - \pi_\phi^T(\lambda) p_2 = \sum_{i=0}^{\eta} p_{1,i}\psi_i(\lambda) - \sum_{i=0}^{\epsilon} p_{2,i}\phi_i(\lambda),$$

assuming η and ϵ are the number of columns of $L_\psi(\lambda)$ and $L_\phi(\lambda)$.

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However, $\mathcal{L}(\lambda)$ is a linearization for a polynomial of grade $\epsilon + \eta$, while we have degree $\max\{\epsilon, \eta\}$. This gives us many infinite eigenvalues.

Infinite eigenvalues

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- We can easily characterize their complete eigenstructure. They form a very long Jordan chain at infinity $\rightarrow$ very badly conditioned!
- However, the others eigenvalues can be perfectly conditioned.
- We can also deflate the infinite eigenstructure by the staircase algorithm (since we know the eigenstructure, no rank decision are needed, and the approach is perfectly stable).

Numerical accuracy

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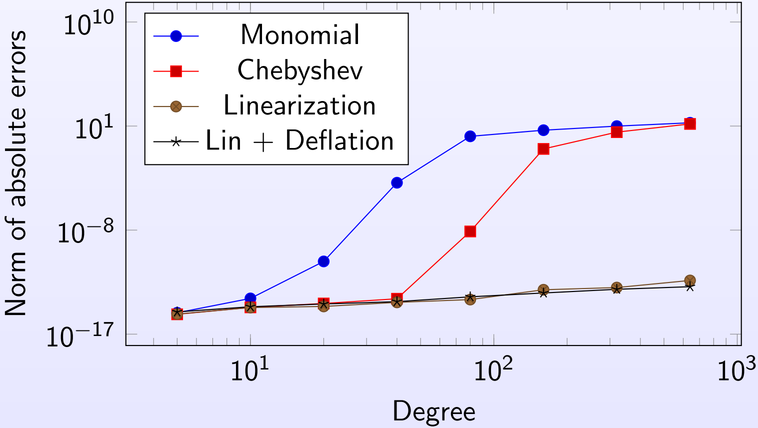

Extensions to rational functions

The same idea can be applied to a slightly more general problem. Find the solutions of:

$$f(\lambda) := \frac{p(\lambda)}{q(\lambda)} + \frac{r(\lambda)}{s(\lambda)} = 0,$$

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This is equivalent to finding the roots of

$$t(\lambda) := p(\lambda)s(\lambda) + q(\lambda)r(\lambda).$$

A "linearization" for rational functions

Similar to the previous case, we have:

$$\mathcal{L}(\lambda) = \begin{bmatrix} ps^T + qr^T & L_\phi(\lambda) \\ L_\psi(\lambda)^T & 0 \end{bmatrix}$$

which is a linearization for $t(\lambda)$. The eigenvalues are solutions of

$$f(\lambda) := \frac{p(\lambda)}{q(\lambda)} + \frac{r(\lambda)}{s(\lambda)} = 0,$$

where $p(\lambda), q(\lambda)$ are expressed in the Φ basis, and $r(\lambda)$ and $s(\lambda)$ in the Ψ basis.

Easy proof

We can easily check what is linearized by $\mathcal{L}(\lambda)$:

$$t(\lambda) = \pi_\phi^T(\lambda)(ps^T + qr^T)\pi_\psi(\lambda) =$$
$$= p(\lambda)s(\lambda) + q(\lambda)r(\lambda).$$

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Notice the the top-left rank 2 block has no term depending on λ.

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Hint: any basis you can think of satisfies the above property (monomial, all orthogonal, Newton, Lagrange, Hermite are all ok).

Another way to write $p(\lambda)$ ...

$$\mathcal{L}(\lambda) = \begin{bmatrix} M(\lambda) & L_\phi(\lambda) \\ L_\psi(\lambda)^T & 0 \end{bmatrix} \quad \text{linearizes} \quad \sum_{i,j} M_{i,j}(\lambda)\psi_i(\lambda)\phi_j(\lambda).$$

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- Not a basis in general.
- However, when ϕ_i and ψ_j this is a generating family for the polynomials of degree up to $\epsilon + \eta$.
- We can exploit the redundancy in a “good way”.

. . . which leads to a natural question

Can we handle more than two bases?

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- . . . and also $j = 2$ (what we have seen until now).
- What about going higher?

Enlarged dual bases

We can deal with this by constructing "enlarged" dual bases.

$$\pi_{\phi\otimes\psi}(\lambda) := \begin{bmatrix} \phi_0(\lambda)\psi_0(\lambda) \\ \vdots \\ \phi_i(\lambda)\psi_j(\lambda) \\ \vdots \\ \phi_\epsilon(\lambda)\psi_\eta(\lambda) \end{bmatrix}$$

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Can we build a suitable $L(\lambda)$ such that $L(\lambda)^T\pi_{\phi\otimes\psi}(\lambda) = 0$? Yes!

Product dual bases

$$L_{k,\phi\otimes\psi}(\lambda)^T = \begin{bmatrix} A \otimes L_{\eta,\psi}(\lambda)^T \\ L_{\epsilon,\phi}(\lambda)^T \otimes w^T \end{bmatrix}, \qquad k := (\epsilon+1)(\eta+1)-1,$$

form a dual basis together with $\pi_{\phi\otimes\psi}(\lambda)$, where A is any invertible matrix and w any vector such that $w^T\pi_{\eta,\psi}(\lambda)$ is a nonzero constant.

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We can just plug this ingredient in our framework and obtain the desired linearizations!

Some examples

- Let $p(\lambda) = \sum_{i=0}^{3} p_i \lambda^i$ a degree 3 polynomial.

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- Choosing $\{\psi_i\} = \{1, \lambda, \lambda^2\}$ and $\{\phi_i\} = \{1\}$ yields Frobenius form:

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- Choosing $\{\psi_i\} = \{\phi_i\} = \{1, \lambda\}$ yields, e.g., the symmetric

$$\mathcal{L}(\lambda) = \left[\begin{array}{cc|c} \lambda p_3 + p_2 & \frac{1}{2} p_1 & 1 \\ \frac{1}{2} p_1 & p_0 & -\lambda \\ \hline 1 & -\lambda & 0 \end{array}\right].$$

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- Note, the dimension could increase (not strong anymore).

Backward stability

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...but QZ will give me back the *exact* eigenvalues of $\widetilde{\mathcal{L}(\lambda)}$:

$$\widetilde{\mathcal{L}(\lambda)} = \begin{bmatrix} W^T + \delta W^T \\ L(\lambda) + \delta L(\lambda) \end{bmatrix} = \begin{bmatrix} \tilde{W}^T \\ \tilde{L}(\lambda) \end{bmatrix}$$

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Many interesting questions for us:

- When do small perturbations to $L(\lambda)$ correspond to "small" perturbations on $\pi(\lambda)$?
- Can I measure the growth factor in the perturbation size?
- Does it work flawlessly also when using two bases (or more?).

To be continued . . .

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- Likely the talk of Piers Lawrence at ILAS 2016 will be about these matters.
- I have a draft in my laptop which seems promising. I am working on it in these days, let me know if you want to discuss something about that!

Multivariate polynomials

What we have seen also naturally fit into the framework of linearizing the bivariate polynomial

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$$p(\lambda, \mu) = \sum_{i=0}^{n} \sum_{j=0}^{n} p_{ij} \lambda^i \mu^j$$

In fact, this can be naturally expressed as

$$p(\lambda, \mu) = \pi(\lambda)^T P \pi(\mu), \qquad P = (p_{ij}),$$

and $\pi(\lambda)$ the usual vector with the monomials.

Visualizing the linearization

The following is a linearization for $p(\lambda, \mu)$:

$$\mathcal{L}(\lambda, \mu) = \begin{bmatrix} P & L(\lambda) \\ L(\mu)^T & 0 \end{bmatrix}$$

in the sense that $\det \mathcal{L}(\lambda, \mu) = p(\lambda, \mu)$.

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One could use this to solve problems of the form

$$\begin{cases} p(\lambda, \mu) = 0 \\ q(\lambda, \mu) = 0 \end{cases}$$

by turning them into a multiparameter eigenvalue problem

$$\begin{cases} \mathcal{L}_1(\lambda, \mu) v_1 = 0 \\ \mathcal{L}_2(\lambda, \mu) v_2 = 0 \end{cases}$$

Issues

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Work on this topic is being carried out by Bor Plestenjak. Our framework easily allows extension to more variables, which seems to be almost optimal in dimension for generic problems.

Conclusions

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Thanks for your attention!