Improved generators for quasiseparable matrices

hal-01264131

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Structured Matrix Days,
Limoges, France
May 10, 2016

Introduction

Two main types of structured matrices

Rank displacement: Hankel, Toeplitz, Vandermonde, Cauchy, etc

- arising from polynomial operations (evaluation, interpolation, etc)
- Rank displacement: rank$(A - PAQ)$ for $P, Q \in \{\text{Diag}, \text{Cyclic shift}\}$
- Structure: rank$(A - PAQ)$ small $\rightsquigarrow A - PAQ = LR$

Rank structure: Tridiagonal, Hessenberg, semiseparable, quasiseparable

- Structure: small rank in the upper/lower triangular part
- no closed form expression

Quasiseparable matrices

$$
\begin{bmatrix}
4 & 2 & 3 & & & & & & & \\
4 & 0 & 1 & 4 & & & & & & \\
4 & 3 & 5 & 6 & 3 & & & & & \\
 & 5 & 1 & 1 & 4 & 5 & & & & \\
 & & 4 & 3 & 1 & 1 & 4 & & & \\
 & & & 4 & 0 & 5 & 1 & 2 & & \\
 & & & & 4 & 3 & 1 & 2 & 6 & \\
 & & & & & 2 & 1 & 3 & 1 & 1 \\
 & & & & & & 1 & 5 & 4 & 3 \\
 & & & & & & & 3 & 0 & 4
\end{bmatrix}^{-1}
=
\begin{bmatrix}
2 & 6 & 1 & 0 & 1 & 4 & 6 & 5 & 2 & 6 \\
6 & 6 & 2 & 5 & 6 & 1 & 6 & 1 & 3 & 1 \\
3 & 2 & 2 & 6 & 4 & 1 & 2 & 2 & 0 & 3 \\
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rank $= 1$ rank $= 1$

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rank $= 2$ rank $= 2$

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rank $= 1$ rank $= 1$

Definition

M $\in \mathrm{K}^{n\times n}$ is (r_L, r_U)-quasiseparable if $\begin{cases} \text{rank}(\mathsf{M}_{k+1..n,1..k}) & \leq & r_L \\ \text{rank}(\mathsf{M}_{1..k,k+1..n}) & \leq & r_U \end{cases}$

Applications

Numerical linear algebra: naturally occur in solving

- generalized eigenvalue problems,
- PDE's, etc

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Complexity

Displacement rank structure:

- Formerly $\tilde{O}(r^2 n)$ time and space [Pan01]
- with fast matrix arithmetic: $\tilde{O}(r^{\omega-1} n)$ time [BJS08]

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Quasiseparable structure

- $O(r^2 n)$ time and space [EG99]
- Seeking to reduce the exponent in r

Structured representation of a quasiseparable matrix

[EG99]

$$\mathsf{M}_{i,j} = \begin{cases} p(i)^T \mathsf{a}(i-1)\ldots\mathsf{a}(j+1)q(j), & 1 \leq j < i \leq n \\ d(i), & 1 \leq i = j \leq n \\ g(i)^T \mathsf{b}(i+1)\ldots\mathsf{b}(i-1)h(j), & 1 \leq i < j \leq n \end{cases}$$

where for $i \in \{1..n-1\}$

- $p(i), q(i) \in \mathrm{K}^{r_L}$, $g(i), h(i) \in \mathrm{K}^{r_U}$,
- $\mathsf{a}(i) \in \mathrm{K}^{r_L \times r_L}$, $\mathsf{b}(i) \in \mathrm{K}^{r_U \times r_U}$,

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Size	$O(nr^2)$
Computing the generator	$O(n^2)$

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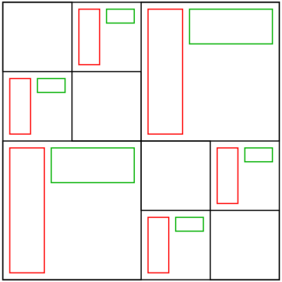
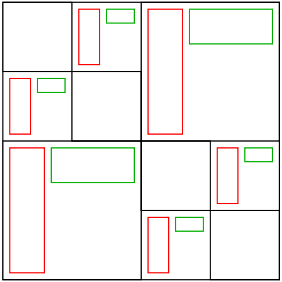
Hierarchically Semi-Separable representation

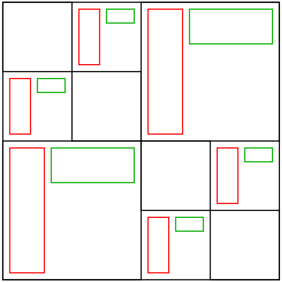


Hierarchically Semi-Separable representation



	[CGP06]
Size	$O(n^2)$ (peak)
Computing the generator Vector apply	$O(n^2)$?

Hierarchically Semi-Separable representation



	[CGP06]	[LHC16]
Size	$O(n^2)$ (peak)	$O(rn \log n)$
Computing the generator	$O(n^2)$	$O(rn \log n)$
Vector apply	?	$O(rn \log n)$ field ops

Contribution

Connecting notions of **quasiseparability** and **rank profile matrix** [DPS15]

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1. Compute the quasiseparable orders in $O(r^{\omega-2}n^2)$
2. Two alternative structured representations
   1. binary tree of PLUQ: $O(rn\log\frac{n}{r})$ space
   2. Bruhat generator: $O(rn)$ space
3. algorithms to compute these representations,
4. algorithms for Quasisep $\times$ vector and Quasisep $\times$ Quasisep

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	[EG99]	[Per16]
Size	$O(r^2n)$	$O(rn)$
Computing the generator	$O(n^2)$	$O(r^{\omega-2}n^2)$
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Vector apply	$O(r^2n)$	$O(rn)$
Quasisep $\times$ Quasisep	$O(r^3n)$	$O(r^{\omega-2}n^2)$

Outline

1. The rank profile matrix
   - Definition
   - Computing the rank profile matrix
2. Computing the orders of quasiseparability
3. More efficient generators

Rank profiles

Definition (Row Rank Profile: RowRP)

Given $A \in \mathsf{K}^{m\times n}, r = \mathsf{rank}(A)$.

informally: *first* r linearly independent rows

formally: lexico-minimal list of r indices of linearly independant rows.

Example

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

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Given $A \in \mathsf{K}^{m\times n}, r = \mathsf{rank}(A)$.

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Example

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Rank = 3
RowRP = {1,2,4}

Rank profiles

Definition (Column Rank Profile: ColRP)

Given $A \in \mathsf{K}^{m\times n}, r = \mathsf{rank}(A)$.

informally: *first* r linearly independent columns

formally: lexico-minimal list of r linearly independant columns.

Example

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Rank = 3
RowRP = {1,2,4}
ColRP = {1,2,3}

Rank profiles

Definition (Column Rank Profile: ColRP)

Given $A \in \mathsf{K}^{m\times n}, r = \mathsf{rank}(A)$.

informally: *first* r linearly independent columns

formally: lexico-minimal list of r linearly independant columns.

Example

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Rank = 3
RowRP = {1,2,4}
ColRP = {1,2,3} → Generic ColRP.

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Generic rank profile $\not\Leftarrow\Rightarrow$ Generic Row RP and Genric ColRP

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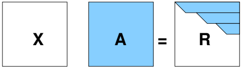
RowRP = ColRP = {1,2}
But $|A_{1,1}| = 0$

Relation to echelon forms

Transformation to echelon form

$\forall A \; \exists X$ non-singular s.t.



Relation to echelon forms:

- ColRP unchanged by left multiplication with an invertible matrix

ColRP $=$ pivot columns in the **row** echelon form

Computing rank profiles

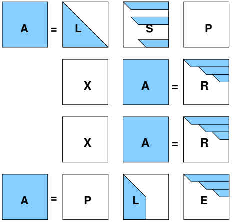
Via Gaussian elimination revealing row echelon forms:

[Ibarra, Moran and Hui 82]

[Keller-Gehrig 85]

[Storjohann 00]

[Jeannerod, P. and Storjohann 13]



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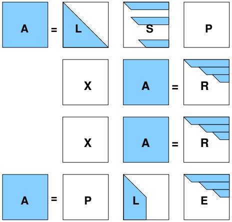
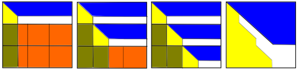


Lessons learned (or what we thought was necessary):

- treat rows in order
- exhaust all columns before considering the next row
- **slab** block splitting (recursive or iterative)



⇝ similar to partial pivoting

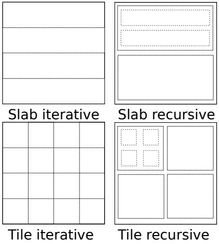
Motivation

Need more flexible blocking

Slab blocking

- leads to inefficient memory access patterns
- is harder to parallelize

Tile blocking instead ?



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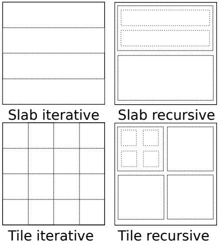


Gathering linear independence invariants

Two ways to look at a matrix (looking left or right):

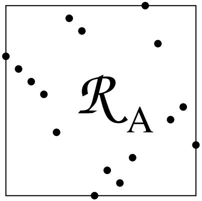
- Row rank profile, column echelon form
- Column rank profile, row echelon form

Unique invariant?

The rank profile Matrix

Theorem

Let $A \in \mathrm{F}^{m\times n}$.
There exists a unique, $m \times n$, rank(A)-sub-permutation matrix $\mathcal{R}^A$ of which every leading sub-matrix has the same rank as the corresponding leading sub-matrix of A.



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Example

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 1 & 3 & 2 & 0 \\ 2 & 5 & 4 & 7 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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Properties of the rank profile matrix

Properties

- *A invertible $\Rightarrow \mathcal{R}^A$ is a permutation matrix*
- *A is square with generic rank profile $\Rightarrow \mathcal{R}^A = I_n$*
- *RowRP(A) = RowSupport$(\mathcal{R}^A)$*
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When does a PLUQ decomposition reveal the rank profile matrix ?

Focus on the pivoting strategy:

- Pivot search
- Permutation to bring the pivot to the main diagonal

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$$A = PLUQ = P \begin{bmatrix} L & 0 \\ M & I_{m-r} \end{bmatrix} \qquad \begin{bmatrix} I_r & \\ & 0 \end{bmatrix} \qquad \begin{bmatrix} U & V \\ & I_{n-r} \end{bmatrix} Q$$

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$$A = PLUQ = P\begin{bmatrix} L & 0 \\ M & I_{m-r} \end{bmatrix} P^T P \begin{bmatrix} I_r & \\ & 0 \end{bmatrix} Q Q^T \begin{bmatrix} U & V \\ & I_{n-r} \end{bmatrix} Q$$

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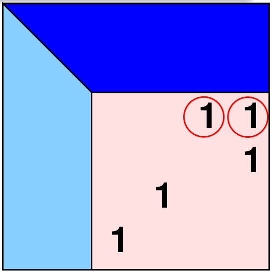
Pivoting and permutation strategies

Pivot Search

Pivot's (i, j) position minimizes some pre-order:

Row order: any non-zero on the first non-zero row



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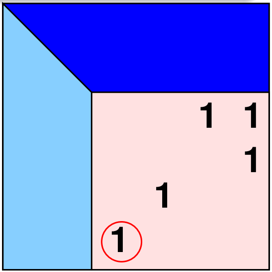

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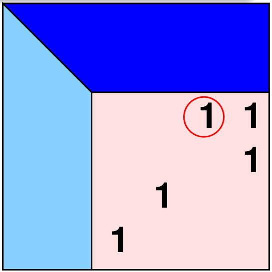
Lex order: first non-zero on the first non-zero row



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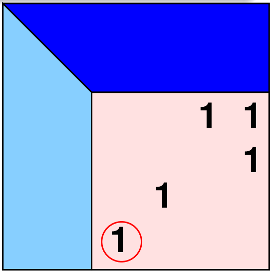

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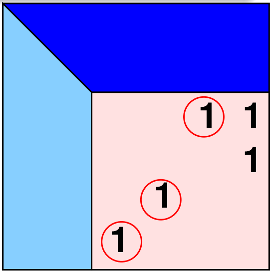

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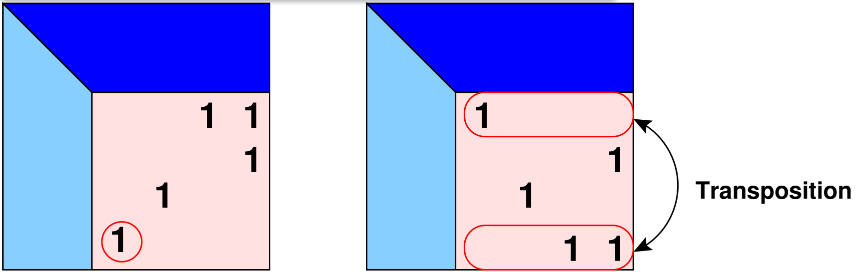
- Transpositions



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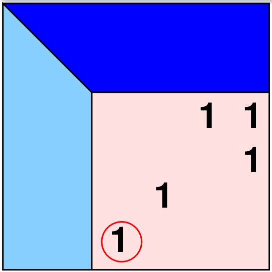
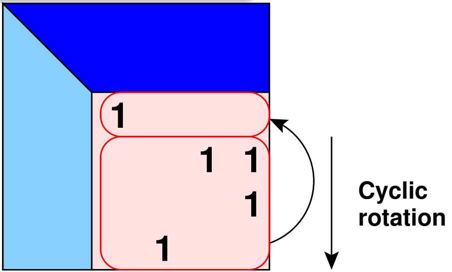
- Transpositions
- Cyclic Rotations





Pivoting strategies revealing rank profiles

For any type of PLUQ algorithm: iterative / block iterative / recursive

Search	**Row perm.**	**Col. perm.**	RowRP	ColRP	$\mathcal{R}^A$	**Instance**
Row order Col. order						
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Rev. lex.						
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- RowRP $= \begin{bmatrix} 1 & 2 & \dots & m \end{bmatrix} P \begin{bmatrix} I_r \\ 0 \end{bmatrix}$

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Product	Rotation	Rotation	✓	✓	✓	[DPS13]

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Lexico.	Transposition	Transposition	✓			[Sto00]
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Lexico.	Transposition	Rotation	✓	✓	✓	[DPS15]
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Rev. lex.	Transposition	Transposition		✓		[Sto00]
Rev. lex.	Rotation	Transposition	✓	✓	✓	[DPS15]
Rev. lex.	Rotation	Rotation	✓	✓	✓	[DPS15]
Product	Rotation	Transposition	✓			[DPS15]
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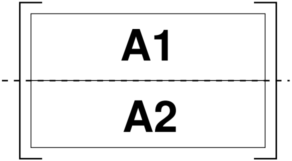
The slab recursive algorithm

Slab Recursive LU [IMH82, KG85, Sto00, JPS13]



1. Split A Row-wise

The slab recursive algorithm

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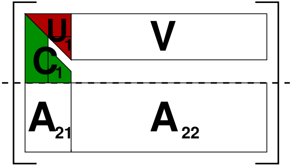

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The slab recursive algorithm

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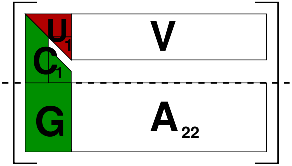

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3. $G \leftarrow A_{21}U_1^{-1}$ (trsm)

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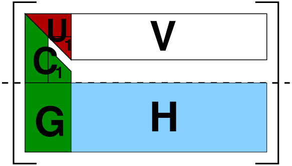

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The slab recursive algorithm

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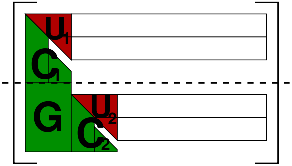

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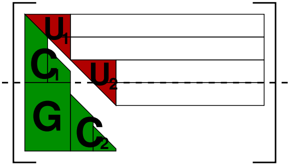

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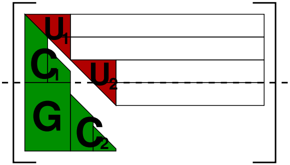

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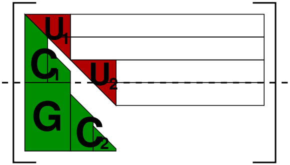
Implements the lexicographic order search.

- Col/Row Transpositions : Computes the ColRP

The slab recursive algorithm

Slab Recursive LU [IMH82, KG85, Sto00, JPS13]



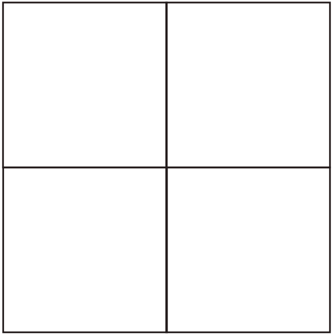
1. Split A Row-wise
2. Recursive call on A_1
3. $G \leftarrow A_{21}U_1^{-1}$ ($\texttt{trsm}$)
4. $H \leftarrow A_{22} - G \times V$ ($\texttt{MM}$)
5. Recursive call on H
6. Row permutations

Implements the lexicographic order search.

- Col/Row Transpositions : Computes the ColRP
- Row Rotations : Computes $\mathcal{R}^A$ [DPS15]

The tiled recursive algorithm

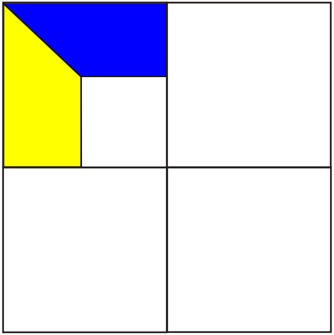
[DPS13]



2×2 block splitting

The tiled recursive algorithm

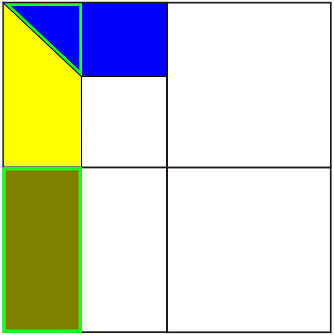
[DPS13]



Recursive call

The tiled recursive algorithm

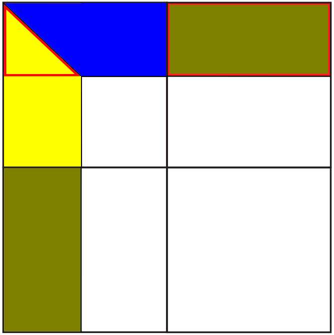
[DPS13]



TRSM: $B \leftarrow BU^{-1}$

The tiled recursive algorithm

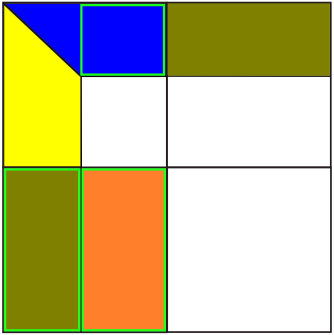
[DPS13]



TRSM: $B \leftarrow L^{-1}B$

The tiled recursive algorithm

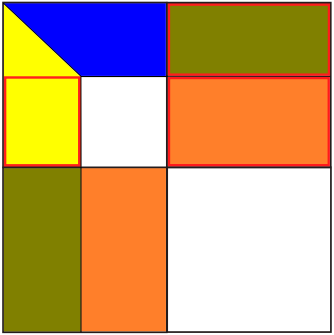
[DPS13]



MatMul: $C \leftarrow C - A \times B$

The tiled recursive algorithm

[DPS13]



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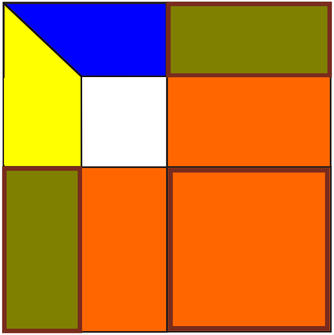
[DPS13]



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The tiled recursive algorithm

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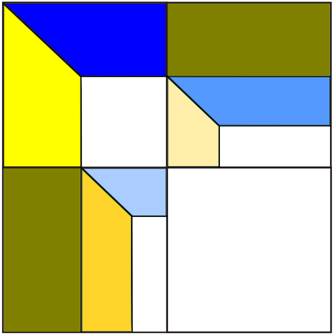
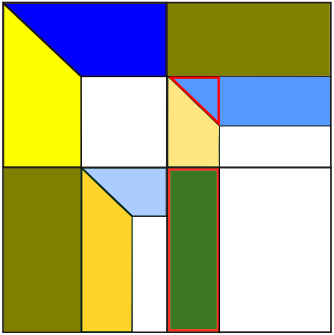


2 independent recursive calls (compatible with the **product order**)

The tiled recursive algorithm

[DPS13]



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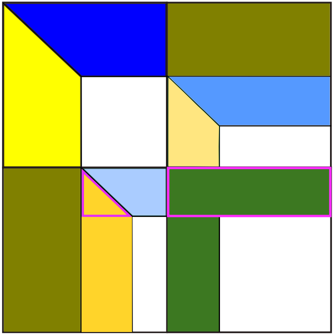
The tiled recursive algorithm

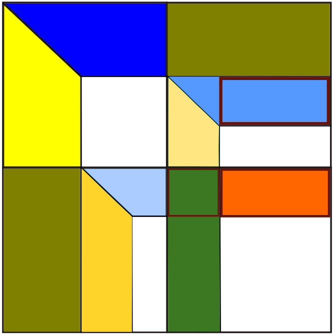
[DPS13]



TRSM: $B \leftarrow L^{-1}B$

The tiled recursive algorithm

[DPS13]



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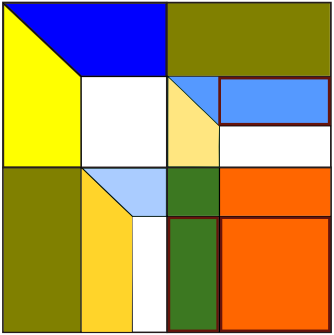
The tiled recursive algorithm

[DPS13]



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The tiled recursive algorithm

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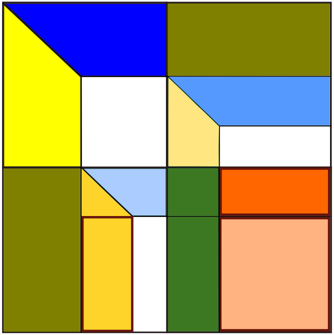

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The tiled recursive algorithm

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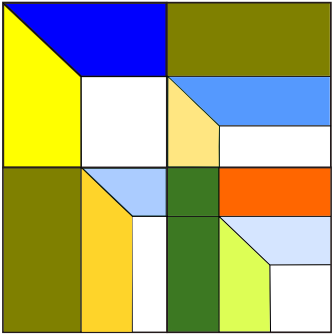
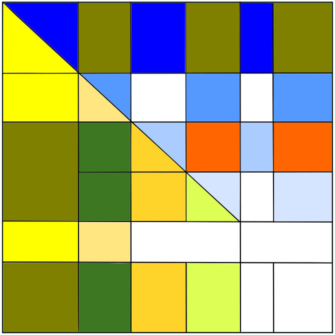


Recursive call

The tiled recursive algorithm

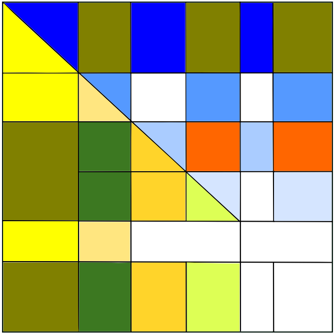
[DPS13]



Puzzle game (block **rotations**)

The tiled recursive algorithm

[DPS13]



- $O(mnr^{\omega-2})$ ($2/3n^3$ for $\omega = 3$)
- fewer modular reductions than slab algorithms
- rank deficiency introduces parallelism

Outline

Finding the quasiseparability orders

Left triangular matrices

Of the form: $\begin{bmatrix} * & * & * & * & 0 \\ * & * & * & 0 & \\ * & * & 0 & & \\ * & 0 & & & \\ 0 & & & & \end{bmatrix}$

Examples: $J_n \times$ Lower, Upper $\times J_n$

$$\begin{bmatrix} 2 & 6 & 1 & 0 & 1 & 4 & 6 & 5 & 2 & 6 \\ 6 & 6 & 2 & 5 & 6 & 1 & 6 & 1 & 3 & 1 \\ 3 & 2 & 2 & 6 & 4 & 1 & 2 & 2 & 0 & 3 \\ 6 & 6 & 2 & 2 & 5 & 1 & 4 & 5 & 5 & 2 \\ 0 & 1 & 6 & 2 & 4 & 4 & 5 & 0 & 3 & 3 \\ 2 & 0 & 5 & 6 & 3 & 1 & 0 & 6 & 2 & 4 \\ 6 & 2 & 6 & 1 & 3 & 4 & 6 & 5 & 2 & 6 \\ 1 & 1 & 5 & 5 & 2 & 1 & 3 & 4 & 5 & 4 \\ 0 & 1 & 6 & 2 & 4 & 4 & 3 & 3 & 2 & 3 \\ 1 & 1 & 5 & 5 & 2 & 1 & 3 & 4 & 5 & 6 \end{bmatrix}$$

Finding the quasiseparability orders

Left triangular matrices

Of the form: $\begin{bmatrix} * & * & * & * & 0 \\ * & * & * & 0 & \\ * & * & 0 & & \\ * & 0 & & & \\ 0 & & & & \end{bmatrix}$

Examples: $J_n \times$ Lower, Upper $\times J_n$

$$\begin{bmatrix} 0 &&&&&&&&& \\ 6 & 0 &&&&&&&& \\ 3 & 2 & 0 &&&&&&& \\ 6 & 6 & 2 & 0 &&&&&& \\ 0 & 1 & 6 & 2 & 0 &&&&& \\ 2 & 0 & 5 & 6 & 3 & 0 &&&& \\ 6 & 2 & 6 & 1 & 3 & 4 & 0 &&& \\ 1 & 1 & 5 & 5 & 2 & 1 & 3 & 0 && \\ 0 & 1 & 6 & 2 & 4 & 4 & 3 & 3 & 0 & \\ 1 & 1 & 5 & 5 & 2 & 1 & 3 & 4 & 5 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 6 & 1 & 0 & 1 & 4 & 6 & 5 & 2 & 6 \\ & 0 & 2 & 5 & 6 & 1 & 6 & 1 & 3 & 1 \\ && 0 & 6 & 4 & 1 & 2 & 2 & 0 & 3 \\ &&& 0 & 5 & 1 & 4 & 5 & 5 & 2 \\ &&&& 0 & 4 & 5 & 0 & 3 & 3 \\ &&&&& 0 & 0 & 6 & 2 & 4 \\ &&&&&& 0 & 5 & 2 & 6 \\ &&&&&&& 0 & 5 & 4 \\ &&&&&&&& 0 & 3 \\ &&&&&&&&& 0 \end{bmatrix}$$

Finding the quasiseparability orders

Left triangular matrices

Of the form: $\begin{bmatrix} * & * & * & * & 0 \\ * & * & * & 0 & \\ * & * & 0 & & \\ * & 0 & & & \\ 0 & & & & \end{bmatrix}$

Examples: $J_n \times$ Lower, Upper $\times J_n$

$$\begin{bmatrix} 1 & 1 & 5 & 5 & 2 & 1 & 3 & 4 & 5 & 0 \\ 0 & 1 & 6 & 2 & 4 & 4 & 3 & 3 & 0 & \\ 1 & 1 & 5 & 5 & 2 & 1 & 3 & 0 & & \\ 6 & 2 & 6 & 1 & 3 & 4 & 0 & & & \\ 2 & 0 & 5 & 6 & 3 & 0 & & & & \\ 0 & 1 & 6 & 2 & 0 & & & & & \\ 6 & 6 & 2 & 0 & & & & & & \\ 3 & 2 & 0 & & & & & & & \\ 6 & 0 & & & & & & & & \\ 0 & & & & & & & & & \end{bmatrix}, \begin{bmatrix} 6 & 2 & 5 & 6 & 4 & 1 & 0 & 1 & 6 & 0 \\ 1 & 3 & 1 & 6 & 1 & 6 & 5 & 2 & 0 & \\ 3 & 0 & 2 & 2 & 1 & 4 & 6 & 0 & & \\ 2 & 5 & 5 & 4 & 1 & 5 & 0 & & & \\ 3 & 3 & 0 & 5 & 4 & 0 & & & & \\ 4 & 2 & 6 & 0 & 0 & & & & & \\ 6 & 2 & 5 & 0 & & & & & & \\ 4 & 5 & 0 & & & & & & & \\ 3 & 0 & & & & & & & & \\ 0 & & & & & & & & & \end{bmatrix}$$

Finding the quasiseparability orders

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$$\begin{bmatrix} 1 & 1 & 5 & 5 & 2 & 1 & 3 & 4 & 5 & 0 \\ 0 & 1 & 6 & 2 & 4 & 4 & 3 & 3 & 0 & \\ 1 & 1 & 5 & 5 & 2 & 1 & 3 & 0 & & \\ 6 & 2 & 6 & 1 & 3 & 4 & 0 & & & \\ 2 & 0 & 5 & 6 & 3 & 0 & & & & \\ 0 & 1 & 6 & 2 & 0 & & & & & \\ 6 & 6 & 2 & 0 & & & & & & \\ 3 & 2 & 0 & & & & & & & \\ 6 & 0 & & & & & & & & \\ 0 & & & & & & & & & \end{bmatrix}, \begin{bmatrix} 6 & 2 & 5 & 6 & 4 & 1 & 0 & 1 & 6 & 0 \\ 1 & 3 & 1 & 6 & 1 & 6 & 5 & 2 & 0 & \\ 3 & 0 & 2 & 2 & 1 & 4 & 6 & 0 & & \\ 2 & 5 & 5 & 4 & 1 & 5 & 0 & & & \\ 3 & 3 & 0 & 5 & 4 & 0 & & & & \\ 4 & 2 & 6 & 0 & 0 & & & & & \\ 6 & 2 & 5 & 0 & & & & & & \\ 4 & 5 & 0 & & & & & & & \\ 3 & 0 & & & & & & & & \\ 0 & & & & & & & & & \end{bmatrix}$$

Finding the quasiseparability orders

Left triangular matrices

Of the form: $\begin{bmatrix} * & * & * & * & 0 \\ * & * & * & 0 & \\ * & * & 0 & & \\ * & 0 & & & \\ 0 & & & & \end{bmatrix}$

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$$\begin{bmatrix} 1 & 1 & 5 & 5 & 2 & 1 & 3 & 4 & 5 & 0 \\ 0 & 1 & 6 & 2 & 4 & 4 & 3 & 3 & 0 & \\ 1 & 1 & 5 & 5 & 2 & 1 & 3 & 0 & & \\ 6 & 2 & 6 & 1 & 3 & 4 & 0 & & & \\ 2 & 0 & 5 & 6 & 3 & 0 & & & & \\ 0 & 1 & 6 & 2 & 0 & & & & & \\ 6 & 6 & 2 & 0 & & & & & & \\ 3 & 2 & 0 & & & & & & & \\ 6 & 0 & & & & & & & & \\ 0 & & & & & & & & & \end{bmatrix}, \begin{bmatrix} 6 & 2 & 5 & 6 & 4 & 1 & 0 & 1 & 6 & 0 \\ 1 & 3 & 1 & 6 & 1 & 6 & 5 & 2 & 0 & \\ 3 & 0 & 2 & 2 & 1 & 4 & 6 & 0 & & \\ 2 & 5 & 5 & 4 & 1 & 5 & 0 & & & \\ 3 & 3 & 0 & 5 & 4 & 0 & & & & \\ 4 & 2 & 6 & 0 & 0 & & & & & \\ 6 & 2 & 5 & 0 & & & & & & \\ 4 & 5 & 0 & & & & & & & \\ 3 & 0 & & & & & & & & \\ 0 & & & & & & & & & \end{bmatrix}$$

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$$\left[\begin{array}{cccccccccc} 1 & 1 & 5 & 5 & 2 & 1 & 3 & 4 & 5 & 0 \\ 0 & 1 & 6 & 2 & 4 & 4 & 3 & 3 & 0 & \\ 1 & 1 & 5 & 5 & 2 & 1 & 3 & 0 & & \\ 6 & 2 & 6 & 1 & 3 & 4 & 0 & & & \\ 2 & 0 & 5 & 6 & 3 & 0 & & & & \\ 0 & 1 & 6 & 2 & 0 & & & & & \\ 6 & 6 & 2 & 0 & & & & & & \\ 3 & 2 & 0 & & & & & & & \\ 6 & 0 & & & & & & & & \\ 0 & & & & & & & & & \end{array}\right], \left[\begin{array}{cccccccccc} 6 & 2 & 5 & 6 & 4 & 1 & 0 & 1 & 6 & 0 \\ 1 & 3 & 1 & 6 & 1 & 6 & 5 & 2 & 0 & \\ 3 & 0 & 2 & 2 & 1 & 4 & 6 & 0 & & \\ 2 & 5 & 5 & 4 & 1 & 5 & 0 & & & \\ 3 & 3 & 0 & 5 & 4 & 0 & & & & \\ 4 & 2 & 6 & 0 & 0 & & & & & \\ 6 & 2 & 5 & 0 & & & & & & \\ 4 & 5 & 0 & & & & & & & \\ 3 & 0 & & & & & & & & \\ 0 & & & & & & & & & \end{array}\right]$$

Quasiseparability orders on the rank profile matrix

$$
\begin{bmatrix}
2 & 2 & 1 & 0 & 2 & 1 & 1 & 1 & 0 & 0\\
2 & 2 & 1 & 0 & 2 & 0 & 1 & 2 & 0 & 0\\
2 & 2 & 1 & 0 & 2 & 2 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
2 & 2 & 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0\\
2 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & & & \\
0 & 0 & 0 & 0 & 1 & & & & & \\
0 & 1 & 0 & 0 & & & & & & \\
0 & 0 & 0 & & & & & & & \\
0 & 0 & 1 & & & & & & & \\
0 & & & & & & & & &
\end{bmatrix}
$$

Rank $= 1$ Quasiseparability order $= 1$

Quasiseparability orders on the rank profile matrix

$$
\begin{bmatrix}
2 & 2 & 1 & 0 & 2 & 1 & 1 & 1 & 0 & 0 \\
2 & 2 & 1 & 0 & 2 & 0 & 1 & 2 & 0 & 0 \\
2 & 2 & 1 & 0 & 2 & 2 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 2 & 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\
2 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & & & \\
0 & 0 & 0 & 0 & 1 & & & & & \\
0 & 1 & 0 & 0 & & & & & & \\
0 & 0 & 0 & & & & & & & \\
0 & 0 & 1 & & & & & & & \\
0 & & & & & & & & &
\end{bmatrix}
$$

Rank $= 2$ Quasiseparability order $= 2$

Quasiseparability orders on the rank profile matrix

$$\begin{bmatrix} 2 & 2 & 1 & 0 & 2 & 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 & 2 & 0 & 1 & 2 & 0 & 0 \\ 2 & 2 & 1 & 0 & 2 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & & & \\ 0 & 0 & 0 & 0 & 1 & & & & & \\ 0 & 1 & 0 & 0 & & & & & & \\ 0 & 0 & 0 & & & & & & & \\ 0 & 0 & 1 & & & & & & & \\ 0 & & & & & & & & & \end{bmatrix}$$

Rank $= 2$ Quasiseparability order $= 2$

Quasiseparability orders on the rank profile matrix

$$
\begin{bmatrix}
2 & 2 & 1 & 0 & 2 & 1 & 1 & 1 & 0 & 0 \\
2 & 2 & 1 & 0 & 2 & 0 & 1 & 2 & 0 & 0 \\
2 & 2 & 1 & 0 & 2 & 2 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 2 & 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\
2 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & & & \\
0 & 0 & 0 & 0 & 1 & & & & & \\
0 & 1 & 0 & 0 & & & & & & \\
0 & 0 & 0 & & & & & & & \\
0 & 0 & 1 & & & & & & & \\
0 & & & & & & & & &
\end{bmatrix}
$$

Rank $= 1$ Quasiseparability order $= 2$

Quasiseparability orders on the rank profile matrix

$$\begin{bmatrix} 2&2&1&0&2&1&1&1&0&0\\ 2&2&1&0&2&0&1&2&0&0\\ 2&2&1&0&2&2&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 2&2&1&0&2&0&0&0&0&0\\ 2&2&1&0&0&0&0&0&0&0\\ 2&1&0&0&0&0&0&0&0&0\\ 1&2&0&0&0&0&0&0&0&0\\ 2&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0 \end{bmatrix}, \begin{bmatrix} 1&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&\\ 0&0&0&0&0&0&0&0&&\\ 0&0&0&0&0&0&0&&&\\ 0&0&0&0&0&0&1&&&\\ 0&0&0&0&1&&&&&\\ 0&1&0&0&&&&&&\\ 0&0&0&&&&&&&\\ 0&0&1&&&&&&&\\ 0&&&&&&&&& \end{bmatrix}$$

Rank $= 1$ Quasiseparability order $= 2$

Major difficulty

Complexity in $O(n^2 r^{\omega-2})$ where $r = \text{rank}(A) \gg r_L, r_U$. But

Quasiseparability orders on the rank profile matrix

$$
\begin{bmatrix}
2 & 2 & 1 & 0 & 2 & 1 & 1 & 1 & 0 & 0\\
2 & 2 & 1 & 0 & 2 & 0 & 1 & 2 & 0 & 0\\
2 & 2 & 1 & 0 & 2 & 2 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
2 & 2 & 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0\\
2 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & & & \\
0 & 0 & 0 & 0 & 1 & & & & & \\
0 & 1 & 0 & 0 & & & & & & \\
0 & 0 & 0 & & & & & & & \\
0 & 0 & 1 & & & & & & & \\
0 & & & & & & & & &
\end{bmatrix}
$$

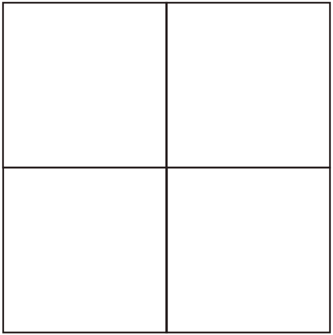
Rank $= 1$ Quasiseparability order $= 2$

Major difficulty

Complexity in $O(n^2 r^{\omega-2})$ where $r = \text{rank}(A) \gg r_L, r_U$. But

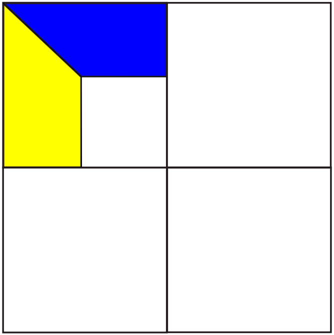
- only a few pivots ($O(r_L, r_U)$) near the top left corner
- numerous pivots lie near the anti-diagonal $\rightsquigarrow$ cheaper elimination

Computing the left-triangular part of a rank profile matrix



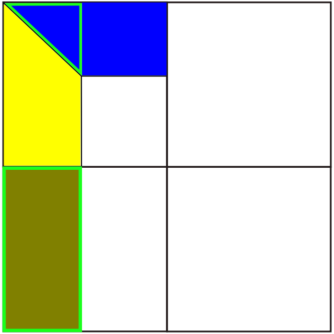
2×2 block splitting

Computing the left-triangular part of a rank profile matrix



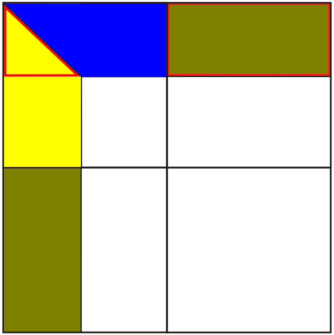
Recursive call

Computing the left-triangular part of a rank profile matrix



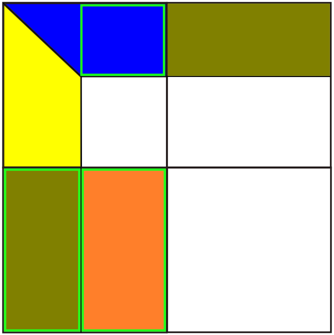
TRSM: $B \leftarrow BU^{-1}$

Computing the left-triangular part of a rank profile matrix



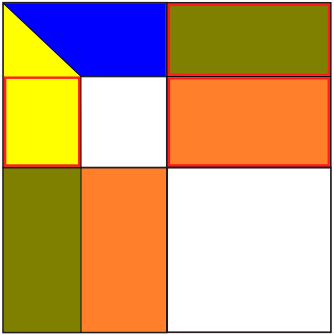
TRSM: $B \leftarrow L^{-1}B$

Computing the left-triangular part of a rank profile matrix



`MatMul`: $C \leftarrow C - A \times B$

Computing the left-triangular part of a rank profile matrix



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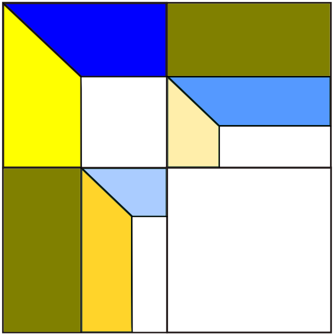

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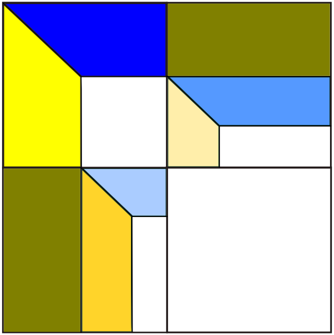

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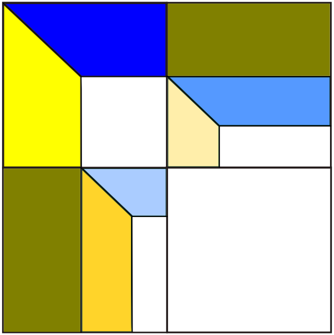

2 recursive calls : not possible as no longer left-triangular

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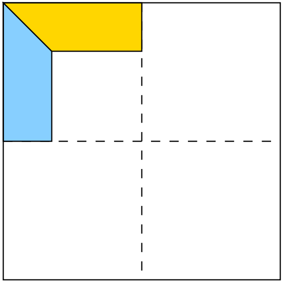
- $O(n^2 s^{\omega-2})$
- in place

Outline

The binary tree of PLUQ generator

Principle: *à la* Hierarchically Semiseparable representations

- Split in 4 quadrants
- Store the top left corner as a PLUQ decomposition: $O(nr)$



The binary tree of PLUQ generator

Principle: *à la* Hierarchically Semiseparable representations

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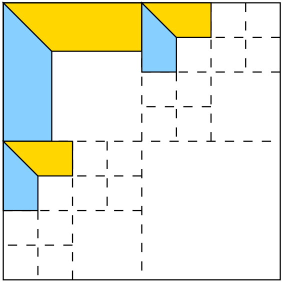
- Apply recursively on the anti-diagonal blocks



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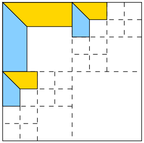

For a quasiseparability order s:

- $s \geq$ all ranks of the PLUQ

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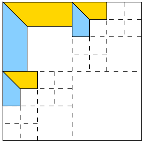

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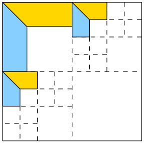

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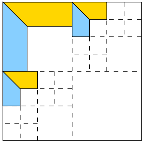

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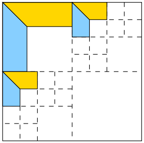

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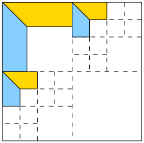

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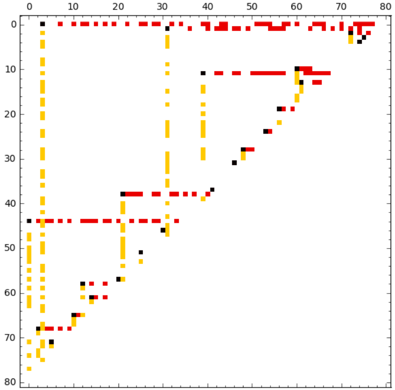
- ✓QuasiSep × QuasiSep in $O(n^2 s^{\omega-2})$

The Bruhat generator

Idea: *There can not be too many pivots near and far from the anti-diagonal.*

From a PLUQ decomposition revealing the rank profile matrix E:

$$\mathcal{L} = \text{LeftTriang}(P \begin{bmatrix} L & 0 \end{bmatrix} Q),$$
$$\mathcal{E} = \text{LeftTriang}(E),$$
$$\mathcal{U} = \text{LeftTriang}(P \begin{bmatrix} U \\ 0 \end{bmatrix} Q).$$



The Bruhat Generator

For a quasiseparability order s:

- ✓Size: $O(ns)$

Lemma

There is less than ns non zero elements in $\mathcal{L}$.

Proof.

$\mathcal{L}$ is left triangular of QS order s.
The k-th row of $\mathcal{L}$ has $\leq s$ pivots above $\rightsquigarrow \leq s$ non-zeros. □

The Bruhat Generator

For a quasiseparability order s:

- ✓Size: $O(ns)$
- ✓Generator: $A = \text{LeftTriang}(\mathcal{L}\mathcal{E}^T\mathcal{U})$

$$A = P \begin{bmatrix} L & 0 \end{bmatrix} Q Q^T \begin{bmatrix} U \\ 0 \end{bmatrix} Q$$

The Bruhat Generator

For a quasiseparability order s:

- ✓Size: $O(ns)$
- ✓Generator: $A = \text{LeftTriang}(\mathcal{L}\mathcal{E}^T\mathcal{U})$

$$\mathcal{A} = \text{LeftTriang}(\mathcal{L}Q^T \begin{bmatrix} U \\ 0 \end{bmatrix} Q)$$

Lemma

If $A = BU$ with U upper triangular, then
$\text{LeftTriang}(A) = \text{LeftTriang}(\text{LeftTriang}(B)U)$

The Bruhat Generator

For a quasiseparability order s:

- ✓Size: $O(ns)$
- ✓Generator: $A = \mathsf{LeftTriang}(\mathcal{L}\mathcal{E}^T\mathcal{U})$

$$\mathcal{A} = \mathsf{LeftTriang}(\mathcal{L}E^T P \begin{bmatrix} U \\ 0 \end{bmatrix} Q)$$

$$E = P \begin{bmatrix} I_r \\ 0 \end{bmatrix} Q$$

The Bruhat Generator

For a quasiseparability order s:

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If $A = LB$ with L lower triangular, then
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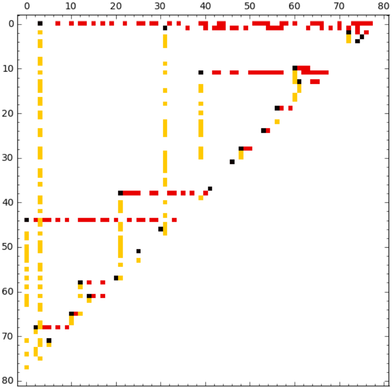
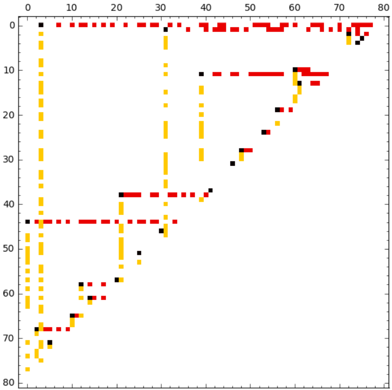


- ✓Computed in $O(n^2 s^{\omega-2})$
  ⇝ Adapted from the tile recursive PLUQ algo



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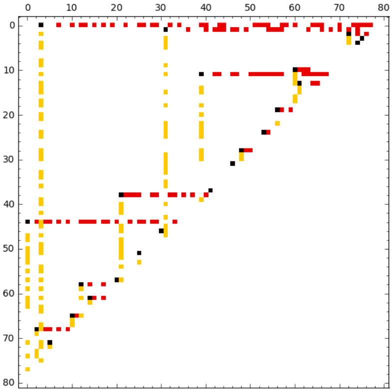
- ✓QuasiSep $\times$ Vector in $O(\text{Size}) = O(ns)$



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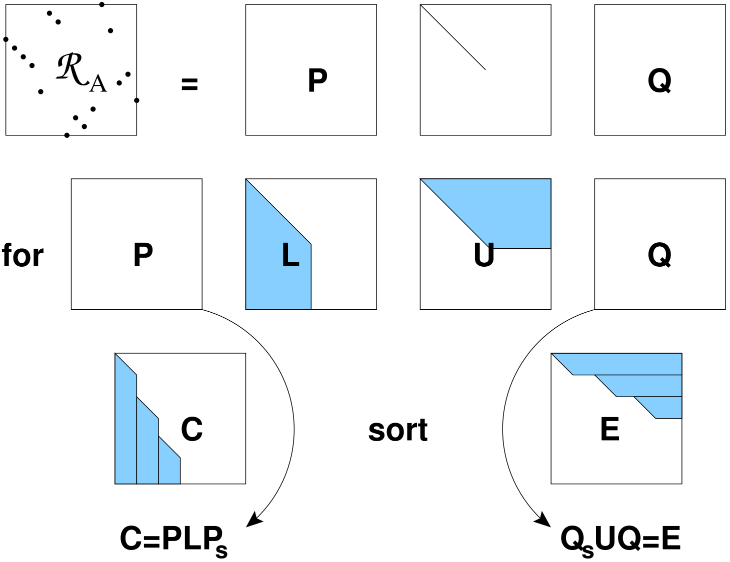
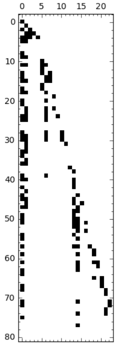
- ✓QuasiSep $\times$ Vector in $O(\text{Size}) = O(ns)$
- ✗Scattered⇝ no fast QuasiSep $\times$ QuasiSep

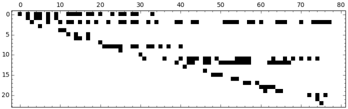
Row and column echelon forms from PLUQ



The compact Bruhat Generator

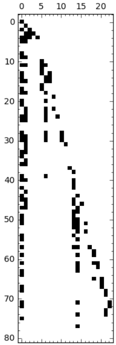
- $\mathcal{C} = \mathcal{LQ}$: Col. permutation to column echelon form
- $\mathcal{R} = \mathcal{PU}$: Row permutation to row echelon form
- $A = \text{LeftTriang}(\mathcal{CSR})$ for an $r \times r$ permutation S (generalized Bruhat decomposition[MH07]).

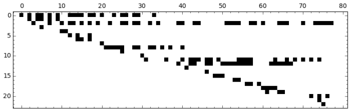




Major difficulty

Again $r = \text{rank}(A) \gg s$





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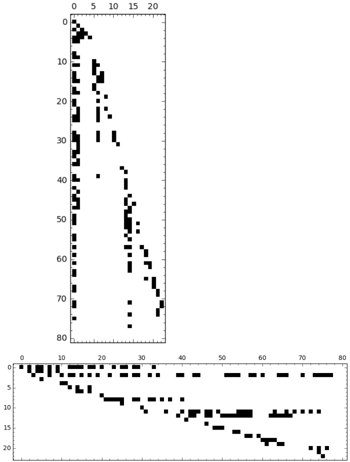

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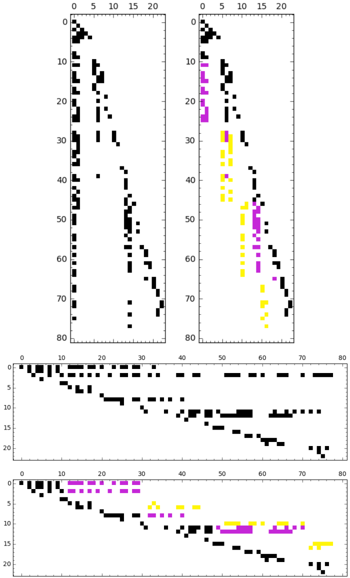

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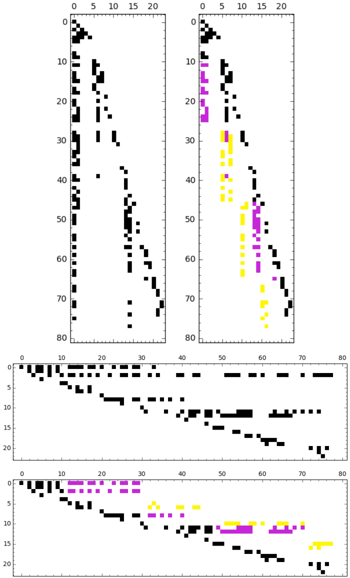

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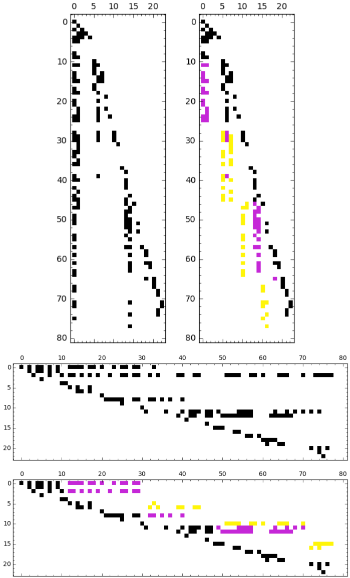
- QuasiSep $\times$ QuasiSep in $O(n^2 s^{\omega-2})$



Multiplying two quasiseparable matrices

Left triangular × Left triangular

$$A \times B = C_A R_A E_A \times C_B R_B E_B.$$

Multiplying two quasiseparable matrices

Left triangular × Left triangular

$$A \times B = C_A(R_A(E_A \times C_B)R_B)E_B.$$

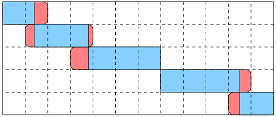
$$E_A \times C_B = (D_{\mathcal{U}_A} + T_{\mathcal{U}_A} S_{\mathcal{U}_A})(D_{\mathcal{L}_B} + S_{\mathcal{L}_B} T_{\mathcal{L}_B})$$

Multiplying two quasiseparable matrices

Left triangular $\times$ Left triangular

$$\mathsf{A} \times \mathsf{B} = \mathsf{C_A}(\mathsf{R_A}(\mathsf{E_A} \times \mathsf{C_B})\mathsf{R_B})\mathsf{E_B}.$$

$\mathsf{E_A} \times \mathsf{C_B} = (\mathsf{D}_{\mathcal{U}_\mathsf{A}} + \mathsf{T}_{\mathcal{U}_\mathsf{A}}\mathsf{S}_{\mathcal{U}_\mathsf{A}})(\mathsf{D}_{\mathcal{L}_\mathsf{B}} + \mathsf{S}_{\mathcal{L}_\mathsf{B}}\mathsf{T}_{\mathcal{L}_\mathsf{B}})$
$\rightsquigarrow$ products of block (sub)-diagonal $r_A \times n$ and $r_B \times n$



$\rightsquigarrow O(s^{\omega-1}n)$

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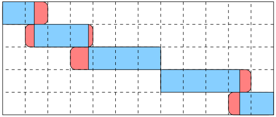

$\rightsquigarrow O(s^{\omega-1}n)$
But: shuffling by permutations: loose structure $\rightsquigarrow$ only $O(n^2 s^{\omega-2})$

Multiplying two quasiseparable matrices

Quasiseparable $\times$ Quasiseparable

$$A \times B = (L_A + D_A + U_A)(L_B + D_B + U_B).$$

$\times$	Lower	Upper
Lower	$J_n \times \text{Left} \times J_n \times \text{Left}$	$J_n \times \text{Left} \times \text{Left} \times J_n$
Upper	$\text{Left} \times J_n \times J_n \times \text{Left}$	$\text{Left} \times J_n \times \text{Left} \times J_n$

$\rightsquigarrow$ reduce to the complexity of Left-triangular $\times$ Left-triangular

Perspectives

Improve QS $\times$ QS complexity

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- [**SY14**] propose a $\tilde{O}(n^2 + r^\omega)$ probabilistic algorithm computing the row rank profile
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- Does it applies to computing
  - the QS orders ?
  - a (compact) Bruhat generator ?

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