Graph and Mean-Field Limits for Interacting Particle Systems on Weighted Deterministic and Random Graphs

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In collaboration with N. Pouradier Duteil, D. Poyato

Introduction

Collective dynamics models

Social dynamics model

$$\frac{d}{dt}x_i(t) = \frac{1}{N}\sum_{j=1}^{N} a_{ij}\,(x_j(t) - x_i(t)),$$

where:

- $x_i \in \mathbb{R}^d$ is the state variable (opinion, position)
- $a_{ij} \in \mathbb{R}$ is the **interaction coefficient**.

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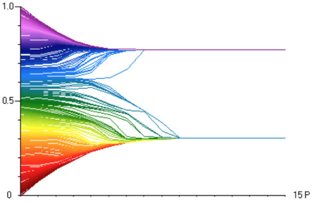
Hegselmann-Krause dynamics

$$\frac{d}{dt}x_i = \frac{1}{N}\sum_{j=1}^{N} a(\|x_i - x_j\|)(x_j - x_i), \quad x_i \in \mathbb{R}^d, \quad i \in \{1, \dots, N\} \tag{HK}$$

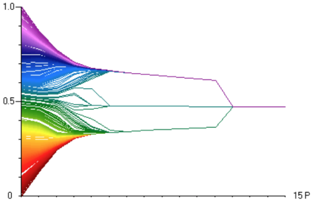
with $a_{ij} = a(\|x_i - x_j\|)$ where $a : \mathbb{R}^+ \to \mathbb{R}^+$ is the *influence function*.

Two types of questions

- **Self-organization**: emergence of well organized group patterns.



Clustering



Consensus

[Hegselmann and Krause, '02]

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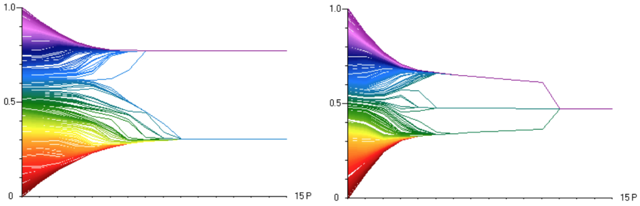

Clustering Consensus

[Hegselmann and Krause, '02]

- **Large Population Limit**: N the number of agents goes to infinity.

The classical approach : The mean-field limit

- **No longer** follow each agent's **individual trajectory**,
- the population is represented by its **probability density**,
- the **limit measure** $\mu_t(x)$ represents the density of agents with opinion x at time t.

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HK model: macroscopic

$$\partial_t \mu_t + \nabla \cdot (V[\mu_t]\mu_t) = 0 \qquad V[\mu_t](x) = \int_{\mathbb{R}^d} a(\|x-y\|)(y-x)d\mu_t(y).$$

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- **Limitation**: *Indistinguishability* of the particles ⇒ reduces the span of models that can be studied.

The new approach : The graph limit

The ℓ-nearest-neighbor interactions model

$$\frac{d}{dt}x_i = \frac{1}{N}\sum_{j=i-\ell}^{i+\ell}(x_j - x_i) \qquad \text{with } \ell = \lfloor rN \rfloor, r \in [0,1] \qquad (\ell\text{-nearest})$$

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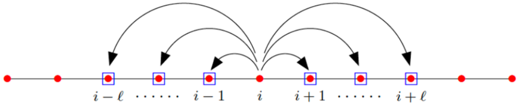
- (ℓ-nearest) : system of ODE on **graph** $G_N =< V(G_N), E(G_N) >$ with

$$V(G_N) = \{1, 2, \ldots, N\} \qquad E(G_N) = \{(i,j) \in \{1,2,\ldots,N\}^2 |\ 0 < dist(i,j) \leq \ell\}$$

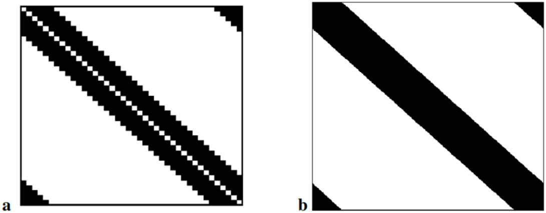
where $dist(i,j) = \min\{|i-j|, N - |i-j|\}$.



Scheme of the ℓ-nearest-neighbor interactions [Biccari, Ko, Zuazua, '19]

- Let $w^{G_N} : [0,1]^2 \to \{0,1\}$

$$w^{G_N}(\xi,\zeta) = 1 \qquad \text{if } (i,j) \in E(G_N) \text{ and } (\xi,\zeta) \in \left[\frac{i-1}{N}, \frac{i}{N}\right) \times \left[\frac{j-1}{N}, \frac{j}{N}\right).$$



Plot of the support of the function w^{G_N} representing the adjacency matrix of the ℓ-nearest-neighbor graph (a) and that of its limit W (b) [Medvedev, '13].

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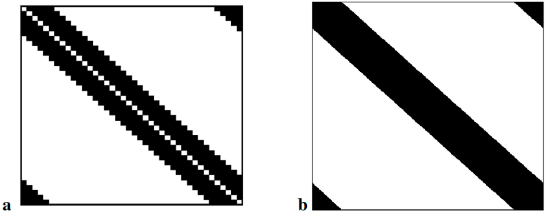

Plot of the support of the function w^{G_N} representing the adjacency matrix of the ℓ-nearest-neighbor graph (a) and that of its limit W (b) [Medvedev, '13].

- $\{w^{G_N}\}$ converges to the $\{0,1\}$-valued function $w(\xi,\zeta) = \chi_{[0,r]}(|\xi - \zeta|)$.

The graph limit *(or the continuum limit)*

Let $I = [0,1]$, $I_1^N := [0, \frac{1}{N})$ and $\forall i \in \{1, \dots, N\}$, $I_i^N := [\frac{i-1}{N}, \frac{i}{N})$. Let $w : I^2 \to \mathbb{R}$ a *graphon* on I^2.

Define a sequence of weighted graphs $G_N =< \{1, \dots, N\}, \{1, \dots, N\}^2, \bar{w}^N >$ with:

$$\bar{w}_{ij}^N = N^2 \iint_{I_i^N \times I_j^N} w(\xi, \zeta) d\xi \, d\zeta.$$

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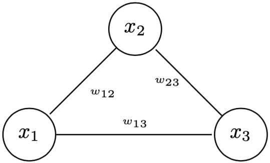
The nonlinear heat equation on G_N

$$\frac{d}{dt} x_i = \frac{1}{N} \sum_{j=1}^{N} (\bar{w}^N)_{ij} \phi(x_j - x_i), \quad x_i \in \mathbb{R}^d, \quad i \in \{1, \ldots, N\}$$



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Theorem [Medvedev, '13]: Graph Limit

If $w \in L^\infty(I)$, it holds

$$\|x - x_N\|_{C([0,T];L^2(I))} \xrightarrow[N \to +\infty]{} 0$$

where x is the solution to the integro-differential equation

$$\partial_t x(t,\xi) = \int_I w(\xi,\zeta)\, \phi(x(t,\zeta) - x(t,\xi)) d\zeta.$$

The mean-field limit

◇ **The exchangeable particle system**

$$\frac{d}{dt}x_i = \frac{1}{N}\sum_{j=1}^{N}\phi(x_j - x_i)$$

The exchangeable mean-field limit

$$\partial_t \mu_t(x) + \nabla_x \cdot \left(\left(\int_{\mathbb{R}^d} \phi(y-x)\mu_t(dy) \right) \mu_t(x) \right) = 0$$

The mean-field limit

◇ **The non-exchangeable particle system**

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$$\partial_t \mu_t^\xi(x) + \nabla_x \cdot \left(\left(\int_I \int_{\mathbb{R}^d} w(\xi,\zeta)\phi(y-x)\mu_t^\zeta(dy)d\zeta \right) \mu_t^\xi(x) \right) = 0$$

- Kaliuzhnyi-Verbovetskyi, Medvedev, '18
- Chiba, Medvedev, '19
- Gkogkas, Kuehn, 20
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◇ More details and **links between the two approaches**
⇒ **Review paper** (A., Pouradier Duteil, '24)

The different systems/equations

- The **microscopic dynamics:**

$$\frac{d}{dt}x_i = \frac{1}{N}\sum_{j=1}^{N} w_{ij}\phi(x_j - x_i)$$

- The **graph limit** equation:

$$\partial_t x(t,\xi) = \int_I w(\xi,\zeta)\,\phi(x(t,\zeta) - x(t,\xi))d\zeta.$$

- The **non-exchangeable mean-field limit** equation:

$$\partial_t \mu_t^{\xi}(x) + \nabla_x \cdot \left(\left(\int_I \int_{\mathbb{R}^d} w(\xi,\zeta)\phi(y-x)\mu_t^{\zeta}(dy)d\zeta\right)\mu_t^{\xi}(x)\right) = 0$$

From one system/equation to another

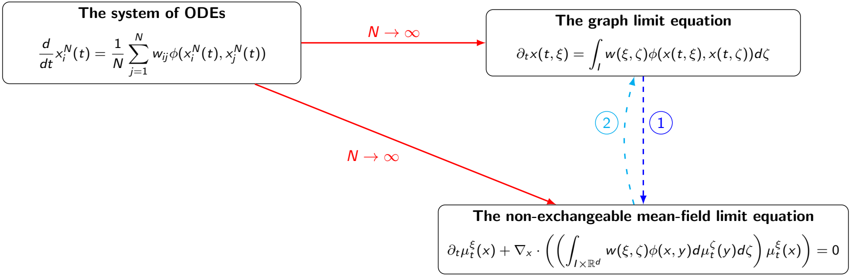

Figure: Links between the different equations.

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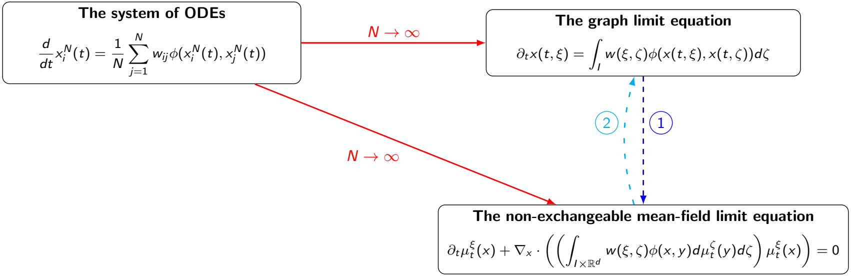

Figure: Links between the different equations.

• The red arrows corresponds to **large population limits**, respectively **graph limit** and **non-exchangeable mean-field limit**.

From graph limit to non-exchangeable limit (A., Pouradier Duteil, '24)

• Let $x(t,\xi)$ denote the **solution** to the **graph limit equation**. Let $\overline{\mu}_t$ denote a **"continuous" empirical measure** defined by

$$\overline{\mu}_t(\xi, x) = \int_I \delta_{x(t,\zeta)}(x)\delta_\zeta(\xi)d\zeta.$$

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- For all test functions $f \in C^\infty(I \times \mathbb{R}^d)$,

$$\begin{aligned} &\frac{d}{dt}\int_{I\times\mathbb{R}^d} f(\xi,x)d\overline{\mu}_t(\xi,x)d\xi = \frac{d}{dt}\int_I f(\xi, x(t,\xi))d\xi \\ = \quad &\int_I \nabla_x f(\xi, x(t,\xi)) \cdot \left(\int_I w(\xi,\zeta)\phi(x(t,\xi), x(t,\zeta))d\zeta\right) d\xi \\ = \quad &\int_{I\times\mathbb{R}^d} \nabla_x f(\xi,x) \cdot \left(\int_{I\times\mathbb{R}^d} w(\xi,\zeta)\phi(x,y)d\overline{\mu}_t(\zeta,y)d\zeta\right) d\overline{\mu}_t(\xi,x)d\xi, \end{aligned}$$

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$\Longrightarrow \overline{\mu}_t(\xi,x)$ **solution** of the **Vlasov equation**

$$\partial_t \mu_t^\xi(x) + \nabla_x \cdot \left(\left(\int_{I\times\mathbb{R}^d} w(\xi,\zeta)\phi(x,y)d\mu_t^\zeta(y)d\zeta \right) \mu_t^\xi(x) \right) = 0$$

From the non-exchangeable mean-field limit to the graph limit (d=1)

We denote

$$\bar{x}(t,\xi) := \int_{\mathbb{R}} x \, d\mu_t^{\xi}(x).$$

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Hypothesis

We suppose that

$$\phi(x,y) = (\lambda_1 x + \lambda_2 y),$$

with $(\lambda_1, \lambda_2) \in \mathbb{R}^2$.

Example: the original **Hegselmann-Krause** for which the interation corresponds to $w(\xi,\zeta)(y-x)$.

We obtain

$$\begin{aligned}
\partial_t \bar{x}(t,\xi) &= \int_{\mathbb{R}} \left(\int_{I\times\mathbb{R}} w(\xi,\zeta)(\lambda_1 x + \lambda_2 y) d\mu_t^{\zeta}(y) d\zeta \right) \, d\mu_t^{\xi}(x) \\
&= \int_I w(\xi,\zeta) \left(\lambda_1 \int_{\mathbb{R}} x d\mu_t^{\xi}(x) + \lambda_2 \int_{\mathbb{R}} y d\mu_t^{\zeta}(y) \right) d\zeta \\
&= \int_I w(\xi,\zeta) \left(\lambda_1 \bar{x}(t,\xi) + \lambda_2 \bar{x}(t,\zeta) \right) d\zeta \\
&= \int_I w(\xi,\zeta) \phi(\bar{x}(t,\xi), \bar{x}(t,\zeta)) d\zeta.
\end{aligned}$$

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• **Obtaining a closed equation** in the general **(nonlinear)** case: **still open** (for further comments, see Paul, Trélat, '22).

Purpose of the talk

Discussion around **three variants** of the previous model:

- **adaptive dynamical** networks,
- **random weighted** graphs,
- **higher-order** interactions.

References:

- *Mean-field and graph limits for collective dynamics models with time-varying weights*, A., Pouradier Duteil, '21,
- *Graph limit for interacting particle systems on weighted random graphs*, A., Pouradier Duteil, '23,
- *Large-population limits of non-exchangeable particle systems*, A., Pouradier Duteil, '24,
- *Mean-field limit of non-exchangeable multi-agent system over hypergraphs with unbounded rank*, A., Pouradier Duteil, Poyato, '24.

ADAPTIVE DYNAMICAL NETWORK

Adaptive dynamical network

• **Real-life interactions**: not only are **relationships influence our opinions**, but our opinions also exert a **reciprocal effect**, inducing **alterations in the network structure** of our relationships.

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Definition

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General form:

$$\begin{cases} \dfrac{d}{dt} x_i(t) = f_i(x_i(t), t) + \displaystyle\sum_{j=1}^{N} w_{ij}(t)\phi\left(x_i(t), x_j(t), t\right) & \text{for all } i \in \{1, \cdots, N\}, \\ \dfrac{d}{dt} w_{ij}(t) = h_{ij}(w^N(t), x^N(t), t), \end{cases}$$

where $x^N = (x_i)_{1 \le i \le N}$ and $w^N = (w_{ij})_{1 \le i,j \le N}$

Weight-varying opinion dynamics (A. Pouradier Duteil, '21)

Opinion dynamics with time-varying influence

$$\begin{cases} \frac{d}{dt}x_i(t) = \frac{1}{N}\sum_{j=1}^{N} m_j(t)\phi(x_j(t) - x_i(t)) \\ \frac{d}{dt}m_i(t) = \psi_i(m(t), x(t)) \end{cases} \qquad (\mathrm{D}_N)$$

where:

- $x_i \in \mathbb{R}^d$ is the state variable (opinion, position)
- $m_i \in \mathbb{R}^+$ is the agent's weight
- $N = \sum_{i=1}^{N} m_i(0)$ is the (initial) total weight of the system
- ϕ is the interaction function (often, $\phi(x_j - x_i) = a(\|x_i(t) - x_j(t)\|)(x_j(t) - x_i(t)))$
- ψ_i dictate the weight dynamics. We suppose $\sum_i \psi_i \equiv 0$.

The model viewed on a graph

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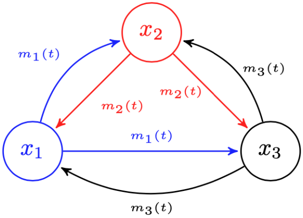

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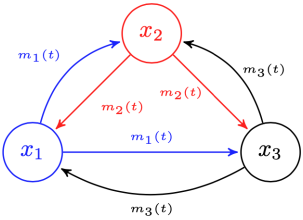

- The edge weights depend on time $m_i(t)$.

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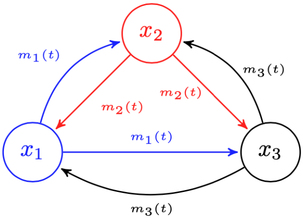
$$\begin{cases} \frac{d}{dt}x_i(t) = \frac{1}{N}\sum_{j=1}^{N} m_j(t)\phi(x_j(t) - x_i(t)) \\ \frac{d}{dt}m_i(t) = \psi_i(m(t), x(t)) \end{cases}$$



- The edge weights depend on time $m_i(t)$.
- Their evolution is coupled with the evolution of the nodes $x_i(t)$.

Continuous model: well-posedness

Hypothesis (H1)

The interaction function ϕ *satisfies* $\phi(0) = 0$ *and* $\phi \in \mathrm{Lip}(\mathbb{R}^d; \mathbb{R})$, *with* $\|\phi\|_{\mathrm{Lip}} = L_\phi$.

Hypothesis (H2)

$$\begin{cases} \|\psi(\cdot, x_1, m_1) - \psi(\cdot, x_2, m_1)\|_{L^2(I)} \leq L_\psi \|x_1 - x_2\|_{L^2(I)} \\ \|\psi(\cdot, x_1, m_1) - \psi(\cdot, x_1, m_2)\|_{L^2(I)} \leq L_\psi \|m_1 - m_2\|_{L^2(I)}. \end{cases}$$

and

$$|\psi(\xi, x, m)| \leq C_\psi(1 + \|m\|_{L^\infty(I)}).$$

Theorem [A., Pouradier Duteil, '21]

Let $x_0 \in L^\infty(I; \mathbb{R}^d)$ and $m_0 \in L^\infty(I; \mathbb{R})$. Then for any $T > 0$, there exists a unique solution $(x, m) \in \mathcal{C}([0, T]; L^\infty(I; \mathbb{R}^d \times \mathbb{R}))$ to the *Graph Limit Equation*

$$\begin{cases} \partial_t x(\xi, t) = \int_I m(\zeta, t)\phi(x(\xi, t) - x(\zeta, t))d\zeta; & x(\cdot, 0) = x_0 \\ \partial_t m(\xi, t) = \psi(\xi, x(\cdot, t), m(\cdot, t)); & m(\cdot, 0) = m_0. \end{cases} \tag{GL}$$

From discrete to continuous

From $(x_i^N(t))_{i\in\{1,\ldots,N\}}$ and $(m_i^N(t))_{i\in\{1,\ldots,N\}}$, we define

$$\begin{cases} x_N(\xi,t) = P_c^N(x^N(t)) := \sum_{i=1}^N x_i^N(t)\mathbf{1}_{[\frac{i-1}{N},\frac{i}{N})}(\xi) \\ m_N(\xi,t) = P_c^N(m^N(t)) := \sum_{i=1}^N m_i^N(t)\mathbf{1}_{[\frac{i-1}{N},\frac{i}{N})}(\xi). \end{cases}$$

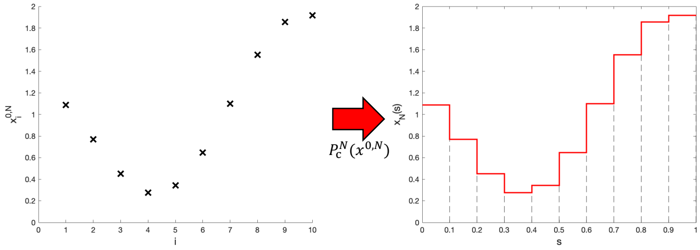


Illustration of the transformation P_c^N for $N = 10$ and $x^{0,N} \in \mathbb{R}$.

Key idea: equivalence of discrete and continuous formulations

Let $x_0 \in L^\infty(I;\mathbb{R}^d)$ and $m_0 \in L^\infty(I;\mathbb{R})$ satisfying $\int_I m_0(s)ds = 1$.
$(x^N, m^N) \in \mathcal{C}([0,T];\mathbb{R}^d)^N \times \mathcal{C}([0,T];\mathbb{R})^N$ satisfy

$$\begin{cases} \frac{d}{dt} x_i^N(t) = \frac{1}{N}\sum_{j=1}^N m_j^N(t)\phi(x_j^N(t) - x_i^N(t)), \\ \frac{d}{dt} m_i^N(t) = \psi_i^{(N)}(m^N(t), x^N(t)), \end{cases} \tag{D_N}$$

with initial conditions $x_i^N(0) = P_{\mathrm{d}}^N(x_0)_i$, $m_i(0) = P_{\mathrm{d}}^N(m_0)_i$,

if and only if $x_N = P_{\mathrm{c}}^N(x^N)$ and $m_N = P_{\mathrm{c}}^N(m^N)$ satisfy

$$\begin{cases} \partial_t x_N(\xi,t) = \int_I m_N(\zeta,t)\,\phi(x_N(\zeta,t) - x_N(\xi,t))\,d\zeta, \\ \partial_t m_N(\xi,t) = N\int_{\frac{1}{N}\lfloor \xi N\rfloor}^{\frac{1}{N}(\lfloor \xi N\rfloor+1)} \psi(\zeta, x_N(\cdot,t), m_N(\cdot,t))\,d\zeta, \end{cases} \tag{C_N}$$

with initial conditions $x_N(\cdot,0) = P_{\mathrm{c}}^N(P_{\mathrm{d}}^N(x_0))$ and $m_N(\cdot,0) = P_{\mathrm{c}}^N(P_{\mathrm{d}}^N(m_0))$.

Mean-field limit for the classical HK model

Being a solution to

$$\frac{d}{dt}x_i^N = \frac{1}{N}\sum_{j=1}^{N}\phi(x_j^N - x_i^N), \quad i \in \{1,\ldots,N\}. \tag{HK}$$

is equivalent to the *empirical measure*

$$\nu^N(t,x) := \frac{1}{N}\sum_{i=1}^{N}\delta(x - x_i^N(t)).$$

being a solution to **the non-local transport equation**

$$\partial_t \nu_t(x) + \nabla\cdot(V[\nu_t]\nu_t) = 0$$

where $V[\nu_t] = \int_{\mathbb{R}^d}\phi(y-x)d\nu_t(y)$.

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Theorem: Convergence in Wasserstein distance

If $\exists \nu_0 \in \mathcal{P}(\mathbb{R}^d)$ s. t. $\lim_{N\to\infty} W(\nu_0^N, \nu_0) = 0$, then $\forall t \in [0,T], \lim_{N\to\infty} W(\nu_t^N, \nu_t) = 0$.

Generalization of the empirical measure

Consider our microscopic model with time-varying weights:

$$\begin{cases} \dfrac{d}{dt}x_i^N(t) = \dfrac{1}{N}\displaystyle\sum_{j=1}^{N} m_j^N(t)\phi(x_j^N(t) - x_i^N(t)), \\ \dfrac{d}{dt}m_i^N(t) = \psi_i^{(N)}(m^N(t), x^N(t)). \end{cases} \tag{D_N}$$

We define a new *empirical measure* by

$$\mu^N(t,x) := \frac{1}{N}\sum_{i=1}^{N} m_i^N(t)\delta(x - x_i^N(t)).$$

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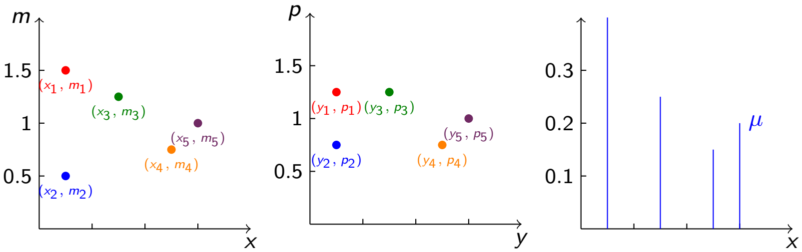
Notice that μ^N is invariant by

- relabeling of the indices,
- grouping of the agents: for every $(x^N, m^N) \in (\mathbb{R}^d)^N \times \mathbb{R}^N$, for every $J \subset \{1, \dots, N\}$, such that $x_i^N = x_J$ for all $i \in J$,

$$\frac{1}{N}\sum_{i=1}^{N} m_i^N \delta(x - x_i^N) = \frac{1}{N}\left[\left(\sum_{i\in J} m_i^N\right)\delta(x - x_J) + \sum_{i\in\{1,\dots,N\}\setminus J} m_i^N \delta(x - x_i^N)\right].$$

Indistinguishability (illustration)



Example: (x^5, m^5) and (y^5, p^5) correspond to the same empirical measure $\mu^5 \in \mathcal{P}(\mathbb{R})$.
Left: (x^5, m^5) with $x^5 = (0.5, 0.5, 1.5, 2.5, 3)$ and $m^5 = (1.5, 0.5, 1.25, 0.75, 1)$.
Center: (y^5, p^5) with $y^5 = (0.5, 0.5, 1.5, 2.5, 3)$ and $p^5 = (1.25, 0.75, 1.25, 0.75, 1)$.
Right: Empirical measure $\mu^5 = \frac{1}{5}(2\,\delta_{0.5} + 1.25\,\delta_{1.5} + 0.75\,\delta_{2.5} + \delta_3)$.

Indistinguishability (definition)

Definition

We say that system (D_N) preserves *indistinguishability* if for all $J \subset \{1, \ldots, N\}$, for all initial conditions $(x^0, m^0) \in \mathbb{R}^{dN} \times \mathbb{R}^N$ and $(y^0, p^0) \in \mathbb{R}^{dN} \times \mathbb{R}^N$ satisfying

$$\begin{cases} x_i^0 = y_i^0 = x_j^0 = y_j^0 & \text{for all } (i,j) \in J^2 \\ x_i^0 = y_i^0 & \text{for all } i \in \{1, \ldots, N\} \\ m_i^0 = p_i^0 & \text{for all } i \in J^c \\ \sum_{i \in J} m_i^0 = \sum_{i \in J} p_i^0, \end{cases}$$

the solutions $t \mapsto (x(t), m(t))$ and $t \mapsto (y(t), p(t))$ to system (D_N) with respective initial conditions (x^0, m^0) and (y^0, p^0) satisfy for all $t \geq 0$,

$$\begin{cases} x_i(t) = y_i(t) = x_j(t) = y_j(t) & \text{for all } (i,j) \in J^2 \\ x_i(t) = y_i(t) & \text{for all } i \in \{1, \ldots, N\} \\ m_i(t) = p_i(t) & \text{for all } i \in J^c \\ \sum_{i \in J} m_i(t) = \sum_{i \in J} p_i(t). \end{cases}$$

Special class of weight dynamics and mean-field limit

$$\begin{cases} \dfrac{d}{dt} x_i^N(t) = \dfrac{1}{N} \displaystyle\sum_{j=1}^{N} m_j^N(t) \phi(x_j^N(t) - x_i^N(t)), \\ \dfrac{d}{dt} m_i^N(t) = m_i \psi(x_i, \mu_N). \end{cases}$$

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$$\begin{cases} \frac{d}{dt}x_i^N(t) = \frac{1}{N}\sum_{j=1}^N m_j^N(t)\phi(x_j^N(t) - x_i^N(t)), \\ \frac{d}{dt}m_i^N(t) = m_i\psi(x_i, \mu_N). \end{cases}$$

Let $f \in \mathcal{C}_c^\infty(\mathbb{R}^d)$,

$$\begin{aligned} &\frac{d}{dt}\int f(x)d\mu_N(x) \\ &= \frac{d}{dt}\left[\frac{1}{N}\sum_{i=1}^N m_i f(x_i)\right] = \frac{1}{N}\sum_{i=1}^N \frac{d}{dt}m_i f(x_i) + \frac{1}{N}\sum_{i=1}^N m_i \frac{d}{dt}x_i \cdot \nabla f(x_i) \\ &= \frac{1}{N}\sum_{i=1}^N m_i\psi(x_i, \mu_N)f(x_i) + \frac{1}{N^2}\sum_{i=1}^N\sum_{j=1}^N m_i m_j^N(t)\phi(x_j^N(t) - x_i^N(t)) \cdot \nabla f(x_i) \\ &= \int \psi(x, \mu_N)f(x)d\mu_N(x) + \int\int \phi(y - x) \cdot \nabla f(x)d\mu_N(x)d\mu_N(y). \end{aligned}$$

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Hence the equation

$$\partial_t\mu_t(x) + \nabla \cdot (V[\mu_t](x)\mu_t(x)) = h[\mu_t](x)$$

with $h[\mu](x) = \psi(x, \mu)\mu(x)$ and $V[\mu](x) = \int \phi(y - x)d\mu(y)$.

Subordination of the Mean-Field Equation to the Graph Limit Equation

$$\psi_i^{(N)}(x,m) = m_i(t)\frac{1}{N^k}\sum_{j_1=1}^{N}\cdots\sum_{j_k=1}^{N} m_{j_1}(t)\cdots m_{j_k}(t)S(x_i(t), x_{j_1}(t),\cdots x_{j_k}(t)). \qquad \text{(S)}$$

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Proposition [A., Pouradier Duteil, '21]

Let $(x,m)\in\mathcal{C}([0,T];L^2(I;\mathbb{R}^d))\times\mathcal{C}([0,T];L^2(I;\mathbb{R}))$such that

$$\begin{cases} \partial_t x(\xi,t) = \int_I m(\zeta,t)\phi(x(\xi,t)-x(\zeta,t))d\zeta \\ \partial_t m(\xi,t) = m(\xi)\int_{I^k} m(\xi_1)\cdots m(\xi_k)\, S(x(\xi),x(\xi_1),\cdots,x(\xi_k))\, d\xi_1\cdots d\xi_k \end{cases} \qquad \text{(GL)}$$

Let $\tilde{\mu}\in\mathcal{P}(\mathbb{R}^d)$ be defined by

$$\tilde{\mu}_t(x) := \int_I m(\xi,t)\delta(x-x(\xi,t))d\xi.$$

Then $\tilde{\mu}$ satisfies the transport equation with source

$$\partial_t\mu_t(x) + \nabla\cdot(V[\mu_t](x)\mu_t(x)) = h[\mu_t](x). \qquad \text{(MFL)}$$

Theorem [A., Pouradier Duteil, '21]

Let $x_0 \in L^\infty(I;\mathbb{R}^d)$ and $m_0 \in L^\infty(I;\mathbb{R}^d)$. Let $(x^N, m^N) \in \mathcal{C}([0,T];\mathbb{R}^d)^N \times \mathcal{C}([0,T];\mathbb{R})^N$ satisfy the ODE system with initial condition $x^{0,N} = P_{\mathrm{d}}^N(x_0)$ and $m^{0,N} = P_{\mathrm{d}}^N(m_0)$ for the special class of weight dynamics. Let μ^N be the **empirical measure** associated with (x^N, m^N), i.e. for all $t \in [0,T]$,

$$\mu_t^N(x) := \frac{1}{N}\sum_{i=1}^N m_i^N(t)\delta(x - x_i^N(t)).$$

Secondly, let (x, m) be the **solutions to the graph limit system** for these weight dynamics and initial conditions given by $x(0,\cdot) = x_0$ and $m(0,\cdot) = m_0$. Let

$$\tilde{\mu}_t(x) := \int_I m(t,\xi)\delta(x - x(t,\xi))d\xi.$$

Then, for all test function $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$, and all $t \in [0,T]$, it holds

$$\lim_{N\to\infty} \int_{\mathbb{R}^d} \varphi(x)(d\mu_t^N(x) - d\tilde{\mu}_t(x)) = 0.$$

Idea

We have, for all test function $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} \varphi(x) d\mu_t^N(x) = \int_{\mathbb{R}^d} \varphi(x) d\tilde{\mu}_t^N(x),$$

where $\tilde{\mu}_t^N \in \mathcal{P}(\mathbb{R}^d)$ is the measure defined by

$$\tilde{\mu}_t^N(x) := \int_I m_N(t,\xi)\delta(x - x_N(t,\xi))d\xi.$$

Example *"The least influenced gain influence"*

Denote by $e_{j\to i} = m_j\phi(x_i - x_j)$ the influence of j on i. Let e_i represent the **total group influence on i**:

$$e_i = \sum_{j=1}^{N} e_{j\to i} = \sum_{j=1}^{N} m_j \|\phi(x_i - x_j)\|.$$

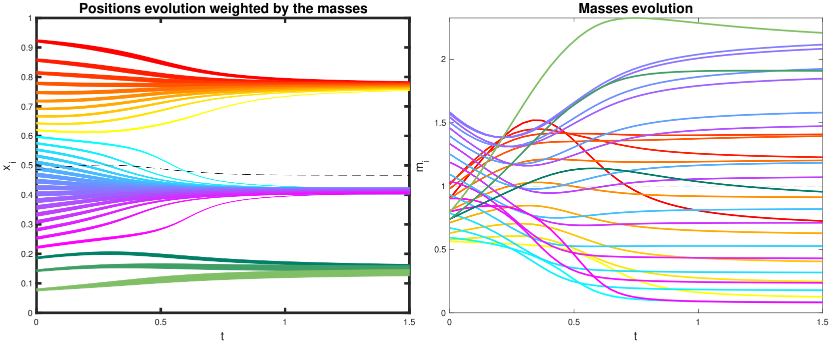
Denoting by $\overline{e}$ the **weighted average of the total group influence**

$$\overline{e} = \sum_{k=1}^{N} \frac{m_k}{N} e_k = \sum_{k=1}^{N}\sum_{j=1}^{N} \frac{m_k}{N} m_j \|\phi(x_k - x_j)\|,$$

we consider the mass dynamics:

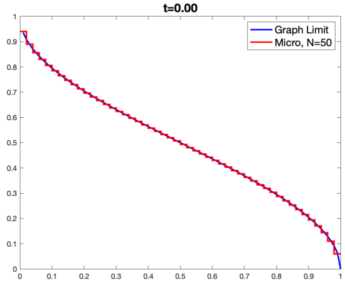
$$\psi_i(x,m) = \frac{1}{N} m_i \left(\overline{e} - e_i\right) = \frac{1}{N} m_i \left(\frac{1}{N}\sum_{k=1}^{N}\sum_{j=1}^{N} m_k m_j \|\phi(x_l - x_j)\| - \sum_{j=1}^{N} m_j \|\phi(x_i - x_j)\|\right).$$

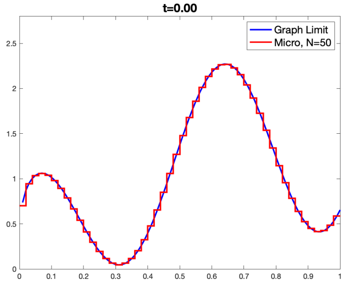
Example *"The least influenced gain influence"*: microscopic system



Evolution of opinions (left) and weights (right) for the microscopic model (D_N) with $N = 30$.

Example *"The least influenced gain influence"*: Graph Limit





Evolution of the solutions to (D_N) and (GL) represented as functions from $I = [0, 1]$ to $\mathbb{R}$.

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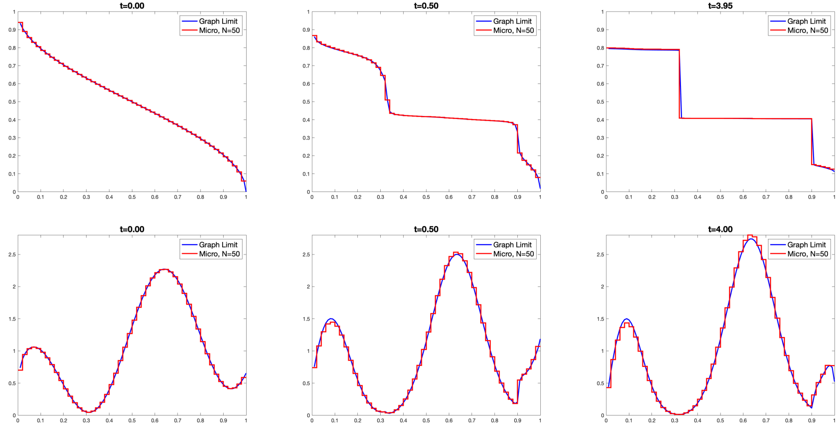

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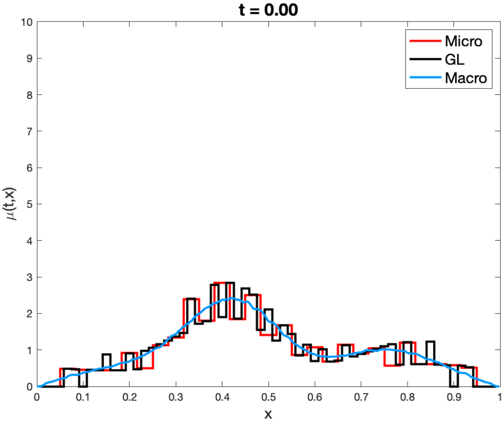
Example *"The least influenced gain influence"*: Mean-Field Limit



Evolution of the solutions to (D_N), (GL) and (MFL) represented as measures on $\mathbb{R}$.

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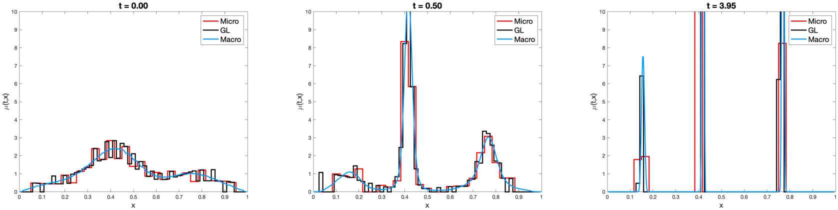

Evolution of the solutions to (D_N), (GL) and (MFL) represented as measures on $\mathbb{R}$.

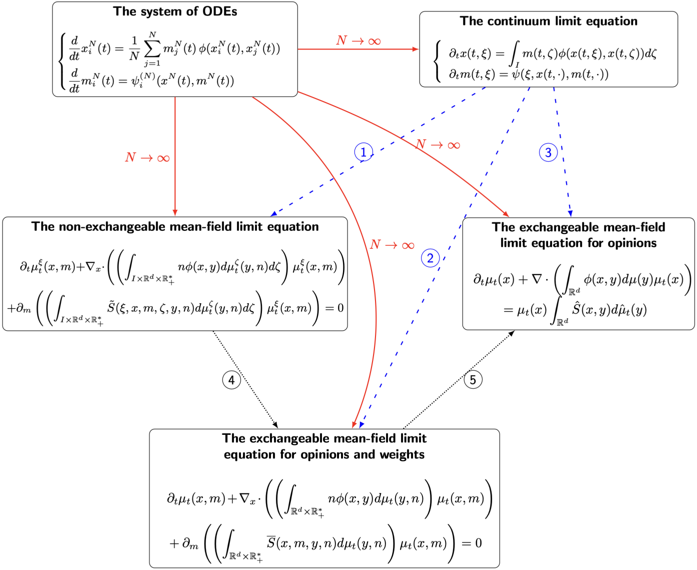

Figure: Links between the different equations (A., Pouradier-Duteil, '24)

Other results

- The setting of **Kuramoto-type model** (Gkogkas, Kuehn, Xu, '23)

$$\begin{cases} \dfrac{d}{dt}x_i = \omega_i(x_i, t) + \dfrac{1}{N}\displaystyle\sum_{j=1}^{N} w_{ij}\phi\left(x_i, x_j\right) \qquad \text{for all } i \in \{1, \cdots, N\} \\ \dfrac{d}{dt}w_{ij} = -\varepsilon\left(w_{ij} + H(x_i, x_j)\right) \end{cases}$$

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- Generalization of the **evolving-weight dynamics** (Throm, '23)

$$\begin{cases} \dfrac{d}{dt}x_i = \omega_i(x,t) + \dfrac{1}{N}\displaystyle\sum_{j=1}^{N} w_{ij}\phi(x_i,x_j) & \text{for all } i \in \{1,\cdots,N\} \\ \dfrac{d}{dt}w_{ij} = \psi_{ij}^{(N)}(x(t),w(t)) \end{cases} \tag{1}$$

WEIGHTED RANDOM GRAPHS

About random graphs

- **Random graph:** a graph which is generated by a random process.

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- **Example 1:** **Erdos-Rényi graph**: the edge between a pair of distinct nodes is inserted with probability p.

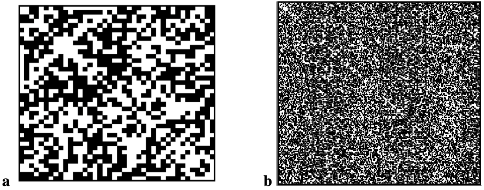

Figure: Pixel pictures of the Erdos-Rényi graph with $N = 40$ and $p = 0.5$ (left), $N = 600$ and $p = 0.5$ (right) [Medvedev, 2014]

About random graphs

- **Random graph:** a graph which is generated by a random process.

- **Example 2 : Small world graph**: replacing a random set of the local connections by randomly chosen long-range ones.

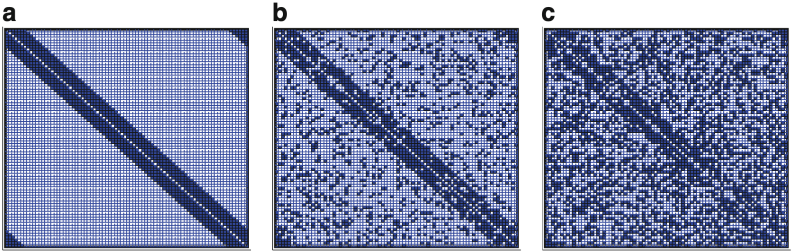

Figure: Pixel pictures of the Small world graph, p starts at 0 and increases from left to right [Medvedev, 2014]

Dynamical systems on W-random graph

• Let $\overline{\xi} = (\xi_1, \xi_2, \xi_3, \dots)$ and $\overline{\xi}^N = (\xi_1, \xi_2, \dots, \xi_N)$ where $\xi_i, i \in \mathbb{N}$ are i.i.d. random variables with $\mathcal{L}(\xi_1) = \mathcal{U}(I)$.

Definition [Medvedev, '14]

A **W-random graph** on N nodes generated by the random sequence $\overline{\xi}$, denoted $G_N = \mathbb{G}(\overline{\xi}_N, W)$ is such that the edges of G_N are **selected at random** and

$$\mathbb{P}((i,j) \in E(G_N)) = W(\xi_i, \xi_j) \text{ for each } (i,j) \in \{1, \dots, N\}^2 \text{ for } i \neq j.$$

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Dynamical systems on W-random graph

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$$\frac{d}{dt} x_i^N(t) = \frac{1}{N} \sum_{j=1}^{N} \sigma_{ij} \phi(x_j^N(t) - x_i^N(t))$$

with $\mathcal{L}(\sigma_{ij} | \overline{\xi}) = \mathcal{B}(W(\xi_i, \xi_j))$.

Random graph limit

Dynamical systems on W-random graph

$$\frac{d}{dt}x_i^N(t) = \frac{1}{N}\sum_{j=1}^{N}\sigma_{ij}\phi(x_j^N(t) - x_i^N(t)) \qquad (\tilde{S}_N^{\mathrm{r-r}})$$

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Medvedev obtains the **convergence** to

The random graph limit equation

$$\partial_t x(\xi,t) = \int_I W(\xi,\zeta)\phi(x(\zeta,t) - x(\xi,t))d\zeta. \qquad (C)$$

Random graph limit

Theorem [Medvedev, '14]: Random Graph Limit

Suppose $W \in \mathcal{W}_0$, a class of symmetric measurable function on I^2 with values on I. ϕ is a **Lipschitz continuous function** on $\mathbb{R}$ and $g \in L^\infty(I)$. Let $T > 0$ and suppose that the solution of (C) $x(\xi, \zeta)$ satisfies the following inequality

$$\min_{t\in[0,T]} \int_I \left\{ \int_I W(\xi,\zeta)\phi(x(\zeta,t) - x(\xi,t))^2 d\zeta - \left(\int_I W(\xi,\zeta)\phi(x(\zeta,t) - x(\xi,t)d\zeta \right)^2 \right\} \geq c_1$$

for some positive constant c_1. Then, the solution of $(\tilde{S}_N^{\mathrm{r-r}})$ and (C) satisfy the following relation

$$\lim_{N\to+\infty} \mathbb{P}\{N^{1/2} \sup_{t\in[0,T]} \|x^N(t) - \mathbf{P}_{\bar{\xi}^N} x(\xi,t)\|_{2,N} \leq C\} = 1$$

for some constant $C > 0$ with $\mathbf{P}_{\bar{\xi}^N} x(\xi,t) = (x(\xi_1^N, t), x(\xi_2^N, t), \ldots, x(\xi_N^N, t))$ and

$$(x,y)_N := \frac{1}{N}\sum_{i=1}^N x_i y_i, \text{ and the corresponding norm } \|x\|_{2,N} := \sqrt{(x,x)_N}.$$

Weighted random graph

Example [Garlaschelli, '09]

A **weighted random graph** model in which the **probability of drawing an edge** of discrete weight $w \in \mathbb{N}$ between vertices i and j is given by

$$\mathbb{P}(\sigma_{ij}^N = w) = q_{ij}(w) = p^w(1-p).$$

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Lack of a general framework !

Definition [A., Pouradier Duteil, '23]

A **q-weighted random graph** on N nodes generated by the random sequence $\bar{\xi}$, denoted G_N, is such that the weight of an edge of G_N is randomly attributed. More precisely, the **law for the weight of the edge** (i,j) is $q(\xi_i, \xi_j, .)$ where

$$q: \quad I \times I \quad \to \quad \mathcal{P}(\mathbb{R}_+)$$

$$(\xi, \zeta) \quad \mapsto \quad q(\xi, \zeta; .).$$

The decision of the attribution of the weight of a pair $(i,j) \in \{1, \ldots, N\}^2$ is made **independently** from the decision for other pairs.

Examples

- **W-random graph** (Medvedev, '14): **Generate** between any two nodes (ξ, ζ) an edge (of weight 1) **with probability W**(ξ, ζ).

$$q(\xi, \zeta; \cdot) = (1 - W(\xi, \zeta))\delta_0 + W(\xi, \zeta)\delta_1, \qquad \text{for all } \xi, \zeta \in \mathbb{R}.$$

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- **Erdös-Rényi weighted random graph** (Garlaschelli, 09): **Generate** between any two nodes **an edge with weight w** $\in \mathbb{N}$, with probability $p^w(1-p)$.

$$q(\xi, \zeta; \cdot) = (1 - p) \sum_{i=0}^{+\infty} p^i \delta_i, \qquad \text{for all } \xi, \zeta \in \mathbb{R}.$$

Weighted random graph limit

• Let $\overline{\xi} = (\xi_1, \xi_2, \xi_3, \dots)$ and $\overline{\xi}^N = (\xi_1, \xi_2, \dots, \xi_N)$ where $\xi_i, i \in \mathbb{N}$ are i.i.d. random variables with $\mathcal{L}(\xi_1) = \mathcal{U}(I)$.

Dynamical systems on q-weighted random graph

$$\begin{cases} \frac{d}{dt} x_i^N(t) = \frac{1}{N} \sum_{j=1}^{N} \sigma_{ij} \phi(x_j^N(t) - x_i^N(t)), \\ x_i^N(0) = g(\xi_i^N), \qquad i \in \{1, \dots, N\} \end{cases} \tag{$S_N^{\mathrm{r-r}}$}$$

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We prove the convergence towards the continuum limit

The weighted random graph limit equation

$$\begin{cases} \partial_t x(\xi, t) = \int_I \left(\int_{\mathbb{R}_+} w q(\xi, \zeta; dw) \right) \phi(x(\zeta, t) - x(\xi, t)) d\zeta \\ x(\xi, 0) = g(\xi), \quad \xi \in I, \end{cases} \tag{C_2}$$

Our result

Hypothesis 1

Let $\phi \in L^\infty(\mathbb{R})$ be bounded and Lipschitz continuous, with $\|\phi\|_{\mathrm{Lip}} := L$ and $\|\phi\|_{L^\infty(\mathbb{R})} := K$.

Hypothesis 2

There exists $M > 0$ such that for all $(\xi, \zeta) \in I^2$, for all $k \in \{1, \cdots, 4\}$,

$$\left(\int_{\mathbb{R}_+} w^k q(\xi, \zeta; dw) \right)^{1/k} \leq M,$$

i.e. the first four moments of the probability measure $q(\xi, \zeta; \cdot)$ are bounded uniformly in ξ and ζ.

Our result

Theorem [A., Pouradier Duteil, 2023]: Weighted Random Graph Limit

Let ϕ satisfy Hypothesis 1, let $g \in L^{\infty}(I)$ and let q be a weighted random graph law satisfying Hypothesis 2. Then, as N goes to infinity, **solution x^N to the discrete system** $(S_N^{\mathrm{r-r}})$ **converges** to the **solution x of the continuous model** (C_2). More precisely,

$$\mathbb{P}\left[\sup_{t\in[0,T]} \|x^N(t) - \mathbf{P}_{\overline{\xi}^N} x(\cdot, t)\|_{2,N} \geq \frac{C_1(T)}{\sqrt{N}}\right] \leq \frac{\tilde{C}_1}{N}$$

where the constants $C_1(T)$ and $\tilde{C}_1$ are respectively defined by $C_1(T) := \sqrt{T}\sqrt{1 + M^2K^2}e^{(\frac{1}{2}+4ML)T}$ and $\tilde{C}_1 := 3M^4K^4 + 6$.

Numerical Illustration: the weighted Erdös-Rényi random graph

- **Erdös-Rényi weighted random graph** (Garlaschelli, 09): **Generate** between any two nodes **an edge with weight w** $\in \mathbb{N}$, with probability $p^w(1-p)$.

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$$\bar{w}(\xi,\zeta) = \int_{\mathbb{R}^+} wq(\xi,\zeta;dw) = (1-p)\sum_{i=1}^{+\infty} ip^i = \frac{p}{1-p}$$

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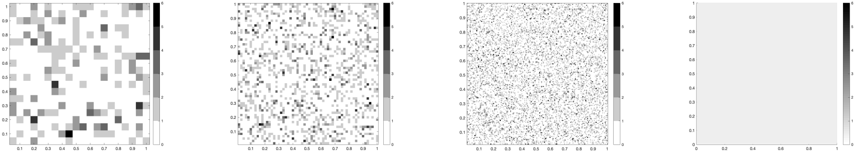
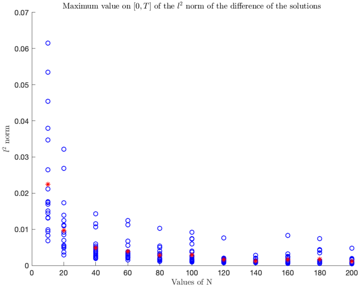


Figure: Left and Centers: Random interaction matrices generated by deterministic sequences for $N = 20$, $N = 60$ and $N = 150$, for the random weighted graphon (44), Right: Corresponding graphon.



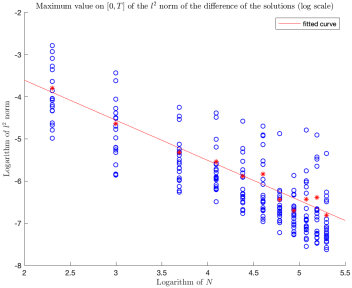


Figure: Convergence of $\sup_{t\in[0,T]} \|x^N(t) - \mathbf{P}_{\overline{\xi}^N} x(\cdot,t)\|_{2,N}$ for different values of N, with 20 runs for each value of N.

Numerical Illustration: Weighted "Small World" network

- **Model for a "small-world" network** (Watts, Strogatz, '98): **Connect each node** with its k **closest neighbors** to form a ring lattice. Then, **rewire each edge at random** with probability p.

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$$q(\xi, \zeta; dw) = \begin{cases} \frac{\rho(\xi,\zeta)}{r} d\lambda_{[0,1]} + (1 - \frac{\rho(\xi,\zeta)}{r})\delta_1 & \text{if } \rho(\xi - \zeta) \leq r \\ d\lambda_{[0,1]} & \text{otherwise} \end{cases} \tag{2}$$

where $\rho(\xi, \zeta) = \min\{|\xi - \zeta|, |\xi - \zeta - 1|, |\zeta - \xi - 1|\}$.

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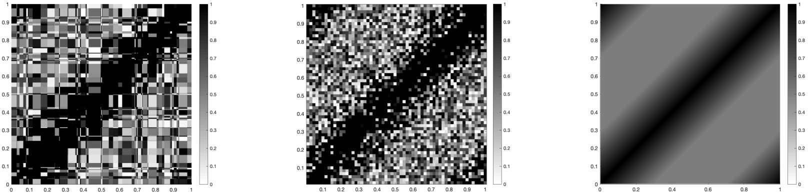

Figure: Values of the random interaction matrices generated from a random sequence (left) and a deterministic sequence (right) according to the random weighted graph law (2) for $N = 60$. Right: Corresponding continuous graphon $(\xi, \zeta) \mapsto \bar{w}(\xi, \zeta)$.

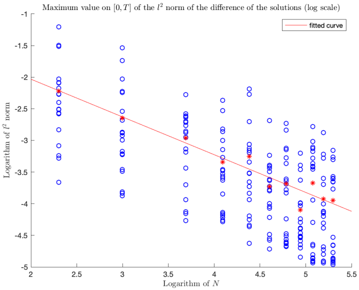

Figure: Convergence of $\sup_{t\in[0,T]} \|x^N(t) - \mathbf{P}_{\overline{\xi}^N} x(\cdot,t)\|_{2,N}$ for different values of N, with 20 runs for each value of N. Case of the random weighted graph law (44).

HIGHER ORDER INTERACTIONS

Hypergraphs

- Many **existing models** focus on **binary interactions**

Hypergraphs

• Many **existing models** focus on **binary interactions** $\neq$ **real-life dynamics** often involve **interactions** within groups containing **more than just two individuals** (virtual group chats, physical meetings . . .)

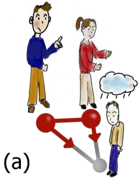

Figure: Higher-order group interactions in social context [Neuhauser et al, 2022]

Hypergraphs

- Hypergraph $H = (V, E)$ where V are the vertices, E the hyperedges.

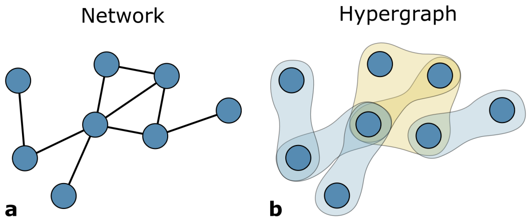

Figure: Pairwise and higher-order interactions [Battiston et al, 2021]

The θ-nearest neighbor example

The unweighted **unbounded ranked hypergraphon**: for all $\ell \in \mathbb{N}$,

$$w_\ell(\xi_0, \xi_1, \cdots, \xi_\ell) = \begin{cases} 1 & \text{if} \quad \max_{i,j \in \{0,\cdots,\ell\}} |\xi_i - \xi_j| \leq \theta \\ 0 & \text{otherwise,} \end{cases}$$

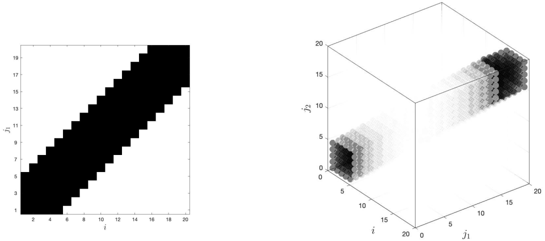

Figure: Pixel representation for $\ell = 1, 2$ with $\theta = 0.3$ and $N = 20$.

Models of multi-agent dynamics on hypergraphs

- Extension of the **Kuramoto-Saraguchi** model on hypergraphs (Skardal, Arenas, '20)

$$\frac{d}{dt}x_i = \sum_{j_1=1}^{N} w_{ij_1}^{N,1}\sin(x_{j_1} - x_i) + \sum_{j_1=1}^{N}\sum_{j_2=1}^{N} w_{ij_1j_2}^{N,2}\sin(2x_{j_1} - x_{j_2} - x_i)$$
$$+ \sum_{j_1=1}^{N}\sum_{j_2=1}^{N}\sum_{j_3=1}^{N} w_{ij_1j_2j_3}^{N,3}\sin(x_{j_1} + x_{j_2} - x_{j_3} - x_i)$$

- **Higher-order opinion dynamics** on a uniform hypergraph of rank 2 (Neuhauser, Lambiotte, Schaub '22)

$$\frac{d}{dt}x_i = \sum_{j_1=1}^{N}\sum_{j_2=1}^{N} w_{ij_1j_2}^{N,2} e^{\lambda|x_{j_1} - x_{j_2}|}\left(\frac{x_{j_1} + x_{j_2}}{2} - x_i\right).$$

Non-exchangeable mean-field limit for higher order case

$$\begin{cases} \dfrac{dX_i^N(t)}{dt} = \displaystyle\sum_{\ell=1}^{N-1} \sum_{j_1,\dots,j_\ell=1}^{N} w_{ij_1\cdots j_\ell}^{\ell,N}\, K_\ell(X_i^N(t), X_{j_1}^N(t), \dots, X_{j_\ell}^N(t)), \\ X_i^N(0) = X_{i,0}^N, \qquad i \in \{1, \cdots, N\}. \end{cases}$$

Mean-field limit of non-exchangeable multi-agent systems over hypergraphs with unbounded rank, A., Pouradier-Duteil, Poyato, In preparation

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Under some **regularity assumptions on the kernel** K_ℓ, **scaling and symmetry conditions on the weights**, then the microscopic dynamics as N goes to ∞ is characterized by **the Vlasov equation**:

$$\begin{cases} \partial_t \mu_t^\xi + \mathrm{div}_x(F_w[\mu_t](\cdot,\xi)\,\mu_t^\xi) = 0, \quad t \geq 0,\ x \in \mathbb{R}^d,\ \xi \in [0,1], \\ \mu_{t=0}^\xi = \mu_0^\xi. \end{cases}$$

where

$$F_w[\mu_t](x,\xi) := \sum_{\ell=1}^{\infty} \int_{[0,1]^\ell} w_\ell(\xi, \xi_1, \ldots, \xi_\ell) \times \left(\int_{\mathbb{R}^{d\ell}} K_\ell(x, x_1, \ldots, x_\ell)\, d\mu_t^{\xi_1}(x_1) \cdots d\mu_t^{\xi_\ell}(x_\ell) \right) d\xi_1, \ldots d\xi_\ell.$$

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Figure: Social graph (http://inicia.org.ar/blog/7-claves-para-hacer-networking/)

Thank you for your attention !