Landau's Currents in Kinetic Theory
Applications to Wave Turbulence

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Unknown: velocity distribution function $f(t,x,v)\geq 0$ satisfying

$$(\partial_t+v\cdot\nabla_x)f(t,x,v)=\mathcal{B}(f(t,x,\cdot))(v)$$

where $\mathcal{B}(f):=\mathcal{B}(f,f)$, with

$$\mathcal{B}(f,g)(v):=(2r)^2\int_{\mathbf{R}^3\times\mathbf{S}^2}(f(v')g(v'_*)-f(v)g(v_*))((v-v_*)\cdot n)_+dv_*dn$$

with $r=$molecular radius and $|n|=1$, and with

$$v':=v-(v-v_*)\cdot nn\,,\qquad v'_*:=v_*+(v-v_*)\cdot nn$$

Thm 1. [Cercignani-Illner-Pulvirenti] If $f\in L^1(\mathbf{R}^3;(1+|v|)^4dv)$, then

$$\mathcal{B}(f)\in L^1(\mathbf{R}^3;(1+|v|)^2dv)\,,\quad \text{and}\quad \int_{\mathbf{R}^3}\mathcal{B}(f)(v)\begin{pmatrix}1\\ v\\ |v|^2\end{pmatrix}dv=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

In his 1936 article on the collision integral for charged particles with Coulomb potential (see also §41 in [Landau-Lifshitz vol. 10]) Landau argued that (at least formally)

$$\int_{\mathbf{R}^3} \mathcal{B}(f)(v)dv = 0 \implies \mathcal{B}(f) = -\nabla_v \cdot J(f)$$

Henceforth $J(f)(v) \in \mathbf{R}^3$ is called a Landau (mass) current.

See also [Villani, M2AN1999] for a more detailed presentation.

By the same formal argument

$$\int_{\mathbf{R}^3} \mathcal{B}(f)(v)|v|^2 dv = 0 \implies \mathcal{B}(f)|v|^2 = -\nabla_v \cdot \Gamma(f)$$

Henceforth $\Gamma(f)(v) \in \mathbf{R}^3$ is called a Landau energy current.

Since $(v, v_*) \mapsto (v', v'_*)$ is an involutive isometry of $\mathbf{R}^6$ for all $n \in \mathbf{S}^2$, it preserves the Lebesgue measure on $\mathbf{R}^6$. Besides (v', v'_*) is invariant under the transformation $n \mapsto -n$, while

$$(v' - v'_*) \cdot n = -(v - v_*) \cdot n$$

Hence, for all $\phi \in C_c^\infty(\mathbf{R}^3)$, setting $2r = 1$ for simplicity

$$\int_{\mathbf{R}^3} \mathcal{B}(f,g)(v)\phi(v)dv$$

$$= \int_{\mathbf{R}^6 \times \mathbf{S}^2} f(v)g(v_*)(\phi(v') - \phi(v))((v - v_*) \cdot n)_+ dvdv_* dn$$

$$= \int_{\mathbf{R}^6 \times \mathbf{S}^2} f(v)g(v_*)\left(\int_0^{(v-v_*)\cdot n} \tfrac{d}{ds}\phi(v-sn)ds\right)((v-v_*)\cdot n)_+ dvdv_* dn$$

Therefore

$$-\int_{\mathbf{R}^3} \mathcal{B}(f,g)(v)\phi(v)dv$$

$$=\int_{\mathbf{R}^6\times\mathbf{S}^2} f(v)g(v_*)\left(\int_0^{(v-v_*)\cdot n} n\cdot\nabla\phi(v-sn)ds\right)((v-v_*)\cdot n)_+ dvdv_*dn$$

$$=\int_{\mathbf{R}^7\times\mathbf{S}^2} f(v)g(v_*)\mathbf{1}_{0<s<(v-v_*)\cdot n} n\cdot\nabla\phi(v-sn)((v-v_*)\cdot n)_+ dsdvdv_*dn$$

$$=\int_{\mathbf{R}^7\times\mathbf{S}^2} f(w+sn)g(w_*+sn)\mathbf{1}_{0<s<(w-w_*)\cdot n} n\cdot\nabla\phi(w)$$

$$\times((w-w_*)\cdot n)_+ dsdwdw_*dn$$

This leads to **Landau's mass current** (up to a divergence-free field)

$$\mathcal{B}(f,g)=-\nabla\cdot J(f,g) \quad \text{with} \quad J(f,g)(v)$$

$$=-\int_{\mathbf{R}^4\times\mathbf{S}^2} \mathbf{1}_{0<s<(v-v_*)\cdot n} f(v+sn)g(v_*+sn)((v-v_*)\cdot n)_+ n dsdv_*dn$$

Schwartz Kernel of J

Set $z := sn$ so that $s = |z|$ and $dsdn = \frac{dz}{|z|^2}$; integrating in the formula giving $J(f,g)$ by substitution shows that

$$J(f,g)(v) = -\int_{(\mathbf{R}^3)^2} f(v+z)g(v_*+z)\mathbf{1}_{0<|z|^2<(v-v_*)\cdot z}\frac{(v-v_*)\cdot z}{|z|^4} z dv_* dz$$

Equivalently, with $v+z=v_1$ and $v_*+z=: v_2$ so that $\frac{\partial(v_1,v_2)}{\partial(z,v_*)}=1$

$$J(f,g)(v) = \int_{(\mathbf{R}^3)^2} \mathcal{A}(v-v_1, v_1-v_2) f(v_1) g(v_2) dv_1 dv_2$$

where

$$\mathcal{A}(\xi,\eta) = -\mathbf{1}_{|\xi|^2+\eta\cdot\xi<0}\frac{\eta\cdot\xi}{|\xi|^4}\xi$$

Observe that

$$|\xi|>|\eta| \implies |\xi|^2 > |\eta\cdot\xi| \implies \mathcal{A}(\xi,\eta)=0$$

Remark. This follows from energy conservation

Lemma 2.
(1) Setting $\tau_c z = z + c$, we see that

$$J(f \circ \tau_c, g \circ \tau_c) = J(f, g) \circ \tau_c \,, \qquad c \in \mathbf{R}^3$$

(2) For each $R \in O_3(\mathbf{R})$

$$J(f \circ R, g \circ R) = R^T J(f, g) \circ R$$

(3) If f is radial, there exists a radial function (or distribution) $j(f)$ that is real-valued and satisfies

$$J(f)(v) = j(f)(|v|)v$$

Proof of (1): Due the fact that J is defined through an integral kernel depending only on $v - v_1$ and $v_1 - v_2$. $\square$

Proof of (2): The integral kernel of J is the form

$$\mathcal{A}(v - v_1, v_1 - v_2) = a[|v - v_1|, (v - v_1) \cdot (v_1 - v_2)](v - v_1) \qquad \square$$

Proof of (3): Since $f = f \circ R$, by $O_3(\mathbf{R})$-equivariance of J

$$R^T J(f)(Rv) = J(f)(v)\,, \quad R \in O_3(\mathbf{R})$$

Specializing this to $v \neq 0$ and $R \in O_3(\mathbf{R})_v \simeq O((\mathbf{R}v)^\perp)$

$$J(f)(v) = RJ(f)(v) \quad \text{so that } v \times J(f)(v) = 0$$

since $O((\mathbf{R}v)^\perp)$ acts transitively on cercles centered at 0 in $(\mathbf{R}v)^\perp$.

Thus $J(f) = j(f)v$, with $j(f)$ real-valued, and $j(f) \circ R = j(f)$ for all $R \in O_3(\mathbf{R})$, so that $j(f)$ is radial. $\square$

Lemma 3.
For each $\lambda > 0$, set $S_\lambda z := \lambda z$; then

$$J(f \circ S_\lambda, g \circ S_\lambda) = \lambda^{-5} J(f,g) \circ S_\lambda$$

<u>Proof:</u> Since $(\xi,\eta) \mapsto \mathcal{A}(\xi,\eta)$ is homogeneous of degree -1,

$$\begin{aligned} J(f \circ S_\lambda, g \circ S_\lambda)(v) &= \int_{\mathbf{R}^6} \mathcal{A}(v-v_1, v_1-v_2) f(\lambda v_1) g(\lambda v_2) dv_1 dv_2 \\ &= \int_{\mathbf{R}^6} \lambda \mathcal{A}(\lambda v - \lambda v_1, \lambda v_1 - \lambda v_2) f(\lambda v_1) g(\lambda v_2) dv_1 dv_2 \\ &= \int_{\mathbf{R}^6} \lambda \mathcal{A}(\lambda v - \bar{v}_1, \bar{v}_1 - \bar{v}_2) f(\bar{v}_1) g(\bar{v}_2) \lambda^{-6} d\bar{v}_1 d\bar{v}_2 \\ &= \lambda^{-5} J(f,g)(\lambda v) \end{aligned}$$

which implies the desired identity. □

Question. Uniqueness of J in the class of vector fields of the form

$$J(f,g)(v) = \int_{(\mathbf{R}^3)^2} \mathcal{A}(v-v_1, v_1-v_2) f(v_1) g(v_2) dv_1 dv_2\,?$$

Lemma 4. Let $\mathcal{A}(\cdot,\eta) \in \mathcal{D}'(\mathbf{R}^3)^3$ for each $\eta \in \mathbf{R}^3$ satisfy

$$\begin{cases} \operatorname{supp}[\mathcal{A}(\cdot,\eta)] \subset \overline{B(0,|\eta|)}\,, & \xi \times \mathcal{A}(\xi,\eta) = 0 \\ \nabla_\xi \cdot \mathcal{A}(\xi,\eta) = 0\,, & \mathcal{A}(\cdot,\eta) \in L^1(B)^3 \end{cases}$$

for some open ball $B \ni 0$. Then, for all $\eta \in \mathbf{R}^3$,

$$\mathcal{A}(\cdot,\eta) = 0 \quad \text{in } \mathcal{D}'(\mathbf{R}^3)^3$$

<u>Proof:</u> Set $\mathcal{A}(\xi,\eta) = a(\xi,\eta)\xi$ for $\xi \neq 0$ and all $\eta \in \mathbf{R}^3$ with real-valued a. The distribution $a(\cdot,\eta)$ is $-d$-homogeneous on $\mathbf{R}^d \setminus \{0\}$, since $\nabla_\xi \cdot \mathcal{A}(\xi,\eta) = 0$, compactly supported, thus $\operatorname{supp}[\mathcal{A}(\cdot,\eta)] \subset \{0\}$. That $\mathcal{A}(\cdot,\eta) \in L^1(B(0,1))^3$ implies the conclusion. □

It is known (consequence of Boltzmann's H Theorem) that, if $f > 0$ is rapidly decaying while $\ln f$ has polynomial growth at ∞

$$\mathcal{B}(f) = 0 \iff f = \mathcal{M}_{(\rho,u,\theta)}(v) := \frac{\rho}{(2\pi\theta)^{3/2}} e^{-|v-u|^2/2\theta}$$

Thm 5. For all $\rho, \theta > 0$ and all $u \in \mathbf{R}^3$, one has

$$J(\mathcal{M}_{(\rho,u,\theta)}) = 0$$

Proof: Set $u = 0$ by Lemma 2 (1) and $\theta = \frac{1}{2}$ by Lemma 3 WLOG, and $\rho = \pi^{3/2}$ since J is quadratic. Then $J(e^{-|\cdot|^2})(v) = \mathfrak{j}(|v|^2)v$ by Lemma 2 (3). Since $|\mathcal{A}(\xi, \eta)| \leq |\eta|/|\xi|^2$

$$|\mathfrak{j}(|v|^2)||v| \leq \int \frac{e^{-|v_1|^2} dv_1}{|v-v_1|^2} \int \tfrac{1}{2}(1 + |v_1 - v_2|^2) e^{-|v_2|^2} dv_2$$

$$\leq \frac{\pi^{3/2}}{2} \int \frac{(\frac{5}{2}+|v_1|^2)}{|v-v_1|^2} e^{-|v_1|^2} dv_1 \leq C \int \frac{e^{-|v_1|^2/2}}{|v-v_1|^2} dv_1$$

By the rearrangement inequality

$$
\begin{aligned}
|\mathrm{j}(|v|^2)|(|v|+|v|^3) \leq & 2C\int \frac{1+|v_1|^2+|v-v_1|^2}{|v-v_1|^2}e^{-|v_1|^2/2}dv_1 \\
\leq & C'\int \frac{e^{-|v_1|^2/4}}{|v_1|^2}dv_1 + C\int e^{-|v_1|^2/2}dv_1
\end{aligned}
$$

This implies that $\mathrm{j}\in L^1(0,+\infty)$. Setting

$$
\mathfrak{J}(|v|^2) := -\int_{|v|^2}^{+\infty}\mathrm{j}(r)dr\,, \qquad \text{so that } \mathfrak{J}\in L^\infty(1,+\infty)
$$

we see that $J(e^{-|\cdot|^2})(v)=\frac{1}{2}\nabla_v\mathfrak{J}(|v|^2)$ and therefore

$$
\Delta_v\mathfrak{J}(|v|^2) = 2\nabla\cdot J(e^{-|\cdot|^2})(v) = 2\mathcal{B}(e^{-|\cdot|^2})(v) = 0
$$

Therefore $v\mapsto\mathfrak{J}(|v|^2)$ is harmonic on $\mathbf{R}^3$ and bounded at infinity. Hence $\mathfrak{J}$ =const. by Liouville's theorem and $J(e^{-|\cdot|^2})=\nabla\mathfrak{J}=0$. $\square$

For all $\phi \in C_c^\infty(\mathbf{R}^3)$, denoting $\phi' := \phi(v')$ and $\phi_* := \phi(v_*)$, one has

$$\int_{\mathbf{R}^3} \mathcal{B}(f)|v|^2\phi dv = \tfrac{1}{2}\int_{\mathbf{R}^6\times\mathbf{S}^2} ff_*(v\cdot n)^2(\phi_*-\phi)((v-v_*)\cdot n)_+ dvdv_*dn$$
$$+\int_{\mathbf{R}^6\times\mathbf{S}^2} ff_*(|v|^2-(v\cdot n)^2+(v_*\cdot n)^2)(\phi'-\phi)((v-v_*)\cdot n)_+ dvdv_*dn$$

The Landau energy current is given by the formula

$$\Gamma(f)=\Gamma_1(f)+\Gamma_2(f)\,,\qquad \text{so that } |v|^2\mathcal{B}(f)=-\nabla\cdot\Gamma(f)$$

where $\Gamma_2(f)$ corresponds to the second integral above, viz.

$$\begin{cases}\Gamma_2(f):=J(|\cdot|^2f,f)+\displaystyle\int_{\mathbf{S}^2}(J_n(f,(\cdot|n)^2f)-J_n((\cdot|n)^2f,f))dn\\ J_n(f,g)(v):=\displaystyle\int_{\mathbf{R}^6}\mathcal{A}(v-v_1,v_1-v_2)\delta_n\left(\tfrac{v-v_1}{|v-v_1|}\right)f(v_1)g(v_2)dv_1dv_2\end{cases}$$

In other words, J_n is the disintegration of J in the direction $n\in\mathbf{S}^2$

Set $Q_\theta := \mathrm{Rot}_\theta \otimes I_3$ and

$$\begin{pmatrix} u_1(\theta) & u_1'(\theta) \\ u_2(\theta) & u_2'(\theta) \end{pmatrix} := Q_\theta \begin{pmatrix} v & v-(v-v_*)\cdot nn \\ v_* & v_*+(v-v_*)\cdot nn \end{pmatrix}$$

In the first term in the decomposition of the integral of $\mathcal{B}(f)|v|^2\phi$

$$\phi_* - \phi = \int_0^{\pi/2} \nabla\phi(u_1(-\theta))\cdot u_2(-\theta)d\theta$$

so that the current corresponding to that term is

$$\Gamma_1(f)(v) := -\int_{\mathbf{R}^3\times\mathbf{S}^2\times(0,\frac{\pi}{2})} ((u_1(\theta)-u_2(\theta))\cdot n)_+ (u_1(\theta)\cdot n)^2 \times f(u_1(\theta))f(u_2(\theta)))v_* dv_* dn d\theta$$

Remark The current $\Gamma_1(f)$ corresponds to the symmetry $v \leftrightarrow v_*$ and $\Gamma_2(f)$ to the pre- to post-collision transform $v' \mapsto v$.

Lemma 6.
(1) For each $R \in O_3(\mathbf{R})$

$$\Gamma(f \circ R) = R^T \Gamma(f) \circ R$$

(2) Denoting $S_\lambda z = \lambda z$ for $z \in \mathbf{R}^3$ and $\lambda > 0$

$$\Gamma(f \circ S_\lambda) = \lambda^{-7} \Gamma(f) \circ S_\lambda$$

Proof of (1): Use the $O_3(\mathbf{R})$-equivariance of J and J_n for Γ_1; for Γ_2 use the commutation of Q_θ with $I_2 \otimes R$. □

Proof of (2): For Γ_2, use the fact that $(v, v_1, \eta) \mapsto \mathcal{A}(v - v_1, \eta) v_1^{\otimes 2}$ is homogeneous of degree 1 on $\mathbf{R}^9$; in the case of Γ_1, use the fact that the map $(v, v_*) \mapsto ((u_1 - u_2) \cdot n)_+ (u_1 \cdot n)^2 v_*$ is homogeneous of degree 4 on $\mathbf{R}^6$ $(9 + 1 = 6 + 4 = 7 + 3...)$. □

Question To find power-law, steady, space-homogeneous solutions of the Boltzmann equation. If they exist, these solutions are called KZ spectra in the context of wave turbulence, by analogy with the K41 theory of fluid turbulence.

Zakharov's approach [Zakharov-Lvov-Falkovich I 1992, chapter 3]
(1) For radial distribution functions, denoting $f(t,v) = F(t,\omega)$ with $\omega = |v|^2$, the space homogeneous Boltzmann equation for hard spheres in dimension 3 is

$$\partial_t F(t,\omega) = 2(\pi r)^2 \int_{(0,+\infty)^3} \frac{\sqrt{\min(\omega,\omega_1,\omega_2,\omega_3)}}{\sqrt{\omega}} \delta(\omega+\omega_3-\omega-\omega_2)$$

$$\times (F(t,\omega_1)F(t,\omega_2) - F(t,\omega)F(t,\omega_3)) d\omega_1 d\omega_2 d\omega_3 =: \mathfrak{B}(F(t,\cdot))(\omega)$$

Think of the r.h.s. as an integral in ω_1,ω_2 with $\omega_3 = \omega_1+\omega_2-\omega$ on $\Delta = \{(\omega_1,\omega_2) \text{ s.t. } \omega_1,\omega_2,\omega_1+\omega_2-\omega > 0\}$

(2) Integrate by substitution using Zakharov's transformations

$$(\omega, \omega_3', \omega_1', \omega_2') = \frac{\omega}{\omega_1}(\omega_1, \omega_2, \omega, \omega_3) \quad \Delta_2 \to \Delta_1 \quad \left|\frac{\partial(\omega_1', \omega_2')}{\partial(\omega_1, \omega_2)}\right| = \frac{\omega^3}{\omega_1^3}$$

$$(\omega, \omega_3', \omega_1', \omega_2') = \frac{\omega}{\omega_3}(\omega_3, \omega, \omega_1, \omega_2) \quad \Delta_3 \to \Delta_1 \quad \left|\frac{\partial(\omega_1', \omega_2')}{\partial(\omega_1, \omega_2)}\right| = \frac{\omega^3}{\omega_3^3}$$

$$(\omega, \omega_3', \omega_1', \omega_2') = \frac{\omega}{\omega_2}(\omega_2, \omega_1, \omega_3, \omega) \quad \Delta_4 \to \Delta_1 \quad \left|\frac{\partial(\omega_1', \omega_2')}{\partial(\omega_1, \omega_2)}\right| = \frac{\omega^3}{\omega_2^3}$$

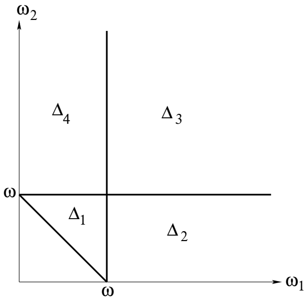

Figure: The domain of integration $\Delta = \Delta_1 \cup \Delta_2 \cup \Delta_3 \cup \Delta_4$

(3) This transforms $\mathfrak{B}(F(t,\cdot))(\omega)$ into an integral on Δ_1:

$$\sqrt{\omega}\mathfrak{B}(|\cdot|^\kappa)(\omega) = \int_{\Delta_1} \sqrt{\min(\omega,\omega_1,\omega_2,\omega_3)}(\omega_1^\kappa\omega_2^\kappa - \omega^\kappa\omega_3^\kappa)$$
$$\times\left(1 + \left(\tfrac{\omega_3}{\omega}\right)^{-\frac{7}{2}-2\kappa} - \left(\tfrac{\omega_1}{\omega}\right)^{-\frac{7}{2}-2\kappa} - \left(\tfrac{\omega_2}{\omega}\right)^{-\frac{7}{2}-2\kappa}\right)$$
$$\times\delta(\omega_1-\omega_2-\omega-\omega_3)d\omega_1 d\omega_2 d\omega_3$$

This argument shows that

$$\left.\begin{array}{l}\kappa = -7/4\\ \kappa = -9/4\end{array}\right\} \implies \left.\begin{array}{l}-\frac{7}{2}-2\kappa = 0\\ -\frac{7}{2}-2\kappa = 1\end{array}\right\} \implies \mathfrak{B}(|\cdot|^\kappa) = 0$$

Questions

(1) Is this compatible with Boltzmann's H Theorem in the equality case? [Boltzmann's H Thm, equality case: for f rapidly decreasing and $\ln f$ with polynomial growth $\mathcal{B}(f) = 0 \implies f =$ Maxwellian]

(2) Are the computations above legitimate? Existence of all the integrals involved in Zakharov's argument?

Thm 7. [Gelfand-Shilov'63 vol. 1 III.3.3] Let $T \in \mathcal{D}'(\mathbf{R}^d \setminus \{0\})$.
(1) If T is homogeneous of degree $\alpha > -d$ on $\mathbf{R}^d \setminus \{0\}$, it has a unique extension $\dot{T} \in \mathcal{D}'(\mathbf{R}^d)$ that is homogeneous of degree α.
(2) A distribution T on $\mathbf{R}^d \setminus \{0\}$ is homogeneous of degree $-d$ iff

$$\nabla \cdot (xT) = 0 \qquad \text{in } \mathcal{D}'(\mathbf{R}^d \setminus \{0\})$$

(3) If $T \in \mathcal{D}'(\mathbf{R}^d \setminus \{0\})$ is homogeneous of degree $-d$, there exists a real number $r_0(T)$ (residue of T at 0) s.t.

$$\nabla \cdot [(xT)\dot{}\,] = r_0(T)\delta_0 \qquad \text{in } \mathcal{D}'(\mathbf{R}^d)$$

(4) A distribution T on $\mathbf{R}^d \setminus \{0\}$ that is homogeneous of degree $-d$ has an homogeneous extension to $\mathbf{R}^d$ if and only if $r_0(T) = 0$, and two such extensions of T to $\mathbf{R}^d$ differ by a multiple of δ_0.

Example: $d = 1$, then $r_0(1/x) = 0$ (take $\text{pv}\frac{1}{x}$) while $r_0(1/|x|) = 2$

The KZ solutions arise from the structural properties of the Landau currents — especially scaling- and $O_3(\mathbf{R})$-equivariance.

Lemma 8. If f is radial and homogeneous of degree θ and if $j(f)$ exists in $\mathcal{D}'(\mathbf{R}^3 \setminus \{0\})$, then $j(f)$ is homogeneous of degree $4+2\theta$.

Proof: For $\lambda > 0$, one has $f \circ S_\lambda = \lambda^\theta f$ so that

$$\lambda^{-5} J(f) \circ S_\lambda = J(f \circ S_\lambda) = J(\lambda^\theta f) = \lambda^{2\theta} J(f)$$

so that $J(f) \circ S_\lambda = \lambda^{5+2\theta} J(f)$, hence $j(f) \circ S_\lambda = \lambda^{4+2\theta} j(f)$. □

Thm 9. If $\kappa \in \mathbf{R}$ is such that $J(|\cdot|^{2\kappa}) \in \mathcal{D}'(\mathbf{R}^3 \setminus \{0\})^3$, then

$$\mathcal{B}(|\cdot|^{2\kappa}) = -\nabla \cdot (J(|\cdot|^{2\kappa})) = 0 \text{ in } \mathcal{D}'(\mathbf{R}^3 \setminus \{0\}) \iff \kappa = -\tfrac{7}{4}$$

Proof: Under the assumption that $J(|\cdot|^{2\kappa}) \in \mathcal{D}'(\mathbf{R}^3 \setminus \{0\})^3$, using Lemma 2 (3) shows that $J(|\cdot|^{2\kappa}) = j(|\cdot|^{2\kappa})v$ in $\mathcal{D}'(\mathbf{R}^3 \setminus \{0\})^3$ where $j(|\cdot|^{2\kappa}) \in \mathcal{D}'(\mathbf{R}^3 \setminus \{0\})$ is a radial distribution.

By Lemma 8, for each $\lambda > 0$

$$j(|\cdot|^{2\kappa}) \circ S_\lambda = \lambda^{4+4\kappa} j(|\cdot|^{2\kappa})$$

On the other hand, by Euler's identity (Thm 7 (2))

$$\mathcal{B}(|\cdot|^{2\kappa}) = -\nabla \cdot (J(|\cdot|^{2\kappa})) = -\nabla \cdot (j(|\cdot|^{2\kappa})v) = 0$$

in $\mathcal{D}'(\mathbf{R}^3 \setminus \{0\})$ iff

$$j(|\cdot|^{2\kappa}) \circ S_\lambda = \lambda^{-3} j(|\cdot|^{2\kappa})$$

Hence $4 + 4\kappa = -3$, or, equivalently, $\kappa = -\frac{7}{4}$. □

Question Does $\mathcal{B}(|\cdot|^{2\kappa}) = 0$ on $\mathbf{R}^3$ and not only on $\mathbf{R}^3 \setminus \{0\}$ for $\kappa = -\frac{7}{4}$? (This remains ambiguous in Zakharov's works.)

For all open $\Omega \ni 0$ with C^1 boundary and outward normal field n

$$r_0[j(|\cdot|^{2\kappa})] = \int_{\partial\Omega} J(|\cdot|^{2\kappa}) \cdot ndS$$

if $J(|\cdot|^{2\kappa})$ is continuous on a neighborhood of $\partial\Omega$, or in the sense of currents in the general case.

Thm. 9' Under the assumptions of Thm 8, $J(|\cdot|^{2\kappa}) \in \mathcal{D}'(\mathbf{R}^3 \setminus \{0\})^3$ has the same flux $\mathfrak{F}$ through all closed surfaces surrounding 0 and

$$\mathcal{B}(|\cdot|^{2\kappa}) = -\mathfrak{F}\delta_0 \text{ in } \mathcal{D}'(\mathbf{R}^3) \qquad \text{for } \kappa = -\tfrac{7}{4}$$

Remark. Obviously, one has $\mathfrak{F} \neq 0$, unless $J(|\cdot|^{2\kappa}) = 0$.
Remark. That KZ profiles are solutions to the steady Boltzmann equation with source term (at 0 or ∞) already appears in [Escobedo-Velasquez, Mem. AMS 741]

Thm 10. If $\theta \in \mathbf{R}$ is such that $\Gamma(|\cdot|^{2\theta}) \in \mathcal{D}'(\mathbf{R}^3 \setminus \{0\})^3$, then

$$|v|^2\mathcal{B}(|\cdot|^{2\theta}) = -\nabla\cdot(\Gamma(|\cdot|^{2\theta})) = 0 \text{ in } \mathcal{D}'(\mathbf{R}^3 \setminus \{0\}) \iff \theta = -\tfrac{9}{4}$$

Proof: If $\theta \in \mathbf{R}$ is such that $\Gamma(|\cdot|^{2\theta}) \in \mathcal{D}'(\mathbf{R}^3 \setminus \{0\})^3$, we deduce from Lemma 2 (3) the existence of $\gamma(|\cdot|^{2\theta}) \in \mathcal{D}'(\mathbf{R}^3 \setminus \{0\})$, a radial, real-valued distribution such that

$$\Gamma(|\cdot|^{2\theta}) = \gamma(|\cdot|^{2\theta})v$$

By Lemma 9 (2), $\gamma(|\cdot|^{2\theta}) \circ S_\lambda = \lambda^{6+4\theta}\gamma(|\cdot|^{2\theta})$, while

$$|v|^2\mathcal{B}(|\cdot|^{2\theta}) = -\nabla\cdot(\gamma(|\cdot|^{2\theta})v) = 0 \text{ in } \mathcal{D}'(\mathbf{R}^3 \setminus \{0\})$$

implies that $\gamma(|\cdot|^{2\theta}) \circ S_\lambda = \lambda^{-3}\gamma(|\cdot|^{2\theta})$ by Thm 7 (2) (since Euler's identity characterizes homogeneous functions). Therefore

$$6 + 4\theta = -3 \iff \theta = -\tfrac{9}{4}$$

As in the case of the mass current $J(|\cdot|^{2\kappa})$ for $\kappa = -\frac{7}{4}$, the energy current $\Gamma(|\cdot|^{2\theta})$ with $\theta = -\frac{9}{4}$ has the same flux $\mathfrak{G}$ through all closed surfaces surrounding 0, and Thm 6 (3) implies that

$$|v|^2\mathcal{B}(|\cdot|^{2\theta})(v) = -\mathfrak{G}\delta_0 \quad \text{in } \mathcal{D}'(\mathbf{R}^3) \qquad \text{for } \theta = -\tfrac{9}{4}$$

where the energy flux is the residue at 0 of the distribution $\gamma(|\cdot|^{2\theta})$

$$\mathfrak{G} = r_0[\gamma(|\cdot|^{2\theta})]$$

Direct vs. inverse energy cascade

In the case of the KZ solution $|\cdot|^{2\theta}$ with $\theta = -\frac{9}{4}$, energy flows from low $|v|$'s to high $|v|$'s if $\mathfrak{G} > 0$, and from high $|v|$'s to low $|v|$'s provided that $\mathfrak{G} < 0$.

Summarizing, we have

(1) written $\mathcal{B}(f) = -\nabla \cdot J(f)$ and $|v|^2\mathcal{B}(f) = -\nabla \cdot \Gamma(f)(v)$
(2) proved that $J(f)(v) = j(f)(|v|)v$ and $\Gamma(f)(v) = \gamma(f)(|v|)v$ when f is a radial distribution function, and
(3) proved that

$$\left.\begin{array}{l} j(|\cdot|^{2\kappa}) \in \mathcal{D}'(\mathbf{R}^3\setminus\{0\}) \\ \mathcal{B}(|\cdot|^{2\kappa}) = 0 \text{ on } \mathbf{R}^3\setminus\{0\} \end{array}\right\} \iff \kappa = -\tfrac{7}{4}$$

$$\left.\begin{array}{l} \gamma(|\cdot|^{2\theta}) \in \mathcal{D}'(\mathbf{R}^3\setminus\{0\}) \\ \mathcal{B}(|\cdot|^{2\theta}) = 0 \text{ on } \mathbf{R}^3\setminus\{0\} \end{array}\right\} \iff \theta = -\tfrac{9}{4}$$

Question Are $j(|\cdot|^{2\kappa})$ and/or $\gamma(|\cdot|^{2\theta})$ in $L^1_{loc}(\mathbf{R}^3\setminus\{0\})$?

Remark. The scalar fields $j(|\cdot|^\alpha)$ and $\gamma(|\cdot|^\alpha)$ **cannot be both** in $L^1_{loc}(\mathbf{R}^3\setminus\{0\})$ for $\alpha = -\frac{7}{2}$ and for $\alpha = -\frac{9}{2}$ (by Thms 9 and 10)

Recall that, if $f(v) = F(\omega)$ with $\omega = |v|^2$, then

$$\mathcal{B}(f)(v) = 2(\pi r)^2 \int_{(0,+\infty)^3} \frac{\sqrt{\min(\omega,\omega_1,\omega_2,\omega_3)}}{\sqrt{\omega}} (F(\omega_1)F(\omega_2) - F(\omega)F(\omega_3)) \times \delta(\omega_1 + \omega_2 - \omega - \omega_3) d\omega_1 d\omega_2 d\omega_3$$

To study the absolute convergence of this integral for $f(v) = |v|^{-7/2}$,

$$\int_{(0,+\infty)^3} \sqrt{\min(\omega,\omega_1,\omega_2,\omega_3)} |\omega_1^{-7/4}\omega_2^{-7/4} - \omega^{-7/4}\omega_3^{-7/4}| \times \delta(\omega_1 + \omega_2 - \omega - \omega_3) d\omega_1 d\omega_2 d\omega_3$$

$$\geq \int_{\substack{\omega_1 < \omega \\ \omega+1 < \omega_2}} \sqrt{\omega_1}(\omega_1^{-7/4}\omega_2^{-7/4} - \omega^{-7/4}(\omega_2 - \omega)^{-7/4}) d\omega_1 d\omega_2$$

$$\geq \underbrace{\int_0^\omega \frac{d\omega_1}{\omega_1^{5/4}}}_{=+\infty} \underbrace{\int_{1+\omega}^\infty \frac{d\omega_2}{\omega_2^{7/4}}}_{=4/3(1+\omega)^{3/4}} - \omega^{-3/4} \underbrace{\int_{\omega+1}^\infty \frac{d\omega_2}{(\omega_2 - \omega)^{7/4}}}_{=4/3}$$

Same argument with $f(v) = |v|^{-9/2}$,

$$\int_{(0,+\infty)^3} \sqrt{\min(\omega,\omega_1,\omega_2,\omega_3)}|\omega_1^{-9/4}\omega_2^{-9/4} - \omega^{-9/4}\omega_3^{-9/4}|$$

$$\times\delta(\omega_1+\omega_2-\omega-\omega_3)d\omega_1 d\omega_2 d\omega_3$$

$$\geq \int_{\substack{\omega_1<\omega \\ \omega+1<\omega_2}} \sqrt{\omega_1}(\omega_1^{-9/4}\omega_2^{-9/4} - \omega^{-9/4}(\omega_2-\omega)^{-9/4})d\omega_1 d\omega_2$$

$$\geq \underbrace{\int_0^{\omega} \frac{d\omega_1}{\omega_1^{7/4}}}_{=+\infty} \underbrace{\int_{1+\omega}^{\infty} \frac{d\omega_2}{\omega_2^{9/4}}}_{=4/5(1+\omega)^{5/4}} - \omega^{-5/4} \underbrace{\int_{\omega+1}^{\infty} \frac{d\omega_2}{(\omega_2-\omega)^{9/4}}}_{=4/5}$$

One could study instead the existence of the currents $J(|\cdot|^{-7/2})$ or $\Gamma(|\cdot|^{-9/2})$. In the former case

$$J(|\cdot|^{-\frac{7}{2}})=\int\frac{dv_1}{|v_1|^{\frac{7}{2}}}\frac{(v-v_1)^{\otimes 2}}{|v-v_1|^4}\int_{(v-v_2)\cdot(v-v_1)<0}\frac{dv_2}{|v_2|^{\frac{7}{2}}}(v_1-v_2)$$

and the inner integral obviously diverges whenever $|v|^2<v\cdot v_1$, since one can let $v_2\to 0$ in the domain of integration.

Conclusion Neither of the KZ solutions for the Boltzmann equation with hard sphere collisions with the exponents $7/4$ or $9/4$ satisfies the locality assumption.

Neither the derivation of KZ inverse power law solutions based on the Zakharov transformation in the collision integral, nor the approach based on the Landau mass flux and the Euler identity are justified in this case.

Let us apply the methods above to the 4-wave collision integral

$$\mathcal{C}(f)(k) = \int_{\mathbf{R}^3\times\mathbf{S}^2} f(k)f(k_*)f(k')f(k'_*)\left(\tfrac{1}{f(k)} + \tfrac{1}{f(k_*)} - \tfrac{1}{f(k')} - \tfrac{1}{f(k'_*)}\right) \times((k-k_*)\cdot n)_+ dk_* dn$$

$$= \frac{1}{2}\int_{(\mathbf{R}^3)^3} f(k)f(k_*)f(k')f(k'_*)\left(\tfrac{1}{f(k)} + \tfrac{1}{f(k_*)} - \tfrac{1}{f(k')} - \tfrac{1}{f(k'_*)}\right) \times\delta(k'+k'_*-k-k_*)\delta(|k'|^2+|k'_*|^2-|k|^2|k_*|^2)dk_*dk'dk'_*$$

If $f>0$ is continuous and rapidly decaying at infinity

$$\int_{\mathbf{R}^3} \mathcal{C}(f)(k)\begin{pmatrix}1\\ k\\ |k|^2\end{pmatrix} dk = 0$$

Remark. This corresponds to cubic NLS; other homogeneities of $\times$-section for the MMT model

(1) The integrand of $\mathcal{C}(f)$ is 0 iff f is a "Rayleigh-Jeans" distribution

$$f(k) = \frac{1}{a + c|k-u|^2}, \qquad a, c > 0, \; u \in \mathbf{R}^3$$

(2) Landau mass current

$$\begin{aligned} J(f)(k) &= -\int_{\mathbf{R}^4 \times \mathbf{S}^2} (f(k+sn-(k-k_*)\cdot nn) + f(k_*+sn+(k-k_*)\cdot nn)) \\ &\qquad \times f(k+sn) f(k_*+sn) \mathbf{1}_{0<s<(k-k_*)\cdot n}(k-k_*)\cdot nn dk_* ds dn \\ &= \int_{(\mathbf{R}^3)^3} \mathfrak{A}(k-k_1, k_1-k_2, k_2-k_3) f(k_1) f(k_2) f(k_3) dk_1 dk_2 dk_3 \end{aligned}$$

where

$$\begin{cases} \mathfrak{A}(\xi,\eta,\zeta) = \mathcal{A}(\xi,\eta)\left(\delta(\eta+\zeta-\frac{\xi\cdot\eta}{|\xi|^2}\xi) + \delta(\zeta+\frac{\xi\cdot\eta}{|\xi|^2}\xi)\right) \\ \mathcal{A}(\xi,\eta) = -\mathbf{1}_{|\xi|^2+\xi\cdot\eta<0}|\xi|^{-4}(\xi\cdot\eta)\xi \end{cases}$$

The mass current J is translation- and $O_3(\mathbf{R})$-equivariant — because the Schwartz kernel of J is a function of $k-k_1, k_1-k_2, k_2-k_3$, and because $\mathfrak{A}(R\xi, R\eta, R\zeta) = R\mathfrak{A}(\xi,\eta,\zeta)$; besides $\mathfrak{A}$ is homogeneous of degree $-1-3=-4$ in $\mathbf{R}^9$. Therefore

$$J(f\circ S_\lambda) = \lambda^{-5} J(f)\circ S_\lambda$$

(3) Since $f\mapsto J(f)$ is cubic, if $J(|\cdot|^{2\kappa})$ is a distribution on $\mathbf{R}^3\setminus\{0\}$

$$J(|\cdot|^{2\kappa}\circ S_\lambda) = J(\lambda^{2\kappa}|\cdot|^{2\kappa}) = \lambda^{6\kappa}J(|\cdot|^{2\kappa}) = \lambda^{-5}J(|\cdot|^{2\kappa})\circ S_\lambda$$

so that $J(|\cdot|^{2\kappa}) = j(|\cdot|^{2\kappa})k$ with $j(|\cdot|^{2\kappa})\in\mathcal{D}'(\mathbf{R}^3\setminus\{0\})$ radial, and

$$j(|\cdot|^{2\kappa})\circ S_\lambda = \lambda^{4+6\kappa}j(|\cdot|^{2\kappa})$$

Then $|\cdot|^{2\kappa}$ is a KZ solution if in the sense of $\mathcal{D}'(\mathbf{R}^3\setminus\{0\})$

$$\mathcal{C}(|\cdot|^{2\kappa}) = -\nabla\cdot(j(|\cdot|^{2\kappa})k) = 0 \iff 4+6\kappa=-3 \iff \kappa=-\tfrac{7}{6}$$

(4) There is an energy current Γ such that $|\cdot|^2\mathcal{C} = \nabla\cdot\Gamma$, and, if $\Gamma(|\cdot|^{2\theta})$ is a distribution on $\mathbf{R}^3\setminus\{0\}$, exactly as for J

$$\Gamma(|\cdot|^{2\theta}) = \gamma(|\cdot|^{2\theta})k\,, \qquad \text{with } \gamma(|\cdot|^{2\theta}) \text{ radial}$$

Besides, $\gamma(|\cdot|^{2\theta})$ has the same degree of homogeneity as $|\cdot|^2 j(|\cdot|^{2\theta})$, i.e. $2+4+6\theta = 6+6\theta$. Therefore θ is a KZ exponent for the energy flux if and only if

$$|\cdot|^2\mathcal{C} = -\nabla\cdot(\gamma(|\cdot|^{2\theta})k) = 0 \text{ on } \mathbf{R}^3\setminus\{0\} \iff 6+6\theta = -3$$
$$\iff \theta = -\tfrac{3}{2}$$

(5) The collision integral $\mathcal{C}(f)$ is absolutely convergent

- for $f(k) = |k|^{-7/3}$ (KZ mass profile), but
- neither for $f(|k|) = |k|^{-3}$ (KZ energy profile)
- nor for $f(k) = (a + c|k-u|^2)^{-1}$ (RJ profile)

See [Collot-Dietert-Germain, ARMA2022]

Conclusion

• The Boltzmann collision integrals in the kinetic theory of gases and in WT can be represented in terms of Landau mass and energy currents, even for anisotropic distribution functions

• The uniqueness of these fluxes has been established under some appropriate gauge conditions

• The KZ exponents are identified by using the Euler identity for homogeneous distributions together with the $O_3(\mathbf{R})$-equivariance and scale invariance of the Landau currents, instead of using Zakharov's transform or Balk's argument [PhysicaD2000]

(Connaughton, Nazarenko and Newell proposed in [PhysicaD2003] a dimensional argument to find KZ exponents, based on *postulating* the existence of mass and energy fluxes in space dimension 1 on the basis of the local conservation laws for these quantities — closer in spirit to K41 theory)

- Unlike in earlier works (from Zakharov's school), KZ spectra satisfy

$$\text{Collision integral} = \rho\delta_0$$

Consistent with the picture of K41 theory with a low frequency source and a high frequency sink — in an asymptotic regime where the wavelength range of the source is shrunk to a point.