# Flow of turbulent-like Rough vector fields

*... with a (tentative) conjecture*

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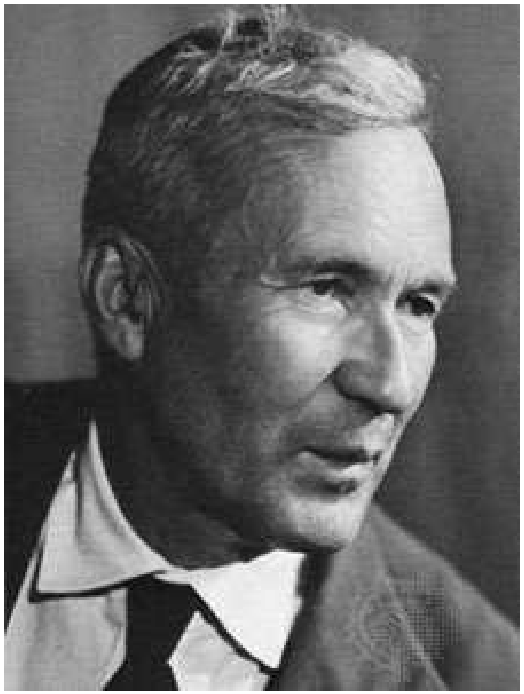
# The Navier-Stokes equations

In three-dimensional space, consider the velocity field $\boldsymbol{u}^\nu(\boldsymbol{x},t)$, where $\boldsymbol{u}^\nu = (u_1^\nu, u_2^\nu, u_3^\nu)$, $\boldsymbol{x} \in \mathbb{R}^3$ and say $t > 0$. Given a (large-scale, divergence-free forcing) $\boldsymbol{f}$, it is solution of

$$\frac{\partial \boldsymbol{u}^\nu}{\partial t} + (\boldsymbol{u}^\nu \cdot \boldsymbol{\nabla})\boldsymbol{u}^\nu = -\frac{1}{\rho}\boldsymbol{\nabla} p^\nu + \nu \Delta \boldsymbol{u}^\nu + \boldsymbol{f} \text{ and } \boldsymbol{\nabla} \cdot \boldsymbol{u}^\nu = 0,$$

where $p^\nu$ is the pressure field, and $\nu$ the kinematic viscosity.



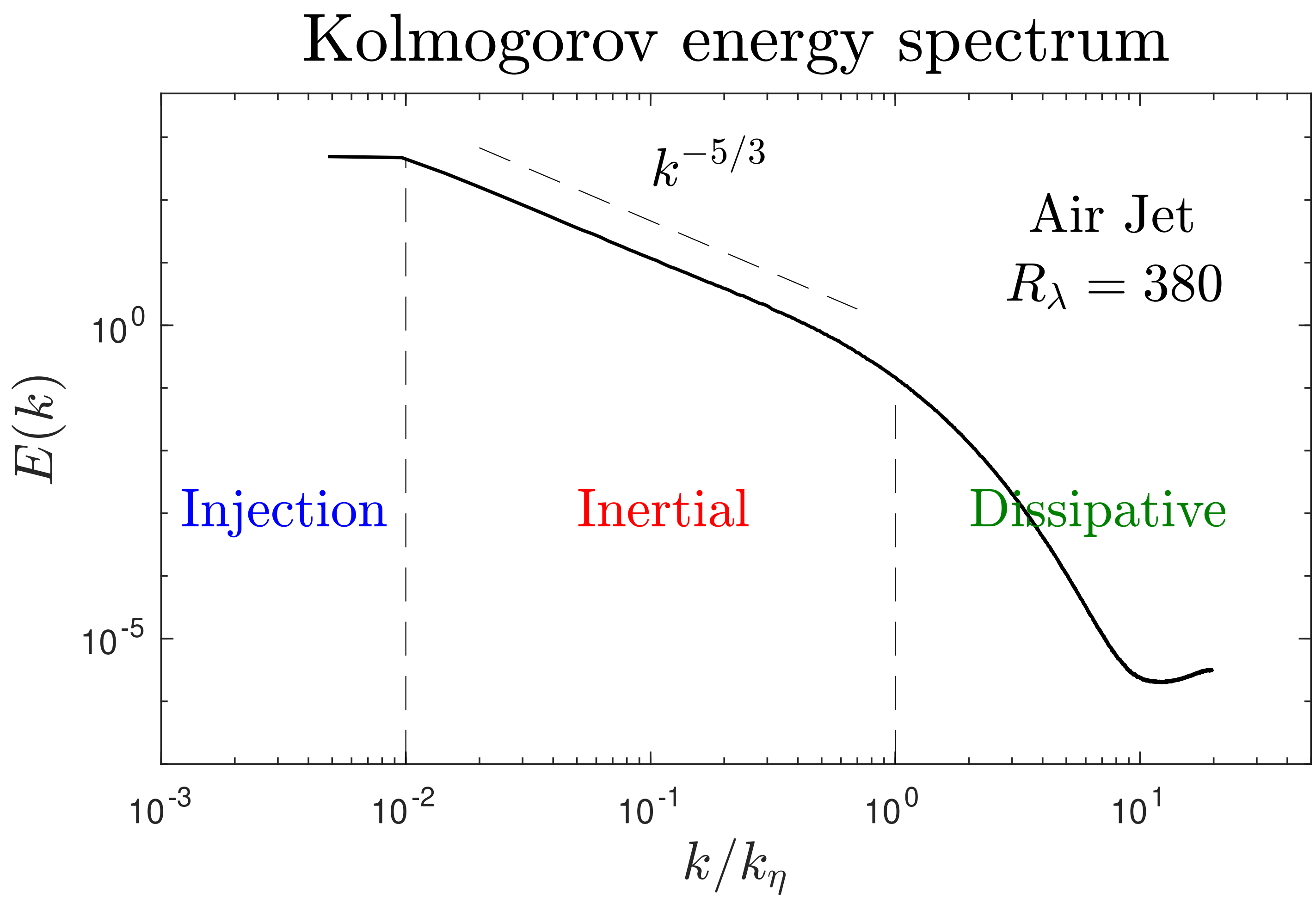
Kolmogorov 1903-1987

"I became interested in turbulent liquid and gas flows at the end of the thirties. From the very beginning it was clear that the theory of random functions of many variables (random fields), whose development only started at that time, must be the underlying mathematical technique. Moreover, I soon understood that there was little hope of developing a pure, closed theory, and because of the absence of such a theory the investigation must be based on hypotheses obtained by processing experimental data."

# *Two-point statistical structure of turbulence*

Define the energy spectrum (Fourier transform of the correlation) as

$$E^\nu(k) = \int e^{-2i\pi k\ell}\langle u^\nu(x)u^\nu(x+\ell)\rangle\, d\ell$$

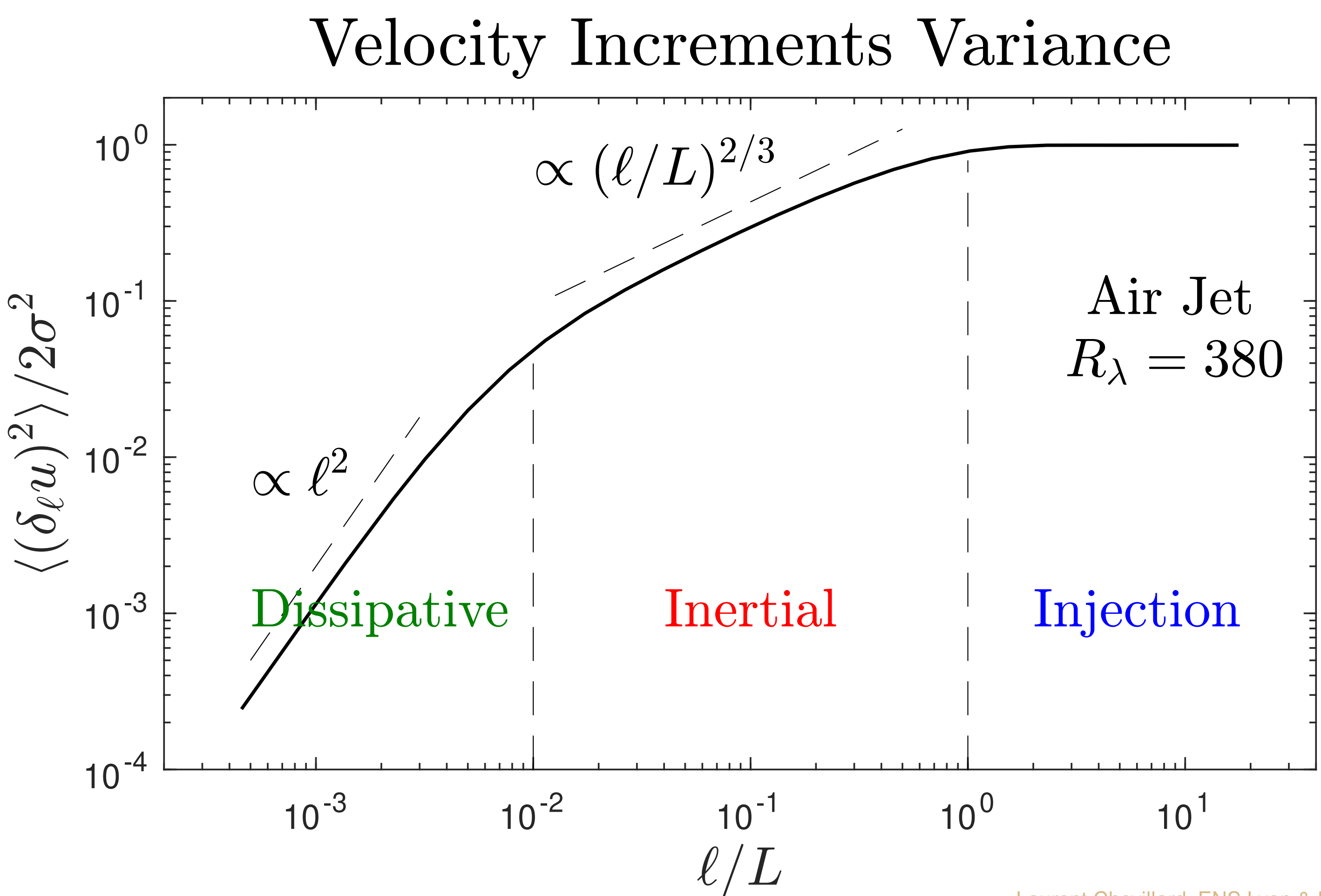


# *Two-point statistical structure of turbulence*

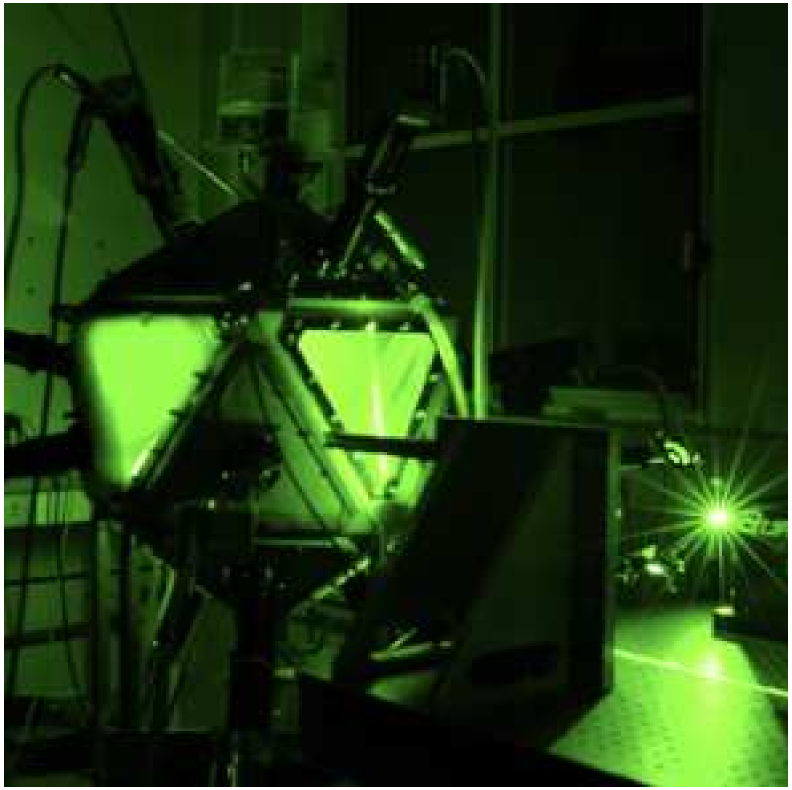
In an equivalent way, define the velocity increment as

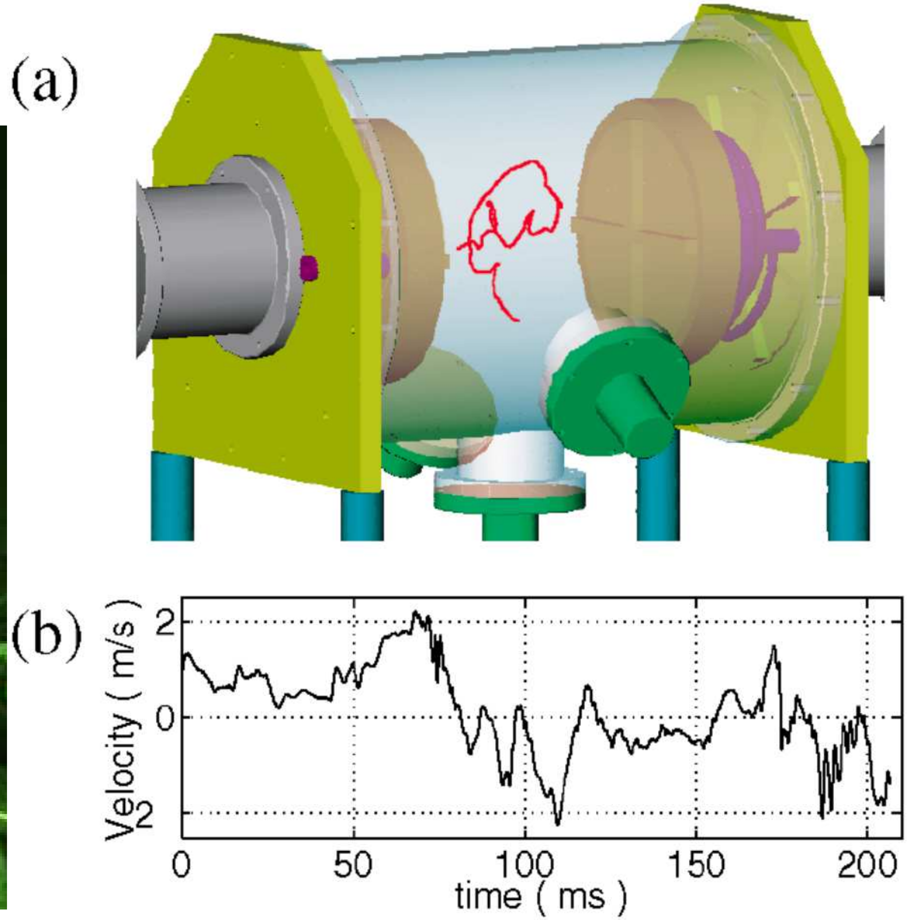
$$\delta_\ell u^\nu(x) = u^\nu(x+\ell) - u^\nu(x),$$

and remark that $\langle(\delta_\ell u^\nu)^2\rangle = 2\sigma^2 - 2\langle u^\nu(x)u^\nu(x+\ell)\rangle$.



# *The Lagrangian picture*



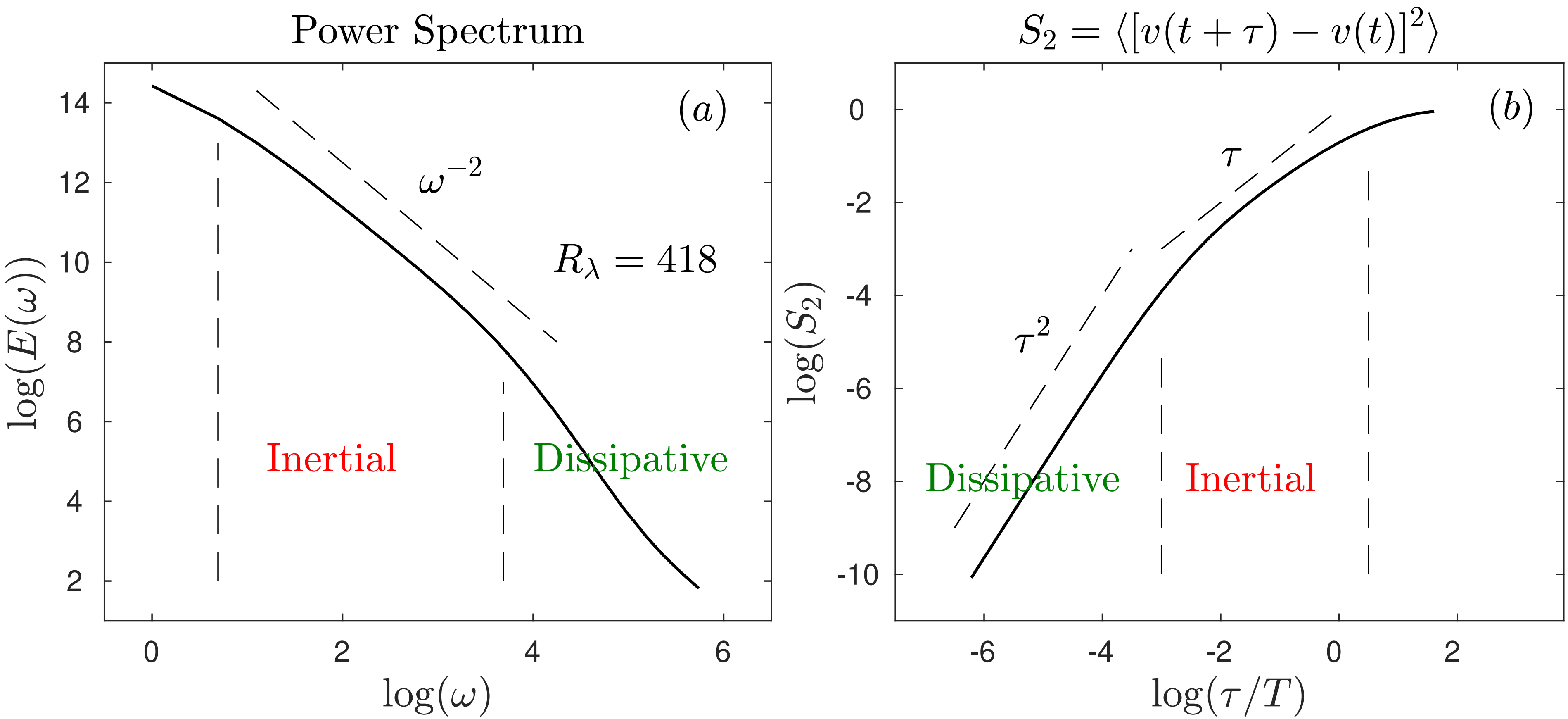



Yeung (97), Mordant et al. (02), Mordant et al. (04), Bourgoin-Volk

Flow equations $v^{\nu}(t) \equiv \frac{dX^{\nu}(t)}{dt} = u^{\nu}(X^{\nu}(t), t)$

# *The Lagrangian picture: Multiscale Analysis*

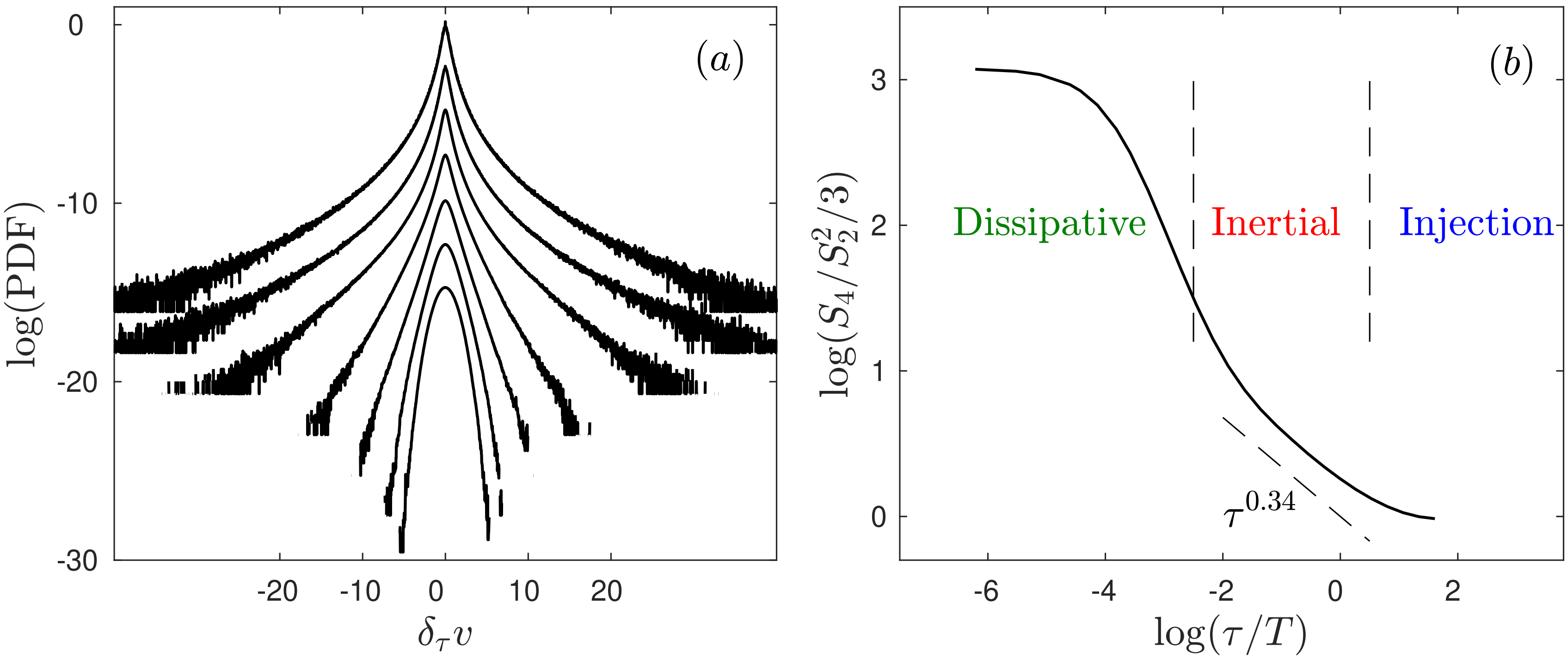
Numerical data from the Hopkins Database: $\mathcal{R}_\lambda = 418$



# *The Lagrangian picture: Multiscale Analysis*

Numerical data from the Hopkins Database: $\mathcal{R}_\lambda = 418$

- The velocity increment: $\delta_\tau v^\nu(t) = v^\nu(t+\tau) - v^\nu(t)$
- We have seen that $\langle(\delta_\tau v^\nu)^2\rangle \propto \tau$ in the inertial range
- We have seen that $\langle(\delta_\tau v^\nu)^2\rangle \approx \tau^2\langle a^2\rangle$ in the dissipative range
- What about high-order statistics? such as Probability density functions (PDF) and Flatness?



# *Asymptotics of phenomenology of fluid turbulence*

Consider (as observed) a homogeneous, isotropic **stationary** solution of the (forced over $L$) Navier and Stokes equations: call it $\mathbf{u}^\nu(x,t)$, with $x \in \mathbb{R}^3$.

- Velocity variance $\sigma^2$ is **finite** and **independent** on viscosity $\nu$, i.e.

$$\overbrace{\lim_{\nu\to 0}\lim_{t\to\infty}\mathbb{E}(|\mathbf{u}^\nu|(t)^2)}^{\text{Eulerian}} = \overbrace{\lim_{\nu\to 0}\lim_{t\to\infty}\mathbb{E}(|\mathbf{v}^\nu(t)|^2)}^{\text{Lagrangian}} = \sigma^2 < +\infty$$

To do so, the fluid develops **Roughness**

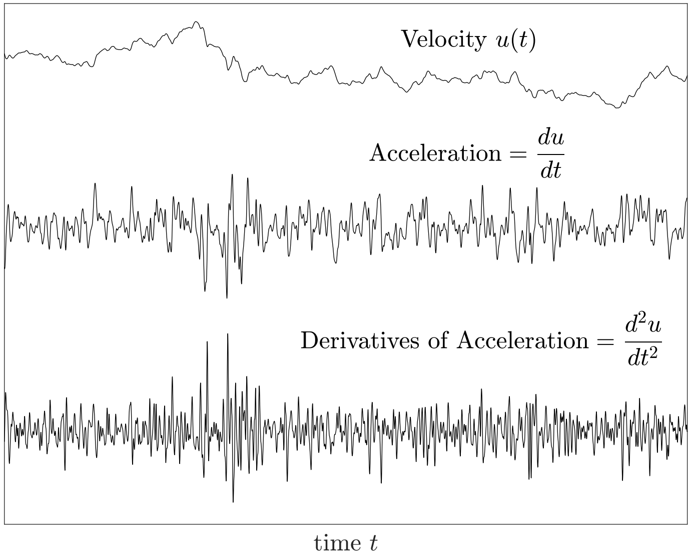
- In the **Eulerian** description, velocity develops $H = 1/3$ Hölder continuity, i.e.

$$\lim_{\nu\to 0}\lim_{t\to\infty}\mathbb{E}\left[|\mathbf{u}^\nu(x+\ell,t) - \mathbf{u}^\nu(x,t)|^2\right] \underset{\tau\to 0}{\propto} \ell^{2/3}.$$

- In the **Lagrangian** description, velocity develops $H = 1/2$ Hölder continuity, i.e.

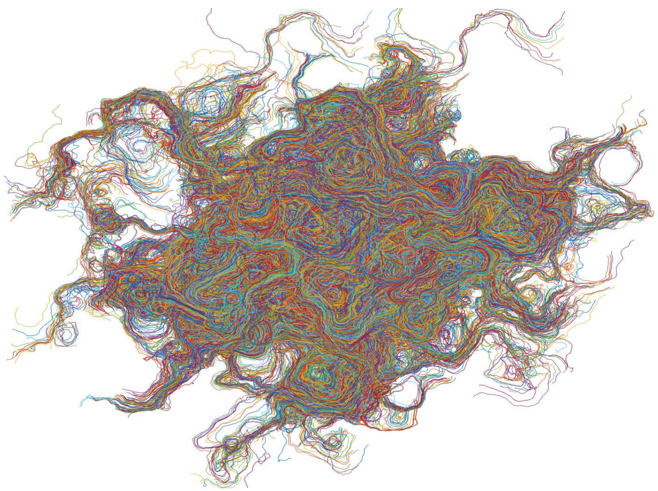
$$\lim_{\nu\to 0}\lim_{t\to\infty}\mathbb{E}\left[|\mathbf{v}^\nu(t+\tau) - \mathbf{v}^\nu(t)|^2\right] \underset{\tau\to 0}{\propto} \tau.$$

# *Contributions on the stochastic modeling of Lagrangian turbulence*




Can we build up an infinitely differentiable and causal random process to mimic fluctuations of Lagrangian velocity at a finite Reynolds number?
→ B. Viggiano, J. Friedrich, R. Volk, M. Bourgoin, RB Cal, L. Chevillard (2020).

What are the minimal ingredients to include in a spatio-temporal random advecting Eulerian field such that induced Lagrangian velocities are realistic of experimental observations?
→ J. Reneuve, L. Chevillard (2020).



# *The Era of Random Fields*

$\rightarrow$ **Fractional Gaussian Fields** are a remarkable (*admirable*) stochastic representation of an asymptotic Eulerian velocity field $u(\mathbf{x})$:

$$u(\mathbf{x}) = \int_{\mathbf{k}\in\mathbb{R}^d} e^{2i\pi \mathbf{k}\cdot\mathbf{x}} \frac{1}{|\mathbf{k}|_L^{H+\frac{d}{2}}} \widehat{W}(d^d k)$$

- $\widehat{W}$ being the Fourier transform of a Gaussian white noise
- $|\mathbf{k}|_L^2 \equiv |\mathbf{k}|^2 + 1/L^2$, a regularized norm over the integral length scale $L$

It is, for $0 < H < 1$,

- a **finite** variance scalar field
- a **statistically homogeneous** random representation of a Hölder continuous function, in the sense that

$$\mathbb{E}(\delta_\ell u)^2 \underset{|\ell|\rightarrow 0}{\propto} |\ell|^{2H}$$

- Easy extension to divergence-free (in a distributional sense) vector field
- Easy numerical representation on the torus $x \in \mathbb{T}^d$

So what about its **flow**?

# *Flow of **Rough** vector fields*

Flow of a vector field $\mathbf{u}(\mathbf{x}, t)$ corresponds to studying the map

$$\mathbf{v} \equiv \dot{\mathbf{X}}(t) = \mathbf{u}(\mathbf{X}(t), t)$$

with initial datum $\mathbf{X}(0) = x$.

- Cauchy-Lipschitz theorem provides global solutions, including existence and unicity.
- DiPerna and Lions provide an extension to vector fields having bounded divergence and some Sobolev type regularity. Further extensions by De Lellis et al. (beyond my understanding).
- No hope to give a meaning to the flow when $\mathbf{u}$ is Hölder continuous!
  See for instance the (deterministic) textbook example $u(x) = |x|^{1/3}$, non unicity of the solution when starting at the origin.

a **Regularization** procedure is needed

When there is no flow, first possible regularization is introducing a white noise in the trajectory, i.e.

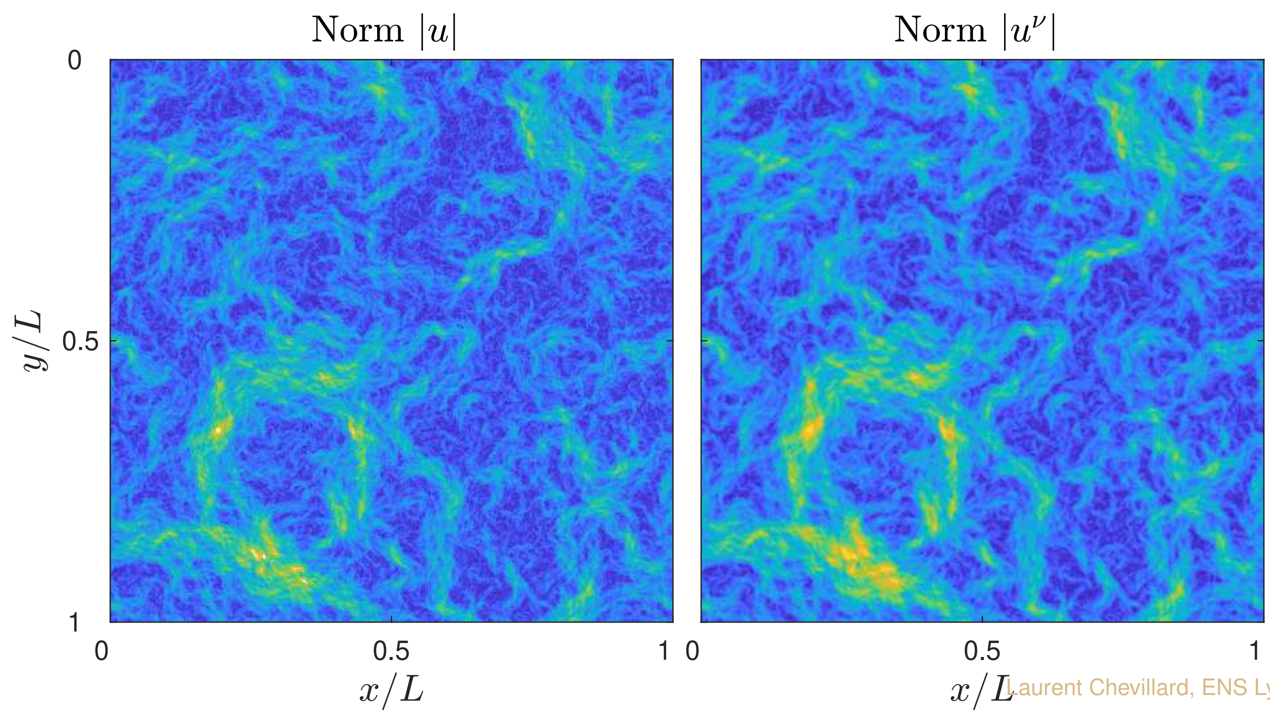
$$dX = u(t, X)dt + dW(t)$$

# *Regularized (by viscosity) Fractional Gaussian Fields*

$$u^{\nu}(\mathbf{x}) = \int_{\mathbf{k}\in\mathbb{R}^d} e^{2i\pi \mathbf{k}\cdot\mathbf{x}} \frac{1}{|\mathbf{k}|_L^{H+\frac{d}{2}}} e^{-|k|\eta_K} \widehat{W}(d^d k)$$

- with $\eta_K(\nu)$ known as the Kolmogorov **dissipative** length scale, and goes to 0 as $\nu \to 0$.
- The random field $u$ is now Lipschitz (actually $\mathcal{C}^{\infty}$)
- Hölder continuity is now obtained in the asymptotic regime, i.e.

$$\lim_{\nu\to 0} \mathbb{E}(\delta_\ell u^{\nu})^2 \underset{|\ell|\to 0}{\propto} |\ell|^{2H}$$



# *Incompressible Spatio-Temporal Fractional Gaussian Field*

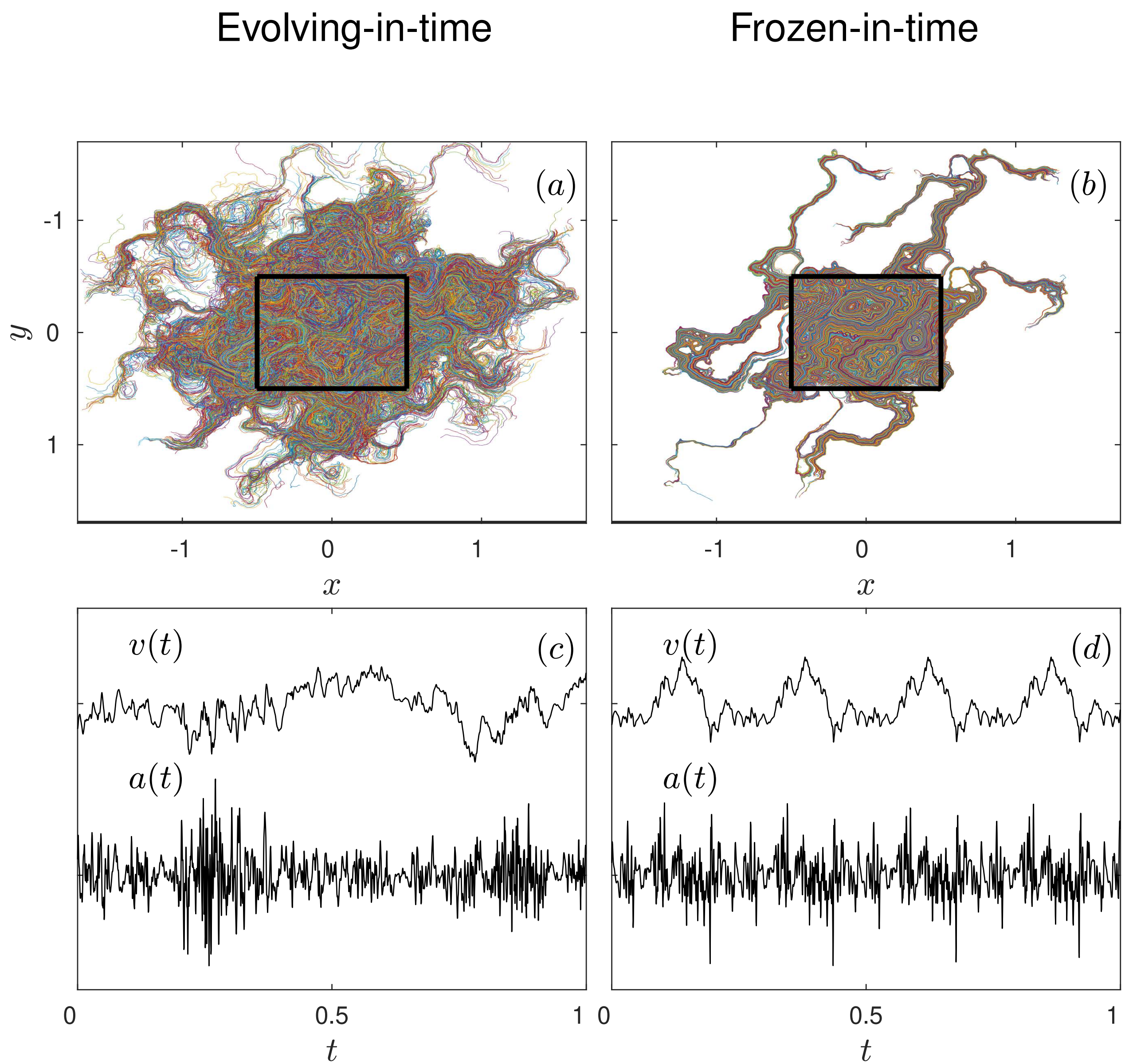
$\rightarrow$ Consider then an incompressible, statistically homogeneous, isotropic and stationary velocity field with proper regularity $H$ in both space and time,

$$\boldsymbol{u}(\boldsymbol{x},t) = \int_{\boldsymbol{y}\in\mathbb{R}^2, s\in\mathbb{R}} \boldsymbol{\mathcal{G}}_H(\boldsymbol{x}-\boldsymbol{y}, t-s) W(d^2y, ds)$$

$$\boldsymbol{\mathcal{G}}_H(\boldsymbol{x},t) = \varphi(\boldsymbol{x},t)\frac{\boldsymbol{x}^\perp}{|\boldsymbol{x}|}||\boldsymbol{x},t||^{H-3/2}$$

- A functional form inspired by the Biot-Savart law.
- $||\boldsymbol{x},t||^2 = |\boldsymbol{x}|^2 + \sigma^2 t^2$ a spatio-temporal norm.
- $\varphi$ a spatio-temporal cut-off function over large (integral) $L$ and $T$ scales.
- $H$ the Holderian regularity, $H \approx 1/3$ for turbulence.
- Keep in mind that this has to be regularized over a small scale $\eta_K$ to ensure differentiability.
- then do funky movies.
- See also alternative (Markovian) propositions by Chaves-Gawędzki-Horvai-Kupiainen-Vergassola (2003).

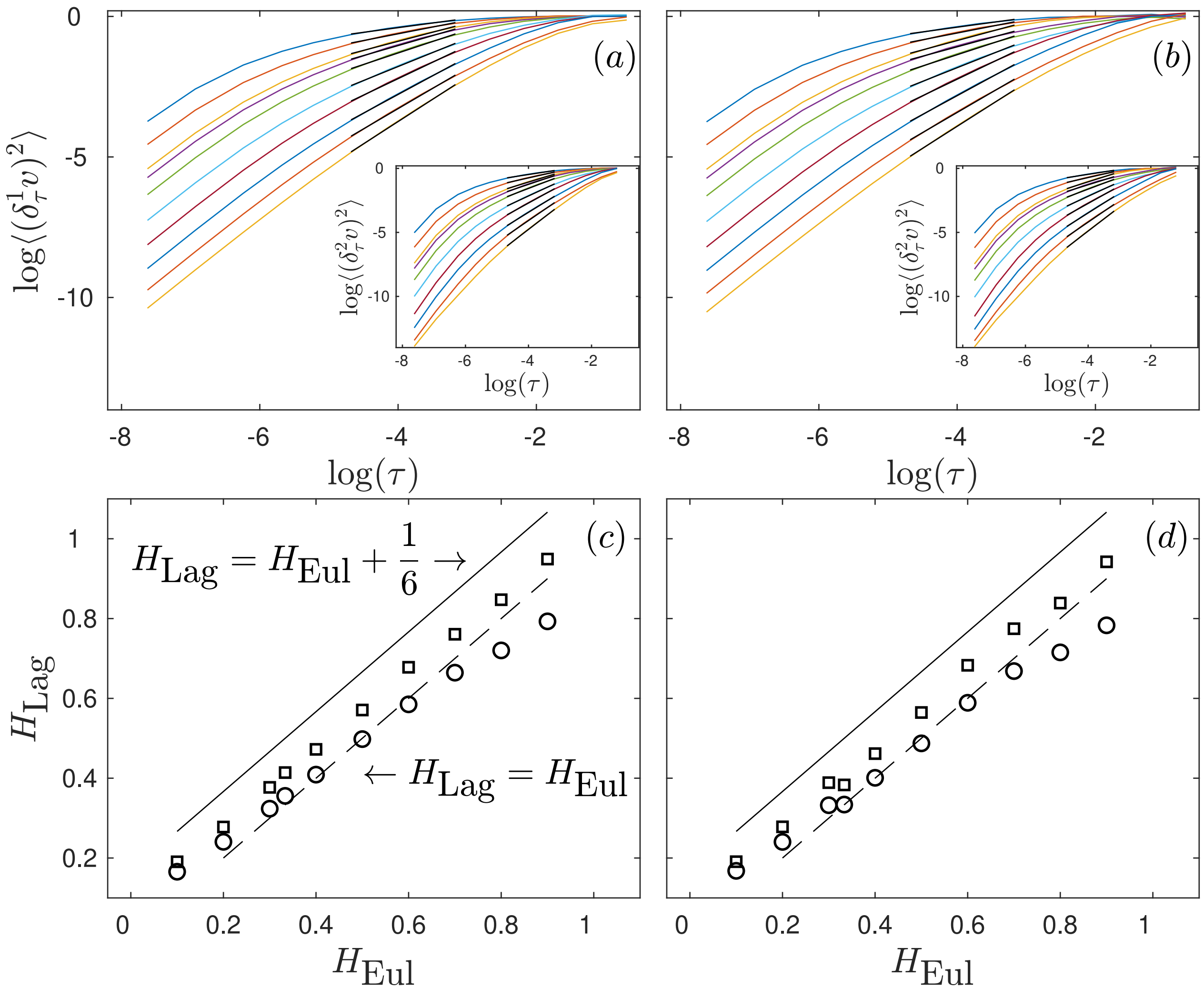
# *Solving the flow equations*



# *and measure the regularity of Lagrangian velocity*

Evolving-in-time

Frozen-in-time



# *Conclusions*

- It remains to understand why and how $\frac{1}{3}$-Eulerian regularity makes a $\frac{1}{2}$-Lagrangian regularity.
- Note also non-Gaussian corrections on $v$ while $u$ is Gaussian.
- See J. Reneuve et al. PRL (2020)
- Go to three-dimensional modeling of the advecting field, plus 1d in time!!

A good conjecture for today (well-posed and difficult!):

Take $u^\nu$ a regularized fractional field as we presented, even time-independent. Consider $0 < H < 1$. Then the regularity of the Eulerian (advective) field sets the regularity in the Lagrangian counterpart, i.e.

$$\lim_{\nu\to 0} \mathbb{E}(\delta_\ell u^\nu)^2 \underset{|\ell|\to 0}{\propto} |\ell|^{2H} \Rightarrow \lim_{\nu\to 0}\lim_{t\to\infty} \mathbb{E}(\delta_\tau v^\nu)^2 \underset{|\tau|\to 0}{\propto} |\tau|^{2H}$$

## *The ensemble of correlated velocities of Gawędzki et al.*

A (Markovian) proposition by Chaves-Gawedzki-Horvai-Kupiainen-Vergassola (2003)

The authors propose a covariance structure for the spatio-temporal velocity field $u_i(t,x)$, for $x \in \mathbb{R}^d$ and $i$ and integer such that $1 \leq i \leq d$. It reads

$$\mathbb{E}\left[u_i(t,x)u_j(t',x')\right] = D_2 \int_{k\in\mathbb{R}^d} e^{2i\pi k\cdot(x-x')} \frac{e^{-|t-t'|D_3(2\pi|k|_L)^{2\beta}}}{(2\pi|k|_L)^{d+2H}} \hat{P}_{ij}(k)dk,$$

where $P_{ij}$ is the projector (Leray) along divergence-free vector fields.
It is a very smart proposition for a **Markovian** incompressible velocity field, that has unique formulation using a Langevin evolution when is assumed a Gaussian framework. Namely

$$d\hat{v}_i(t,k) = -D_3(2\pi|k|_L)^{2\beta}\hat{v}_i(t,k)dt + \sqrt{2D_2D_3}(2\pi|k|_L)^{\beta-d/2-H}d\hat{W}_i(t,k),$$

and then project $\hat{\mathbf{u}}(t,k) = \hat{\mathbf{P}}(k)\hat{\mathbf{v}}(t,k)$.

# *Intermittency in Eulerian fluctuations*

Eulerian longitudinal velocity increments: $\delta_\ell u(x) = u(x+\ell) - u(x)$

Flatness $F = \frac{\langle (\delta_\ell u)^4 \rangle}{\langle (\delta_\ell u)^2 \rangle^2}$