Scattering, random phase and wave turbulence

Turbulent · e · s

Antoine Mouzard DMA

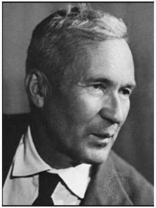
Joint work with Erwan Faou

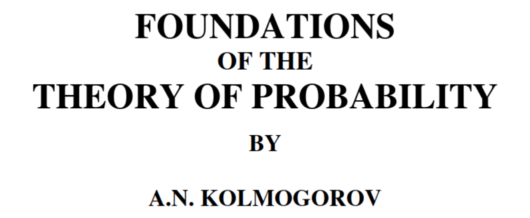
Contents

Nonlinear waves and turbulence

Kolmogorov 1903 – 1987

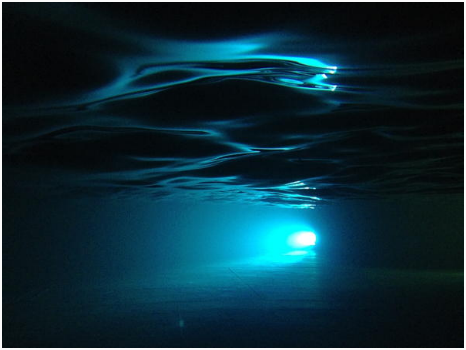
"I became interested in turbulent liquid and gas flows at the end of the thirties. From the very beginning it was clear that the theory of random functions of many variables (random fields), whose development only started at that time, must be the underlying mathematical technique. Moreover, I soon understood that there was little hope of developing a pure, closed theory, and because of the absence of such a theory the investigation must be based on hypotheses obtaining in processing experimental data."





FOUNDATIONS
OF THE
THEORY OF PROBABILITY
BY
A.N. KOLMOGOROV

Wave Turbulence (N. Mordant)



Botlzmann's statistical description for waves

Boltzmann's kinetic theory : interaction of $N \gg 1$ particles of size $\varepsilon \ll 1$ in the scaling $N\varepsilon^{d-1} \sim 1$

Botlzmann's statistical description for waves

Boltzmann's kinetic theory : interaction of $N \gg 1$ particles of size $\varepsilon \ll 1$ in the scaling $N\varepsilon^{d-1} \sim 1$

Wave turbulence : interaction of $N \gg 1$ waves with interaction strength $\varepsilon \ll 1$ in a scaling $\varepsilon L^{\alpha} \sim 1$

$$\partial_t n = \mathcal{T}(n) \tag{WKE}$$

Botlzmann's statistical description for waves

Boltzmann's kinetic theory : interaction of $N \gg 1$ particles of size $\varepsilon \ll 1$ in the scaling $N\varepsilon^{d-1} \sim 1$

Wave turbulence : interaction of $N \gg 1$ waves with interaction strength $\varepsilon \ll 1$ in a scaling $\varepsilon L^{\alpha} \sim 1$

$$\partial_t n = \mathcal{T}(n) \qquad \text{(WKE)}$$

$\rightarrow$ Which interactions ?

$\rightarrow$ How to quantify the number of wave ?

$\rightarrow$ Randomness and statistical description ?

$\rightarrow$ Solution to the kinetic equation ?

Physical ideas

Theory : turbulent cascade, KZ spectrum, large deviations ...

Experiments : shape of the boxes, dispersive relation, forcing ...

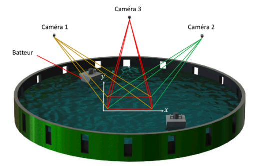
Numerics : simulations for understanding and predictions

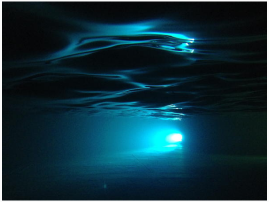
Physical ideas

Theory : turbulent cascade, KZ spectrum, large deviations ...

Experiments : shape of the boxes, dispersive relation, forcing ...

Numerics : simulations for understanding and predictions





Mathematical ideas for cubic NLS

- Growth of H^s norm and turbulent cascade

$$\|u\|^2_{L^2(\mathbb{T}^d)} = \sum_{K\in\mathbb{Z}} |u_K|^2 \quad \text{and} \quad \|u\|^2_{H^s(\mathbb{T}^d)} = \sum_{K\in\mathbb{Z}} \langle K\rangle^s |u_K|^2$$

Mathematical ideas for cubic NLS

- Growth of H^s norm and turbulent cascade

$$\|u\|_{L^2(\mathbb{T}^d)}^2 = \sum_{K\in\mathbb{Z}} |u_K|^2 \quad \text{and} \quad \|u\|_{H^s(\mathbb{T}^d)}^2 = \sum_{K\in\mathbb{Z}} \langle K\rangle^s |u_K|^2$$

- Wave Kinetic Equation for $\eta_K := \mathbb{E}[|u_K|^2]$ with $K \in \mathbb{Z}_L^d := \frac{1}{L}\mathbb{Z}^d$

$$i\partial_t \eta_k = \int_{\substack{k=k_1-k_2+k_3 \\ |k|^2=|k_1|^2-|k_2|^2+|k_3|^2}} n_k n_{k_1} n_{k_2} n_{k_3} \Big(\frac{1}{n_k} - \frac{1}{n_{k_1}} + \frac{1}{n_{k_2}} - \frac{1}{n_{k_3}}\Big)$$

in the continuous limit

Mathematical ideas for cubic NLS

- Growth of H^s norm and turbulent cascade

$$\|u\|^2_{L^2(\mathbb{T}^d)} = \sum_{K\in\mathbb{Z}} |u_K|^2 \quad \text{and} \quad \|u\|^2_{H^s(\mathbb{T}^d)} = \sum_{K\in\mathbb{Z}} \langle K\rangle^s |u_K|^2$$

- Wave Kinetic Equation for $\eta_K := \mathbb{E}[|u_K|^2]$ with $K \in \mathbb{Z}^d_L := \frac{1}{L}\mathbb{Z}^d$

$$i\partial_t \eta_k = \int_{\substack{k=k_1-k_2+k_3 \\ |k|^2=|k_1|^2-|k_2|^2+|k_3|^2}} n_k n_{k_1} n_{k_2} n_{k_3} \Big(\frac{1}{n_k} - \frac{1}{n_{k_1}} + \frac{1}{n_{k_2}} - \frac{1}{n_{k_3}} \Big)$$

in the continuous limit

- Continuous Resonant equation by Faou, Germain and Hani

$$i\partial_t v = \lambda \sum_{\substack{K=K_1-K_2+K_3 \\ |K|^2=|K_1|^2-|K_2|^2+|K_3|^2}} v_{K_1}\overline{v_{K_2}} v_{K_3}$$

indexed by $K_1, K_2, K_3 \in \mathbb{Z}^d_L$

Random initial data

Statistical description for random initial data

- Random Phase (RP) :

$$\varphi(x) = \sum_{K \in \mathbb{Z}^d} \eta_K e^{i\theta_K} e^{iK\cdot x}$$

with $(\theta_K)_{K\in\mathbb{Z}^d}$ i.i.d. uniform random variables on $[0, 2\pi]$

- Random Phase and Amplitudes (RPA) :

$$\varphi(x) = \sum_{K \in \mathbb{Z}^d} \eta_K \xi_K e^{iK\cdot x}$$

with $(\xi_K)_{K\in\mathbb{Z}^d}$ i.i.d. Gaussian complex random variables

Random initial data

Statistical description for random initial data

- Random Phase (RP) :

$$\varphi(x) = \sum_{K \in \mathbb{Z}^d} \eta_K e^{i\theta_K} e^{iK\cdot x}$$

with $(\theta_K)_{K\in\mathbb{Z}^d}$ i.i.d. uniform random variables on $[0, 2\pi]$

- Random Phase and Amplitudes (RPA) :

$$\varphi(x) = \sum_{K \in \mathbb{Z}^d} \eta_K \xi_K e^{iK\cdot x}$$

with $(\xi_K)_{K\in\mathbb{Z}^d}$ i.i.d. Gaussian complex random variables

$\rightarrow$ Propagation of chaos : correlations at $t > 0$?

Random initial data

Statistical description for random initial data

- Random Phase (RP) :

$$\varphi(x) = \sum_{K \in \mathbb{Z}^d} \eta_K e^{i\theta_K} e^{iK\cdot x}$$

with $(\theta_K)_{K\in\mathbb{Z}^d}$ i.i.d. uniform random variables on $[0, 2\pi]$

- Random Phase and Amplitudes (RPA) :

$$\varphi(x) = \sum_{K \in \mathbb{Z}^d} \eta_K \xi_K e^{iK\cdot x}$$

with $(\xi_K)_{K\in\mathbb{Z}^d}$ i.i.d. Gaussian complex random variables

$\rightarrow$ Propagation of chaos : correlations at $t > 0$?

$\neq$ dynamic at the equilibrium given by invariant Gibbs measure

Wave Kinetic Equation and Deng/Hani

Solve the equation

$$i\partial_t u = \Delta u + \lambda |u|^2 u \qquad \text{(NLS)}$$

on the large torus $\mathbb{T}_L^d$ with $L \gg 1$ and $\lambda \ll 1$ for $d \geq 3$

$\rightarrow$ random initial data of the form

$$u_L(x) = L^{-\frac{d}{2}} \sum_{K \in \mathbb{Z}_L^2} \eta(K) \xi_K e^{iK \cdot x}$$

with $(\xi_K)_K$ i.i.d. Gaussian complex random variables

Wave Kinetic Equation and Deng/Hani

Solve the equation

$$i\partial_t u = \Delta u + \lambda |u|^2 u \qquad \text{(NLS)}$$

on the large torus $\mathbb{T}^d_L$ with $L \gg 1$ and $\lambda \ll 1$ for $d \geq 3$

$\rightarrow$ random initial data of the form

$$u_L(x) = L^{-\frac{d}{2}} \sum_{K \in \mathbb{Z}^2_L} \eta(K) \xi_K e^{iK\cdot x}$$

with $(\xi_K)_K$ i.i.d. Gaussian complex random variables

Wave Kinetic Equation for $\eta_K := \mathbb{E}[|u_K|^2]$ with $K \in \mathbb{Z}^d_L$

$$i\partial_t \eta_k = \int_{\substack{k=k_1-k_2+k_3 \\ |k|^2=|k_1|^2-|k_2|^2+|k_3|^2}} n_k n_{k_1} n_{k_2} n_{k_3} \Big(\frac{1}{n_k} - \frac{1}{n_{k_1}} + \frac{1}{n_{k_2}} - \frac{1}{n_{k_3}} \Big)$$

Continuous Resoning equation by Faou, Germain and Hani

Solve the equation

$$i\partial_t u = \Delta u + \lambda |u|^2 u \tag{NLS}$$

on the large torus $\mathbb{T}_L^2$ with $L \gg 1$ and $\lambda \ll 1$

$v(t) = e^{-it\Delta} u(t)$ gives in frequency

$$i\partial_t v_K = \lambda \sum_{K = K_1 - K_2 + K_3} e^{i\omega t} v_{K_1} \overline{v_{K_2}} v_{K_3}$$

with $K_1, K_2, K_3 \in \mathbb{Z}_L^2$ and $\omega = |K|^2 + |K_2|^2 - |K_1|^2 - |K_3|^2$

Continuous Resoning equation by Faou, Germain and Hani

Solve the equation

$$i\partial_t u = \Delta u + \lambda |u|^2 u \qquad \text{(NLS)}$$

on the large torus $\mathbb{T}_L^2$ with $L \gg 1$ and $\lambda \ll 1$

$v(t) = e^{-it\Delta} u(t)$ gives in frequency

$$i\partial_t v_K = \lambda \sum_{K = K_1 - K_2 + K_3} e^{i\omega t} v_{K_1} \overline{v_{K_2}} v_{K_3}$$

with $K_1, K_2, K_3 \in \mathbb{Z}_L^2$ and $\omega = |K|^2 + |K_2|^2 - |K_1|^2 - |K_3|^2$

$\rightarrow$ In the limit $L \gg 1$ and $\lambda \ll 1$, effective resonant system

$$i\partial_t v_K = \lambda \sum_{\substack{K = K_1 - K_2 + K_3 \\ |K|^2 = |K_1|^2 - |K_2|^2 + |K_3|^2}} v_{K_1} \overline{v_{K_2}} v_{K_3}$$

using normal form and number theory with $\lambda L \ll 1$

CR as the effect of exact resonances [FGH]

The trilinear discrete sum

$$\mathcal{T}_K^L(f,g,h) = \sum_{\substack{K=K_1-K_2+K_3 \\ |K|^2=|K_1|^2-|K_2|^2+|K_3|^2}} f_{K_1} g_{K_2} h_{K_3}$$

is of order

$$\frac{2}{\zeta(2)} L^2 \log(L) \int_{\substack{K=k_1-k_2+k_3 \\ |K|^2=|k_1|^2-|k_2|^2+|k_3|^2}} f_{k_1} g_{k_2} h_{k_3}$$

CR as the effect of exact resonances [FGH]

The trilinear discrete sum

$$\mathcal{T}_K^L(f,g,h) = \sum_{\substack{K=K_1-K_2+K_3 \\ |K|^2=|K_1|^2-|K_2|^2+|K_3|^2}} f_{K_1} g_{K_2} h_{K_3}$$

is of order

$$\frac{2}{\zeta(2)} L^2 \log(L) \int_{\substack{K=k_1-k_2+k_3 \\ |K|^2=|k_1|^2-|k_2|^2+|k_3|^2}} f_{k_1} g_{k_2} h_{k_3}$$

The long time behavior of NLS in the limit $\lambda \ll 1$ and $L \gg 1$ with a scaling can be described by the Continuous Resoning equation

$$i\partial_t g = \mathcal{R}(g,g,g) \tag{CR}$$

as an equation on the full space $\mathbb{R}^2$

Deterministic initial data and CR

New family of random initial data

$L \gg 1$ is the period of the oscillating function

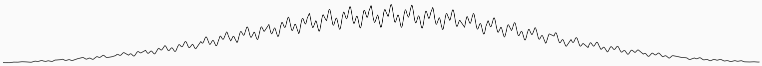
$h \ll 1$ is the spatial truncation parameter

Observation of a large number of waves under the scaling $hL \ll 1$

$$\varphi(x) = \frac{1}{(2\pi)^2} \sum_{K \in \mathbb{Z}_L^2} \eta_K e^{iK\cdot x} e^{-\frac{1}{2}h^2|x|^2} = e^{-\frac{1}{2}h^2|x|^2} F_L(x)$$

$$\varphi_k = \frac{1}{2\pi h^2} \sum_{K \in \mathbb{Z}_L^2} \eta_K e^{-\frac{1}{2h^2}|x|^2}$$

New family of random initial data



$L \gg 1$ is the period of the oscillating function

$h \ll 1$ is the spatial truncation parameter

Observation of a large number of waves under the scaling $hL \ll 1$

$$\varphi(x) = \frac{1}{(2\pi)^2} \sum_{K \in \mathbb{Z}_L^2} \eta_K e^{iK\cdot x} e^{-\frac{1}{2}h^2|x|^2} = e^{-\frac{1}{2}h^2|x|^2} F_L(x)$$

$$\varphi_k = \frac{1}{2\pi h^2} \sum_{K \in \mathbb{Z}_L^2} \eta_K e^{-\frac{1}{2h^2}|x|^2}$$

Can we observe wave turbulence in the limit ?

Nonlinear Schrödinger equation

The equation

$$i\partial_t u = \Delta u + |u|^2 u \qquad \text{(NLS)}$$

on $\mathbb{R}^2$ rewrites in frequency as

$$i\partial_t u_k = \omega_k u_k + \int_{k=\ell-m+j} u_\ell \overline{u_m} u_j$$

Nonlinear Schrödinger equation

The equation

$$i\partial_t u = \Delta u + |u|^2 u \tag{NLS}$$

on $\mathbb{R}^2$ rewrites in frequency as

$$i\partial_t u_k = \omega_k u_k + \int_{k=\ell-m+j} u_\ell \overline{u_m} u_j$$

and in the new variable $v(t) := e^{-it\Delta}u(t)$

$$i\partial_t v_k = \int_{k=\ell-m+j} e^{-it\Delta\omega_{k\ell mj}} v_\ell \overline{v_m} v_j =: R_k(t, v, v, v)$$

with the resonance relation

$$\Delta\omega_{k\ell mj} = |k|^2 - |\ell|^2 + |m|^2 - |j|^2$$

Scattering and resonant manifold

Scattering is a linear behavior for large time of the solution

$$v(t) = e^{-it\Delta} u(t) \underset{t \to +\infty}{\longrightarrow} \psi$$

Scattering and resonant manifold

Scattering is a linear behavior for large time of the solution

$$v(t) = e^{-it\Delta} u(t) \underset{t \to +\infty}{\longrightarrow} \psi$$

This is related to

$$\begin{aligned} R_k(t, u, v, w) &= \int_{k=\ell-m+j} e^{it\Delta\omega_{k\ell mj}} u_\ell \bar{v}_m w_j \\ &= \int_{\mathbb{R}} e^{it\xi} \left(\int_{\substack{k=\ell-m+j \\ \Delta\omega_{k\ell mj}=\xi}} u_\ell \bar{v}_m w_j \right) \mathrm{d}\xi \end{aligned}$$

using the co-area formula

Scattering and resonant manifold

Scattering is a linear behavior for large time of the solution

$$v(t) = e^{-it\Delta} u(t) \underset{t \to +\infty}{\longrightarrow} \psi$$

This is related to

$$\begin{aligned} R_k(t, u, v, w) &= \int_{k=\ell-m+j} e^{it\Delta\omega_{k\ell mj}} u_\ell \bar{v}_m w_j \\ &= \int_{\mathbb{R}} e^{it\xi} \left(\int_{\substack{k=\ell-m+j \\ \Delta\omega_{k\ell mj}=\xi}} u_\ell \bar{v}_m w_j \right) \mathrm{d}\xi \end{aligned}$$

using the co-area formula thus related to

$$S_k(\xi) = \{(\ell, m, j) \in \mathbb{R}^3 \; ; \; k = \ell - m + j \quad \text{and} \quad \Delta\omega_{k\ell mj} = \xi\}$$

Scattering for NLS

The nonlinear Schrödinger equation scatters for initial data in

$$\Sigma = \{\varphi(x) \in H^1(\mathbb{R}^2)\ ;\ |x|\varphi(x) \in L^2(\mathbb{R}^2)\}$$

Carles and Gallagher proved that the scattering operator is analytic in Σ, that is for initial data φ the solution satisfies

$$v_k(t) = \varphi_k + \sum_{n\geq 1} (-i)^n V_k^n(t)$$

with $\|V^n\|_\Sigma \lesssim \|\varphi\|_\Sigma^{2n+1}$,

Scattering for NLS

The nonlinear Schrödinger equation scatters for initial data in

$$\Sigma = \{\varphi(x) \in H^1(\mathbb{R}^2)\ ;\ |x|\varphi(x) \in L^2(\mathbb{R}^2)\}$$

Carles and Gallagher proved that the scattering operator is analytic in Σ, that is for initial data φ the solution satisfies

$$v_k(t) = \varphi_k + \sum_{n\geq 1} (-i)^n V_k^n(t)$$

with $\|V^n\|_\Sigma \lesssim \|\varphi\|_\Sigma^{2n+1}$, $V_k^1(t) = \int_0^t R_k(s,\varphi,\varphi,\varphi)\mathrm{d}s$ and

$$\begin{aligned} V_k^2(t) = 2\int_0^t \int_0^s & R_k(s,\varphi,\varphi,R(s',\varphi,\varphi,\varphi))\mathrm{d}s'\mathrm{d}s \\ + \int_0^t \int_0^s & R_k(s,\varphi,R(s',\varphi,\varphi,\varphi),\varphi)\mathrm{d}s'\mathrm{d}s \end{aligned}$$

Back to our initial data

The function φ is a $2\pi L$-periodic function embbeded in Σ by Gaussian truncation and

$$\|\varphi\|_\Sigma \lesssim \frac{L}{h^2}$$

thus for $u(0) = \epsilon\varphi$ with $\epsilon \ll \frac{L}{h^2}$, we have

$$v(t) = \epsilon\varphi - i\epsilon^3 V^1(t,\varphi) + \mathcal{O}(\epsilon^5)$$

Back to our initial data

The function φ is a $2\pi L$-periodic function embbeded in Σ by Gaussian truncation and

$$\|\varphi\|_\Sigma \lesssim \frac{L}{h^2}$$

thus for $u(0) = \epsilon\varphi$ with $\epsilon \ll \frac{L}{h^2}$, we have

$$v(t) = \epsilon\varphi - i\epsilon^3 V^1(t,\varphi) + \mathcal{O}(\epsilon^5)$$

Can we obtain a limit for

$$V^1(t,\varphi) = \int_0^t e^{-is\Delta}(|e^{is\Delta}\varphi|^2 e^{is\Delta}\varphi)\mathrm{d}s$$

in a suitable timeframe depending on the parameters ?

Coarse grained observables

In frequency, the initial data

$$\varphi_k = \frac{1}{2\pi h^2} \sum_{K \in \mathbb{Z}_L^2} \eta_K e^{-\frac{1}{2h^2}|k-K|^2} \underset{h \to 0}{\longrightarrow} \sum_{K \in \mathbb{Z}_L^2} \eta_K \delta_0(k-K)$$

Coarse grained observables

In frequency, the initial data

$$\varphi_k = \frac{1}{2\pi h^2} \sum_{K\in\mathbb{Z}_L^2} \eta_K e^{-\frac{1}{2h^2}|k-K|^2} \underset{h\to 0}{\longrightarrow} \sum_{K\in\mathbb{Z}_L^2} \eta_K \delta_0(k-K)$$

hence we consider observable

$$\langle v\rangle_{K,\sigma} = \int_{\mathbb{R}^2} e^{-\frac{1}{2\sigma^2}|k-K|^2} \widehat{v}(k)\mathrm{d}k$$

with $h \ll \sigma \ll \frac{1}{L}$

Coarse grained observables

In frequency, the initial data

$$\varphi_k = \frac{1}{2\pi h^2} \sum_{K\in\mathbb{Z}_L^2} \eta_K e^{-\frac{1}{2h^2}|k-K|^2} \underset{h\to 0}{\longrightarrow} \sum_{K\in\mathbb{Z}_L^2} \eta_K \delta_0(k-K)$$

hence we consider observable

$$\langle v\rangle_{K,\sigma} = \int_{\mathbb{R}^2} e^{-\frac{1}{2\sigma^2}|k-K|^2} \widehat{v}(k)\mathrm{d}k$$

with $h \ll \sigma \ll \frac{1}{L}$

$$\int_{\mathbb{R}^2} e^{-\frac{1}{2\sigma^2}|k-K|^2} \widehat{g}_{K_1,h}(k)\mathrm{d}k \simeq e^{-\frac{1}{2\sigma^2}|K_1-K|^2} \simeq \mathbb{1}_{K=K_1}$$

hence

$$\langle \varphi\rangle_{K,\sigma} \simeq \eta_K$$

Identifying the first order term

For $t \ll \frac{1}{\epsilon^2}$, we have

$$(e^{it\Delta} g_{K,\epsilon})(x) \simeq \frac{1}{(2\pi)^2} e^{-\frac{\epsilon^2}{2}|x|^2 + ix\cdot K - it|K|^2} = g_{K,\epsilon}(x) e^{-it|K|^2}$$

Identifying the first order term

For $t \ll \frac{1}{\epsilon^2}$, we have

$$(e^{it\Delta} g_{K,\epsilon})(x) \simeq \frac{1}{(2\pi)^2} e^{-\frac{\epsilon^2}{2}|x|^2 + ix\cdot K - it|K|^2} = g_{K,\epsilon}(x) e^{-it|K|^2}$$

which gives for $t \ll \frac{1}{h}$

$$\langle V^1 \rangle_{K,\sigma} = \int_0^t \int_{\mathbb{R}^2} e^{-\frac{1}{2\sigma^2}|k-K|^2} \mathscr{F}\Big(e^{-is\Delta}\big(|e^{is\Delta}\varphi|^2 e^{is\Delta}\varphi\big)\Big)(k) \mathrm{d}k \mathrm{d}s$$

$$\simeq \frac{1}{(2\pi)^4} \sum_{K = K_1 - K_2 + K_3} \eta_{K_1} \overline{\eta_{K_2}} \eta_{K_3} \int_0^t e^{-is(|K|^2 - |K_1|^2 + |K_2|^2 - |K_3|^2)} \mathrm{d}s$$

Identifying the first order term

For $t \ll \frac{1}{\epsilon^2}$, we have

$$(e^{it\Delta} g_{K,\epsilon})(x) \simeq \frac{1}{(2\pi)^2} e^{-\frac{\epsilon^2}{2}|x|^2 + ix\cdot K - it|K|^2} = g_{K,\epsilon}(x) e^{-it|K|^2}$$

which gives for $t \ll \frac{1}{h}$

$$\langle V^1 \rangle_{K,\sigma} = \int_0^t \int_{\mathbb{R}^2} e^{-\frac{1}{2\sigma^2}|k-K|^2} \mathscr{F}\Big(e^{-is\Delta} \big(|e^{is\Delta}\varphi|^2 e^{is\Delta}\varphi \big) \Big)(k) \mathrm{d}k \mathrm{d}s$$

$$\simeq \frac{1}{(2\pi)^4} \sum_{K = K_1 - K_2 + K_3} \eta_{K_1} \overline{\eta_{K_2}} \eta_{K_3} \int_0^t e^{-is(|K|^2 - |K_1|^2 + |K_2|^2 - |K_3|^2)} \mathrm{d}s$$

thus

$$\langle v(t) \rangle_{K,\sigma} \simeq \varepsilon \varphi_K - i \frac{\epsilon^3}{(2\pi)^4} \sum_{K = K_1 - K_2 + K_3} \eta_{K_1} \overline{\eta_{K_2}} \eta_{K_3} \frac{1 - e^{-it\Delta\omega_{KK_1K_2K_3}}}{i\Delta\omega_{KK_1K_2K_3}}$$

Limit to the kinetic operator CR

For $t \leq L$, we have a convergent Riemann sum

$$\begin{aligned}
&\lim_{L\to\infty} \frac{1}{L^4} \sum_{K=K_1-K_2+K_3} \eta_{K_1}\overline{\eta_{K_2}}\eta_{K_3} \frac{1-e^{-it\Delta\omega_{KK_1K_2K_3}}}{i\Delta\omega_{KK_1K_2K_3}} \\
&\qquad = \int_{K=k_1-k_2+k_3} \eta(k_1)\overline{\eta(k_2)}\eta(k_3) \frac{1-e^{-it\Delta\omega_{Kk_1k_2k_3}}}{i\Delta\omega_{Kk_1k_2k_3}} \mathrm{d}k_1\mathrm{d}k_2\mathrm{d}k_3 \\
&\qquad = \int_{\substack{K=k_1-k_2+k_3 \\ \Delta\omega_{Kk_1k_2k_3}=0}} \eta(k_1)\overline{\eta(k_2)}\eta(k_3)\mathrm{d}k_1\mathrm{d}k_2\mathrm{d}k_3 + o(t^{-1})
\end{aligned}$$

Limit to the kinetic operator CR

For $t \leq L$, we have a convergent Riemann sum

$$
\begin{aligned}
\lim_{L\to\infty} \frac{1}{L^4} \sum_{K=K_1-K_2+K_3} & \eta_{K_1}\overline{\eta_{K_2}}\eta_{K_3} \frac{1-e^{-it\Delta\omega_{KK_1K_2K_3}}}{i\Delta\omega_{KK_1K_2K_3}} \\
&= \int_{K=k_1-k_2+k_3} \eta(k_1)\overline{\eta(k_2)}\eta(k_3) \frac{1-e^{-it\Delta\omega_{Kk_1k_2k_3}}}{i\Delta\omega_{Kk_1k_2k_3}} \mathrm{d}k_1\mathrm{d}k_2\mathrm{d}k_3 \\
&= \int_{\substack{K=k_1-k_2+k_3 \\ \Delta\omega_{Kk_1k_2k_3}=0}} \eta(k_1)\overline{\eta(k_2)}\eta(k_3)\mathrm{d}k_1\mathrm{d}k_2\mathrm{d}k_3 + o(t^{-1})
\end{aligned}
$$

$\to$ open problem for $L \leq t \ll L^2$, this is number theory

Limit to the kinetic operator CR

For $t \leq L$, we have a convergent Riemann sum

$$\lim_{L\to\infty} \frac{1}{L^4} \sum_{K=K_1-K_2+K_3} \eta_{K_1}\overline{\eta_{K_2}}\eta_{K_3} \frac{1-e^{-it\Delta\omega_{KK_1K_2K_3}}}{i\Delta\omega_{KK_1K_2K_3}}$$

$$= \int_{K=k_1-k_2+k_3} \eta(k_1)\overline{\eta(k_2)}\eta(k_3) \frac{1-e^{-it\Delta\omega_{Kk_1k_2k_3}}}{i\Delta\omega_{Kk_1k_2k_3}} \mathrm{d}k_1\mathrm{d}k_2\mathrm{d}k_3$$

$$= \int_{\substack{K=k_1-k_2+k_3 \\ \Delta\omega_{Kk_1k_2k_3}=0}} \eta(k_1)\overline{\eta(k_2)}\eta(k_3)\mathrm{d}k_1\mathrm{d}k_2\mathrm{d}k_3 + o(t^{-1})$$

$\to$ open problem for $L \leq t \ll L^2$, this is number theory

$$\langle v(t)\rangle_{K,\sigma} \simeq \varepsilon\varphi_K - i\frac{\epsilon^3 L^4}{(2\pi)^4}\mathcal{R}_K(\eta)$$

with $\mathcal{R}$ from Faou, Germain and Hani

Limit to the kinetic operator CR

For all time $t \in \mathbb{R}$, we have

$$\Big| \sum_{\substack{K=K_1-K_2+K_3 \\ \Delta\omega_{KK_1K_2K_3}\neq 0}} \eta_{K_1}\overline{\eta_{K_2}}\eta_{K_3} \frac{1-e^{-it\Delta\omega_{KK_1K_2K_3}}}{i\Delta\omega_{KK_1K_2K_3}} \Big| \leq L^{4+\delta}$$

for any $\delta > 0$

Limit to the kinetic operator CR

For all time $t \in \mathbb{R}$, we have

$$\Big| \sum_{\substack{K=K_1-K_2+K_3 \\ \Delta\omega_{KK_1K_2K_3}\neq 0}} \eta_{K_1}\overline{\eta_{K_2}}\eta_{K_3} \frac{1-e^{-it\Delta\omega_{KK_1K_2K_3}}}{i\Delta\omega_{KK_1K_2K_3}} \Big| \leq L^{4+\delta}$$

for any $\delta > 0$ while

$$\lim_{L\to\infty} \frac{\zeta(2)}{2tL^2\log(L)} \sum_{\substack{K=K_1-K_2+K_3 \\ \Delta\omega_{KK_1K_2K_3}\neq 0}} \eta_{K_1}\overline{\eta_{K_2}}\eta_{K_3} \frac{1-e^{-it\Delta\omega_{KK_1K_2K_3}}}{i\Delta\omega_{KK_1K_2K_3}} = \mathcal{R}_K(\eta)$$

as proved by Faou, Germain and Hani

Limit to the kinetic operator CR

For all time $t \in \mathbb{R}$, we have

$$\Big| \sum_{\substack{K=K_1-K_2+K_3 \\ \Delta\omega_{KK_1K_2K_3}\neq 0}} \eta_{K_1}\overline{\eta_{K_2}}\eta_{K_3} \frac{1-e^{-it\Delta\omega_{KK_1K_2K_3}}}{i\Delta\omega_{KK_1K_2K_3}} \Big| \leq L^{4+\delta}$$

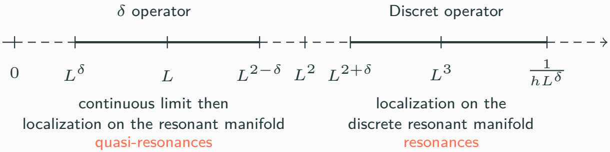
for any $\delta > 0$ while

$$\lim_{L\to\infty} \frac{\zeta(2)}{2tL^2\log(L)} \sum_{\substack{K=K_1-K_2+K_3 \\ \Delta\omega_{KK_1K_2K_3}\neq 0}} \eta_{K_1}\overline{\eta_{K_2}}\eta_{K_3} \frac{1-e^{-it\Delta\omega_{KK_1K_2K_3}}}{i\Delta\omega_{KK_1K_2K_3}} = \mathcal{R}_K(\eta)$$

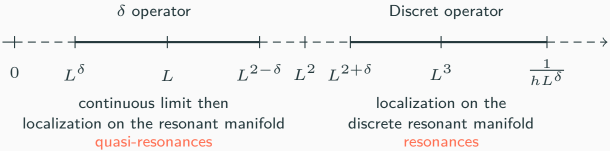
as proved by Faou, Germain and Hani thus

$$\langle v(t)\rangle_{K,\sigma} \simeq \varepsilon\varphi_K - i\frac{2t\epsilon^3L^2\log(L)}{\zeta(2)(2\pi)^4}\mathcal{R}_K(\eta)$$

General picture



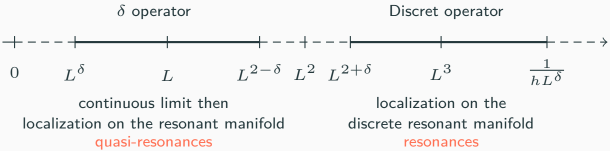
General picture



For $L^\delta \leq t \leq L^{1-\delta}$, we have

$$\langle v(t) \rangle_{K,\sigma} = \varepsilon \varphi_K - i \frac{\varepsilon^3 L^4}{(2\pi)^4} \mathcal{R}_K(\eta) + o(\varepsilon^3 L^4)$$

General picture



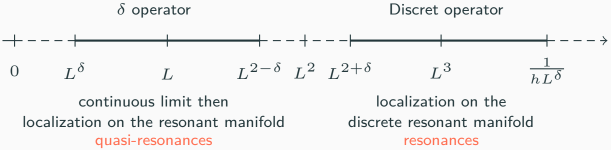
For $L^\delta \leq t \leq L^{1-\delta}$, we have

$$\langle v(t) \rangle_{K,\sigma} = \varepsilon \varphi_K - i \frac{\varepsilon^3 L^4}{(2\pi)^4} \mathcal{R}_K(\eta) + o(\varepsilon^3 L^4)$$

For $L^{2+\delta} \leq t \leq \frac{1}{hL^\delta}$, we have

$$\langle v(t) \rangle_{K,\sigma} = \varepsilon \varphi_K - i \frac{2t\varepsilon^3 L^2 \log(L)}{\zeta(2)(2\pi)^4} \mathcal{R}_K(\eta) + o(\varepsilon^3 t L^2 \log(L))$$

General picture



For $L^{\delta} \leq t \leq L^{1-\delta}$, we have

$$\langle v(t) \rangle_{K,\sigma} = \varepsilon \varphi_K - i \frac{\varepsilon^3 L^4}{(2\pi)^4} \mathcal{R}_K(\eta) + o(\varepsilon^3 L^4)$$

For $L^{2+\delta} \leq t \leq \frac{1}{hL^{\delta}}$, we have

$$\langle v(t) \rangle_{K,\sigma} = \varepsilon \varphi_K - i \frac{2t\varepsilon^3 L^2 \log(L)}{\zeta(2)(2\pi)^4} \mathcal{R}_K(\eta) + o(\varepsilon^3 t L^2 \log(L))$$

scaling : $hL^4 \ll 1$ and $hL \ll \sigma \leq h^{\frac{3}{4}}$

Random phase and WKE

Randomization of the initial data

We consider

$$\varphi(x) = \frac{1}{(2\pi)^2} \sum_{K \in \mathbb{Z}_L^2} \eta_K e^{i\theta_K} e^{iK\cdot x} e^{-\frac{1}{2}h^2|x|^2} = e^{-\frac{1}{2}h^2|x|^2} F_L(x)$$

with $(\theta_K)_{K\in\mathbb{Z}_L^2}$ i.i.d. uniform random variables on $[0, 2\pi]$, that is Random Phase

Randomization of the initial data

We consider

$$\varphi(x) = \frac{1}{(2\pi)^2} \sum_{K \in \mathbb{Z}_L^2} \eta_K e^{i\theta_K} e^{iK\cdot x} e^{-\frac{1}{2}h^2|x|^2} = e^{-\frac{1}{2}h^2|x|^2} F_L(x)$$

with $(\theta_K)_{K\in\mathbb{Z}_L^2}$ i.i.d. uniform random variables on $[0, 2\pi]$, that is Random Phase and the CR operator should be replaced by

$$\mathcal{T}_k(n) = \int_{\substack{k=k_1-k_2+k_3 \\ \Delta\omega_{kk_1k_2k_3}=0}} n(k)n(k_1)n(k_2)n(k_3) \left(\frac{1}{n(k)} - \frac{1}{n(k_1)} + \frac{1}{n(k_2)} - \frac{1}{n(k_3)}\right) \mathrm{d}k_1 \mathrm{d}k_2 \mathrm{d}k_3$$

Randomization of the initial data

We consider

$$\varphi(x) = \frac{1}{(2\pi)^2} \sum_{K \in \mathbb{Z}_L^2} \eta_K e^{i\theta_K} e^{iK\cdot x} e^{-\frac{1}{2}h^2|x|^2} = e^{-\frac{1}{2}h^2|x|^2} F_L(x)$$

with $(\theta_K)_{K\in\mathbb{Z}_L^2}$ i.i.d. uniform random variables on $[0, 2\pi]$, that is Random Phase and the CR operator should be replaced by

$$\mathcal{T}_k(n) = \int_{\substack{k=k_1-k_2+k_3 \\ \Delta\omega_{kk_1k_2k_3}=0}} n(k)n(k_1)n(k_2)n(k_3) \left(\frac{1}{n(k)} - \frac{1}{n(k_1)} + \frac{1}{n(k_2)} - \frac{1}{n(k_3)}\right) \mathrm{d}k_1\mathrm{d}k_2\mathrm{d}k_3$$

Can we observe the Wave Kinetic Equation in the description of $\mathbb{E}\big[|\langle v(t)\rangle_{K,\sigma}|^2\big]$?

Second order expansion

The expansion

$$\langle v(t)\rangle_{K,\sigma} = \varepsilon\varphi_K - i\varepsilon^3 V_K^1(t) - \varepsilon^5 V_K^2(t) + \mathcal{O}(\varepsilon^7)$$

gives

$$\mathbb{E}\big[|\langle v(t)\rangle_{K,\sigma}|^2\big] = \varepsilon^2 E^0_{K,\sigma}(t,\varphi) + \varepsilon^4 E^1_{K,\sigma}(t,\varphi) + \varepsilon^6 E^2_{K,\sigma}(t,\varphi) + \mathcal{O}(\varepsilon^8)$$

with

$$\begin{aligned} E^0_{K,\sigma}(t,\varphi) &= \mathbb{E}\big[|\langle \varphi\rangle_{K,\sigma}|^2\big] \\ E^1_{K,\sigma}(t,\varphi) &= 2\mathbb{E}\big[\mathrm{Re}\big(\langle v\rangle_{K,\sigma}\overline{\langle V^1\rangle}_{K,\sigma}\big)\big] \\ E^2_{K,\sigma}(t,\varphi) &= \mathbb{E}\big[|\langle V^1\rangle_{K,\sigma}|^2\big] + 2\mathbb{E}\big[\mathrm{Re}\big(\langle v\rangle_{K,\sigma}\overline{\langle V^2\rangle}_{K,\sigma}\big)\big] \end{aligned}$$

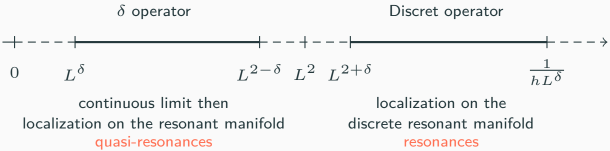
Identifying the second order term

With the same type of computations, we get

$$\langle V^2(t)\rangle_{K,\sigma} \simeq 2 \sum_{\substack{K=K_1-K_2+K_3 \\ K_1=K_4-K_5+K_6}} \eta_{K_4}\overline{\eta_{K_5}}\eta_{K_6}\overline{\eta_{K_2}}\eta_{K_3} \times \int_0^t \int_0^s e^{is\Delta\omega_{KK_1K_2K_3}} e^{-is'\Delta\omega_{K_1K_4K_5K_6}} \mathrm{d}s'\mathrm{d}s + \sum_{\substack{K=K_1-K_2+K_3 \\ K_2=K_4-K_5+K_6}} \eta_{K_1}\overline{\eta_{K_4}}\eta_{K_5}\overline{\eta_{K_6}}\eta_{K_3} \times \int_0^t \int_0^s e^{is\Delta\omega_{KK_1K_2K_3}} e^{-is'\Delta\omega_{K_2K_4K_5K_6}} \mathrm{d}s'\mathrm{d}s$$

which gives the three missing terms to get from CR to WKE

General picture



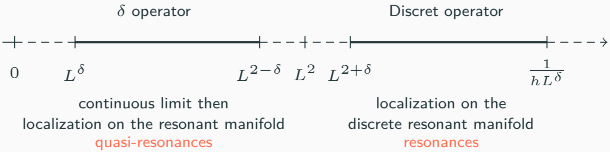
For $L^\delta \leq t \leq L^{2-\delta}$, we have

$$\mathbb{E}\big[|\langle v(t)\rangle_{K,\sigma}|^2\big] = \varepsilon^2|\varphi_K|^2 + \frac{t\varepsilon^6 L^4}{(2\pi)^8}\mathcal{T}_K(\eta) + o(t\varepsilon^6 L^4)$$

For $L^{2+\delta} \leq t \leq \frac{1}{hL^\delta}$, we have

$$\mathbb{E}\big[|\langle v(t)\rangle_{K,\sigma}|^2\big] = \varepsilon^2|\varphi_K|^2 + \frac{2t^2\varepsilon^6 L^2\log(L)}{\zeta(2)(2\pi)^8}\mathcal{T}_K(\eta) + o\big(t^2\varepsilon^6 t L^2\log(L)\big)$$

General picture



For $L^\delta \leq t \leq L^{2-\delta}$, we have

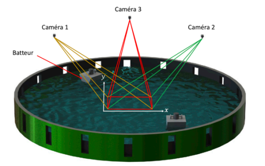
$$\mathbb{E}\big[|\langle v(t)\rangle_{K,\sigma}|^2\big] = \varepsilon^2|\varphi_K|^2 + \frac{t\varepsilon^6 L^4}{(2\pi)^8}\mathcal{T}_K(\eta) + o(t\varepsilon^6 L^4)$$

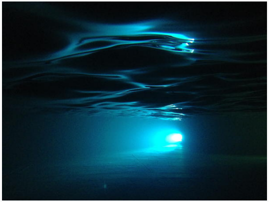
For $L^{2+\delta} \leq t \leq \frac{1}{hL^\delta}$, we have

$$\mathbb{E}\big[|\langle v(t)\rangle_{K,\sigma}|^2\big] = \varepsilon^2|\varphi_K|^2 + \frac{2t^2\varepsilon^6 L^2\log(L)}{\zeta(2)(2\pi)^8}\mathcal{T}_K(\eta) + o\big(t^2\varepsilon^6 t L^2\log(L)\big)$$

and $\mathbb{E}\Big[\langle v(t)\rangle_{K,\sigma}\overline{\langle v(t)\rangle_{K',\sigma}}\Big] \simeq 0$ for $K \neq K'$ (propagation of chaos)

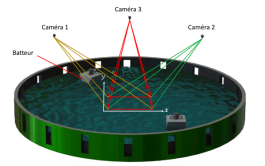
What's next ?

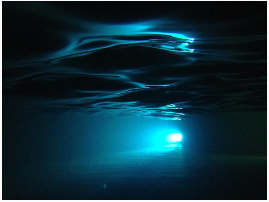





- Forcing and dissipation (with Erwan Faou)
- Numeric simulations (with Quentin Chauleur)
- Other equations
- Law of the solution

What's next ?





- Forcing and dissipation (with Erwan Faou)
- Numeric simulations (with Quentin Chauleur)
- Other equations
- Law of the solution

Thanks for you attention !