Anti-Self-Dual Yang-Mills eq.

4dimensional Wess-Zumino-Witten Models (WZW) as a unified theory of integrable systems

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21 June 2024 @ Bourgogne

- MH, Shan-Chi Huang, Hiroaki Kanno, 2212.11800 Prog. Theor. Exp. Phys. (PTEP) 2023-4, 043B03 ; etc.

§1 Introduction

Anti-Self-Dual (ASD) Yang-Mills (YM) eqs.

- reveal non-perturbative aspects of QFT

  e.g. $$Z(g) = \underbrace{\oslash + \boxslash\, g + \blacktriangle\, g^2 + \cdots}_{\text{perturbation}\left(\substack{\text{positive}\\ \text{powers}}\right)} + \underset{\left(\substack{\text{negative}\\ \text{powers}}\right)}{\underline{\blacklozenge \frac{1}{g} + \blacktriangledown \frac{1}{g^2} + \cdots}}$$

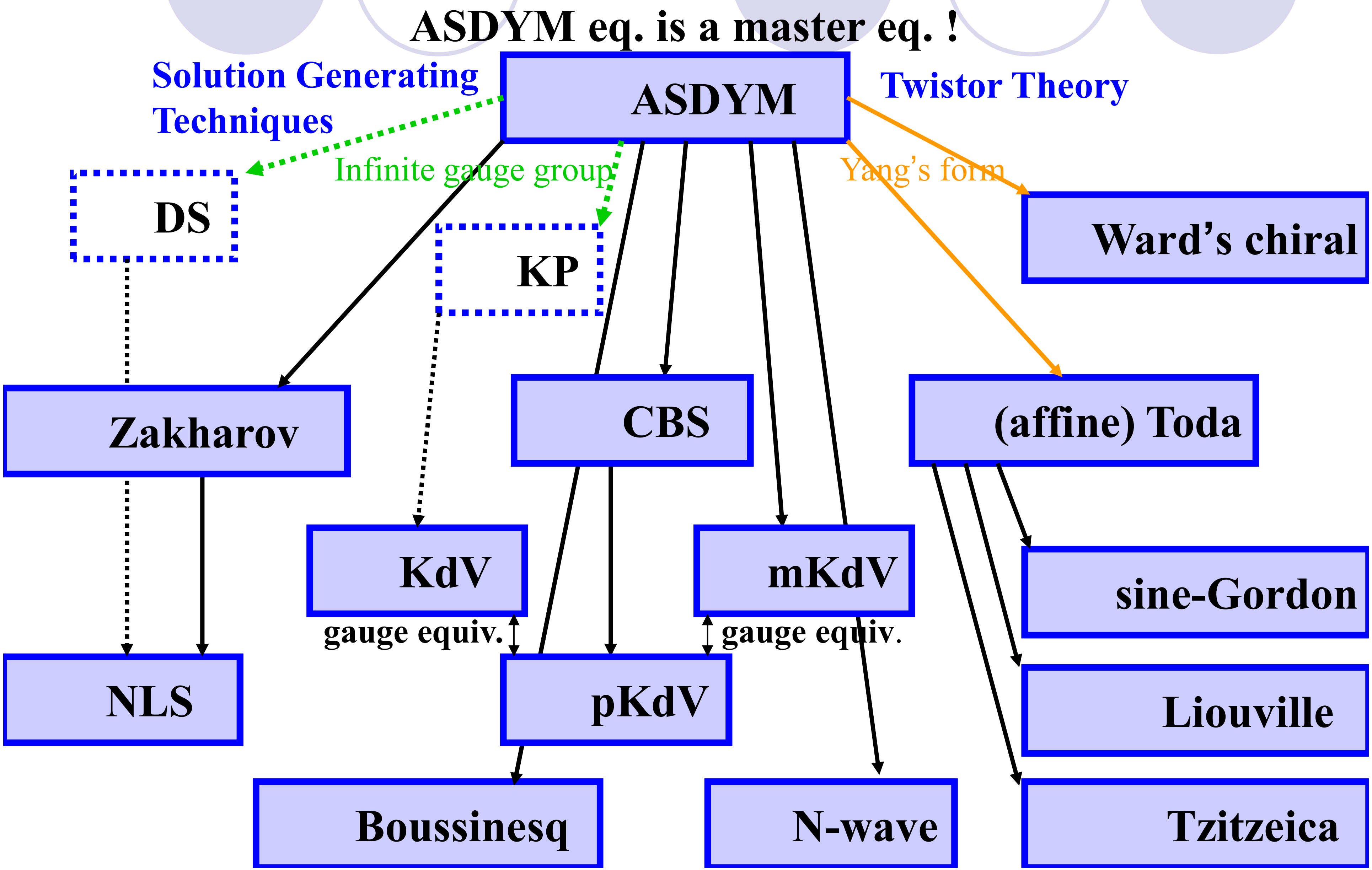
  ['t Hooft, ...]

- Application to geometry

  e.g. Donaldson inv., Nekrasov part. fcn, ...

- "master eq" of integrable systems [R. Ward, ...]

Ward's conjecture: Many (perhaps all?) integrable equations are reductions of the ASDYM eqs.



ASDYM eq. is a master eq. !

Solution Generating Techniques

ASDYM

Twistor Theory

Infinite gauge group

Yang's form

DS

KP

Ward's chiral

Zakharov

CBS

(affine) Toda

KdV

mKdV

sine-Gordon

gauge equiv.

gauge equiv.

NLS

pKdV

Liouville

Boussinesq

N-wave

Tzitzeica

ASDYM eq. (in 4 dim., G_{YM}: gauge gp.) in $\mathbb{E}$ or $\mathbb{U}$

$$\underset{\sim}{*}F_{\mu\nu} = -F_{\mu\nu}$$

Hodge dual

$\mu, \nu \in \{1, 2, 3, 4\}$

$$\underset{\sim\sim\sim}{F_{\mu\nu}} = \partial_\mu \underset{\sim\sim}{A_\nu} - \partial_\nu A_\mu + \underset{\uparrow}{[}A_\mu, A_\nu]$$

field strength

gauge field

commutator

e.g. $G = U(1)$, $F_{\mu\nu} = \text{“}\begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{pmatrix}\text{”}$

$\vec{E}$: electric field, $\vec{B}$: magnetic field

ASDYM eq. (in 4 dim., G_{YM}: gauge gp.)

$$\underset{\text{Hodge dual}}{\underset{\sim}{*}} F_{\mu\nu} = -F_{\mu\nu} \qquad \underbrace{F_{\mu\nu}}_{\text{field strength}} = \partial_\mu \underbrace{A_\nu}_{\text{gauge field}} - \partial_\nu A_\mu + \underset{\substack{\uparrow \\ \text{commutator}}}{[} A_\mu , A_\nu]$$

$$\mu, \nu \in \{1, 2, 3, 4\}$$

$$\Leftrightarrow \quad F_{12} = -F_{34}, \quad F_{13} = -F_{42}, \quad F_{14} = -F_{23}$$

$$\Leftrightarrow \quad F_{zw} = 0, \quad F_{\bar{z}\bar{w}} = 0, \quad F_{z\bar{z}} + F_{w\bar{w}} = 0 \qquad (\text{in } \overset{\substack{\text{later} \\ \downarrow}}{E})$$

ASDYM eq. (in 4 dim., G_{YM} : gauge gp.)

$$\underset{\text{Hodge dual}}{\underset{\sim}{*}}F_{\mu\nu} = -F_{\mu\nu} \qquad \underset{\text{field strength}}{\underset{\sim\sim}{F_{\mu\nu}}} = \partial_\mu \underset{\text{gauge field}}{\underset{\sim\sim}{A_\nu}} - \partial_\nu A_\mu + \underset{\text{commutator}}{\underset{\uparrow}{[}} A_\mu , A_\nu]$$

$\mu, \nu \in \{1, 2, 3, 4\}$

$$\Leftrightarrow \quad F_{12} = -F_{34}, \quad F_{13} = -F_{42}, \quad F_{14} = -F_{23}$$

$$\Leftrightarrow \quad F_{zw} = 0, \quad F_{\bar{z}\bar{w}} = 0, \quad F_{z\bar{z}} + F_{w\bar{w}} = 0 \qquad (\text{in } \overset{\text{later}}{\overset{\downarrow}{\mathbb{E}}})$$

Note A_μ, $F_{\mu\nu}$ take value in $\mathfrak{g}$ (Lie alg. of G_{YM})

e.g. $G_{YM} = U(N)$, A_μ, $F_{\mu\nu}$ are $N \times N$ anti-hermitian.

Reduction to 'KdV from ASDYM $(G_{YM}=SL(2,\mathbb{C}))$

ASDYM : $F_{zw}=0,\ F_{\tilde{z}\tilde{w}}=0,\ F_{z\tilde{z}}-F_{w\tilde{w}}=0$

$$\Bigg\downarrow$$

① $\partial_w-\partial_{\tilde{w}}=0,\ \partial_{\tilde{z}}=0$ (dim. reduction)

② $A_{\tilde{w}}=\begin{pmatrix}0&0\\ \frac{u}{2}&0\end{pmatrix},\ A_{\tilde{z}}=\begin{pmatrix}0&0\\1&0\end{pmatrix},\ A_w=\begin{pmatrix}0&-1\\u&0\end{pmatrix}$

$A_z=\frac{1}{4}\begin{pmatrix}u'&-2u\\u''+2u^2&-u'\end{pmatrix}$ $\qquad u=u(z,\underset{\sim}{x}{}^{\text{''}})\ \underset{\sim\sim\sim}{w+\tilde{w}}$

$\qquad u'=\partial_x u$

$u_{\underset{\overset{\shortparallel}{t}}{z}}-u_{xxx}-\frac{3}{2}uu_x=0$: **KdV eq. !**

cf. P9

(t,x) are real ⇒ not $(++++)$ but $(++--)$

A Unified theory of integrable systems

EoM = ASDYM eq. !

↑

4d CS

↓ [Costello-Yamazaki (-Witen)]

various [Delduc-Lacroix-Magro-Vicedo],
solvable models [Yoshida(K), Sakamoto,
(spin chains, PCM, ⋯) Fukushima, ⋯]
⋯

4d WZW (++−−)

[Ward] ↓ [Mason-Woodhouse]

various
integrable eqs.
(KdV NLS, Toda, ⋯)

A Unified theory of integrable systems

6d meromorphic Chern-Simons (CS)

↙ ↘

[Costello]
[Bittleston-Skinner]

4d CS ← duality? → **4d WZW** (++−−)

↓ [Costello-Yamazaki(-Witten)]

[Ward] ↓ [Mason-Woodhouse]

various solvable models [Delduc-Lacroix-Magro-Vicedo], [Yoshida(K), Sakamoto, Fukushima, …]
…

(spin chains, PCM, …)

various integrable eqs.

(KdV NLS, Toda, …)

Plan of Talk (simple discussion)

§1 Introduction (15 min)

§2 ADHM Construction of Instantons (17 min)

§3 4d WZW model (8 min)

§4 Soliton Solutions of ASDYM (10 min)

§5 Conclusion & Discussion (5 min)

2. Atiyah-Drinfeld-Hitchin-Manin Construction

based on duality for the instanton moduli space

4dim. ASDYang-Mills eq.
(Difficult)

$$F_{z_1\bar{z}_1}+F_{z_2\bar{z}_2}=0$$

$$F_{z_1 z_2}=0$$

N × N PDE

Sol.= instantons
(G=U(N), C_2 =k)

$$A_\mu : N\times N$$

Gauge trf.:

$$A_\mu \mapsto g^{-1}A_\mu g + g^{-1}\partial_\mu g$$

$$g\in U(N)$$

1:1

↔

ADHM eq. (≒0dim. ASDYM)
(Easy)

$$[B_1,B_1^+]+[B_2,B_2^+]+I\,I^+-J^+J=0$$

$$[B_1,B_2]+IJ=0$$

k × k Matrix eqs.

Sol.=ADHM data
(G=`U(k)’)

$$B_{1,2} : k\times k,\quad I : k\times N,\quad J : N\times k$$

Gauge trf.:

$$B_{1,2}\mapsto \tilde{g}^{-1}B_{1,2}\tilde{g},\quad \tilde{g}\in U(k)$$

$$I\mapsto \tilde{g}^{-1}I,\quad J\mapsto J\tilde{g}$$

Fourier-Mukai-Nahm transformation

Beautiful duality between instanton moduli on 4-tori and instanton moduli on the dual tori

4dim. ASDYang-Mills eq.
on a 4-torus

$$F_{z_1\bar{z}_1}+F_{z_2\bar{z}_2}=0$$

$$F_{z_1z_2}=0$$

N × N PDE

$$\underset{F}{\overset{G}{\rightleftarrows}}$$

4dim. ASD Yang-Mills eq.
on the dual torus

$$\widetilde{F}_{\xi_1\bar{\xi}_1}+\widetilde{F}_{\xi_2\bar{\xi}_2}=0$$

$$\widetilde{F}_{\xi_1\xi_2}=0$$

k × k PDE

Sol.=instantons
(G=U(N), C_2 =k)

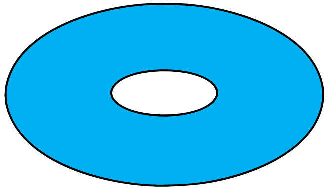
1:1

Sol.=the dual instantons
(G=U(k), C_2=N)

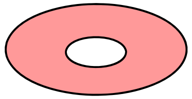
$A_\mu : N\times N$

↕

$\widetilde{A}_\mu : k\times k$



Define the maps F & G,
& G∘F=id. & F∘G=id.



On a 4-torus

On the dual 4-torus

Fourier-Mukai-Nahm transformation

Beautiful duality between instanton moduli on 4-tori and instanton moduli on the dual tori

4dim. ASDYang-Mills eq.
on a 4-torus

$$F_{z_1\bar{z}_1} + F_{z_2\bar{z}_2} = 0$$

$$F_{z_1 z_2} = 0$$

N × N PDE

4dim. ASD Yang-Mills eq.
on the dual torus

$$\widetilde{F}_{\xi_1\bar{\xi}_1} + \widetilde{F}_{\xi_2\bar{\xi}_2} = 0$$

$$\widetilde{F}_{\xi_1\xi_2} = 0$$

k × k PDE

1:1

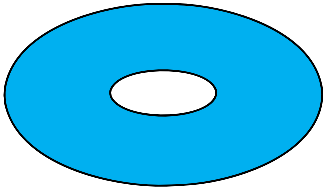
Sol.=instantons
(G=U(N), C₂ =k)

⟷

Sol.=the dual instantons
(G=U(k), C₂=N)

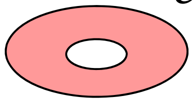
$$A_\mu(x) = \left\langle V, \partial_\mu V \right\rangle_\xi \xleftarrow{\text{map F (Dirac eq.)}} \widetilde{A}_\mu(\xi) : k \times k$$

N×N



$$\nabla^+ V = \bar{e}^\mu \otimes \left(\frac{\partial}{\partial \xi^\mu} + \widetilde{A}_\mu - i x_\mu\right) V = 0$$

$$\bar{e}^\mu := (i\sigma_a, 1_2)$$



$$V : 2k \times \underline{N}$$

Family index thm.

On a 4-torus $: x_\mu$

On the dual 4-torus $: \xi_\mu$

Fourier-Mukai-Nahm transformation

Beautiful duality between instanton moduli on 4-tori and instanton moduli on the dual tori

4dim. ASDYang-Mills eq.
on a 4-torus

$$F_{z_1\bar{z}_1}+F_{z_2\bar{z}_2}=0$$

$$F_{z_1 z_2}=0$$

N × N PDE

4dim. ASD Yang-Mills eq.
on the dual torus

$$\widetilde{F}_{\xi_1\bar{\xi}_1}+\widetilde{F}_{\xi_2\bar{\xi}_2}=0$$

$$\widetilde{F}_{\xi_1\xi_2}=0$$

k × k PDE

1:1

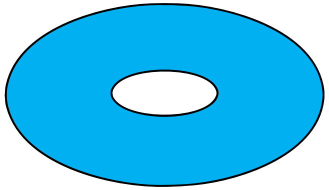
Sol.=instantons
(G=U(N), C_2 =k)

↔

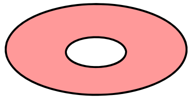
Sol.=the dual instantons
(G=U(k), C_2=N)

$$A_\mu(x):N\times N \xrightarrow{\text{map G (Dirac eq.)}} \widetilde{A}_\mu(\xi)=\left\langle\psi,\widetilde{\partial}_\mu\psi\right\rangle_\xi$$

k × k



$$\bar{e}_\mu D_\mu\psi=\bar{e}^\mu\otimes\left(\frac{\partial}{\partial x^\mu}+A_\mu-i\xi_\mu\right)\psi=0$$



$$\psi:2N\times\underline{k}$$

On a 4-torus : x_μ

Family index thm.

On the dual 4-torus : ξ_μ

Fourier-Mukai-Nahm transformation

Beautiful reciprocity between instanton moduli on 4-tori and instanton moduli on the dual tori

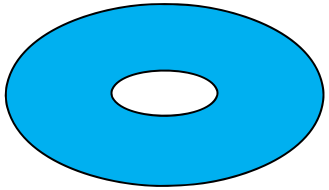
4dim. ASDYang-Mills eq.
on a 4-torus

$$F_{z_1\bar{z}_1}+F_{z_2\bar{z}_2}=0$$
$$F_{z_1z_2}=0$$

N × N PDE

Sol.=instantons
(G=U(N), C_2 =k)

$$A_\mu : N\times N$$



On a 4-torus

Dirac eq.

$$\xrightarrow{G}$$
$$\xleftarrow[F]{}$$

Dirac eq.

1:1
(reciprocity)

⇕

**Define the maps F & G,
& G◦F=id. & F◦G=id.**

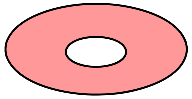
4dim. ASD Yang-Mills eq.
on the dual torus

$$\widetilde{F}_{\xi_1\bar{\xi}_1}+\widetilde{F}_{\xi_2\bar{\xi}_2}=0$$
$$\widetilde{F}_{\xi_1\xi_2}=0$$

k × k PDE

Sol.=the dual instantons
(G=U(k), C_2=N)

$$\widetilde{A}_\mu : k\times k$$



On the dual 4-torus

Fourier-Mukai-Nahm trf. (radii of the torus→∞)

reciprocity between instanton moduli on R^4

and instanton moduli on ``1pt.''

[cf. van Baal, hep-th/9512223]

4dim. ASDYang-Mills eq.

$$F_{z_1\bar{z}_1}+F_{z_2\bar{z}_2}=0$$

$$F_{z_1 z_2}=0$$

N × N PDE

0dim. ASD Yang-Mills eq.

$$\tilde{F}_{\mu\nu} := \cancel{\partial_\mu}\tilde{A}_\nu - \cancel{\partial_\nu}\tilde{A}_\mu + [\tilde{A}_\mu,\tilde{A}_\nu]$$

$$\tilde{F}_{\xi_1\bar{\xi}_1}+\tilde{F}_{\xi_2\bar{\xi}_2}=0$$

$$\tilde{F}_{\xi_1\xi_2}=0$$

Matrix eq. !

k × k ~~PDE~~

1:1

Sol.=instantons
(G=U(N), C_2 =k)

↔

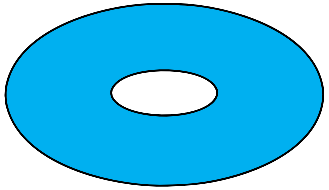
Sol.=``dual instantons''
(G=U(k), ``C_2=N'')

$$A_\mu = V^+\partial_\mu V$$

map F (0dim Dirac eq.)

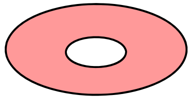
←

$$\tilde{A}_\mu : k\times k$$



$$\nabla^+ V = \bar{e}^\mu \otimes \left(\cancel{\frac{\partial}{\partial\xi^\mu}} + \tilde{A}_\mu - i x_\mu\right)V = 0$$

Matrix eq. !



$$V : 2k\times \underline{N}$$

Linear alg.

On a 4-torus→R^4

On the dual 4-torus→1 pt.

Atiyah-Drinfeld-Hitchin-Manin (ADHM) Construction

based on the following reciprocity

4dim. ASDYang-Mills eq.

$$F_{z_1\bar{z}_1} + F_{z_2\bar{z}_2} = 0$$

$$F_{z_1 z_2} = 0$$

N × N PDE

G(4dim D.eq.)

$$\rightarrow$$

$$\leftarrow$$

F(0dim D.eq.)

ADHM eq. (≒0dim. ASDYM)

$$[B_1, B_1^+] + [B_2, B_2^+] + I\,I^+ - J^+J = 0$$

$$[B_1, B_2] + IJ = 0$$

k × k matrix eq.

Sol.=instantons
(G=U(N), C_2 =k)

$$A_\mu : N \times N$$

1:1

$$\longleftrightarrow$$

Proved in the same way as the Nahm trf.

Sol.=ADHM data
(G=`U(k)’)

$$B_{1,2} : k \times k,$$

$$I : k \times N, \quad J : N \times k$$

Atiyah-Drinfeld-Hitchin-Manin (ADHM) Construction

based on the following reciprocity

4dim. ASDYang-Mills eq.

$$F_{z_1\bar{z}_1}+F_{z_2\bar{z}_2}=0$$

$$F_{z_1 z_2}=0$$

N × N PDE

G(4dim D.eq.) →

← F(0dim D.eq.)

ADHM eq.. (≒0dim. ASDYM)

RHS is in fact $[z_1,\bar{z}_1]+[z_2,\bar{z}_2]$

$$[B_1,B_1^+]+[B_2,B_2^+]+I\,I^+-J^+J=0$$

$$[B_1,B_2]+IJ=0$$

k × k matrix eq.

Sol.=instantons

(G=U(N), C_2 =k)

$$A_\mu : N\times N$$

1:1

↔

Proved in the same way as the Nahm trf.

Sol.=ADHM data

(G=`U(k)’)

$$B_{1,2} : k\times k,$$

$$I : k\times N,\quad J : N\times k$$

ADHM(Atiyah-Drinfeld-Hitchin-Manin) construction

Ex.) Commutative BPST instanton（N=2, k=1）

4dim. ASDYang-Mills eq.

$$F_{z_1\bar{z}_1} + F_{z_2\bar{z}_2} = 0$$

$$F_{z_1 z_2} = 0$$

N × N PDE

BPST instanton
(G=U(2), C_2 =1)

$$A_\mu = \frac{i(x-b)^\nu \eta_{\mu\nu}^{ASD}}{(z-\alpha)^2 + \rho^2}, \quad 2 \times 2$$

$$F_{\mu\nu} = \frac{2i\rho^2}{((z-\alpha)^2+\rho^2)^2} \eta_{\mu\nu}^{ASD}$$

1:1

↔

ADHM eq. (≒0dim. ASDYM)

$$[B_1, B_1^+] + [B_2, B_2^+] + I\,I^+ - J^+ J = 0$$

$$[B_1, B_2] + IJ = 0$$

k × k matrix eq.

Sol.=ADHM data
(G=`U(1)’)

$$B_{1,2} = \alpha_{1,2}, \quad 1 \times 1$$

$$I = (\rho, 0), \ J = \begin{pmatrix} 0 \\ \rho \end{pmatrix}$$

ADHM(Atiyah-Drinfeld-Hitchin-Manin) construction

Ex.) Commutative BPST instanton（N=2, k=1）

4dim. ASDYang-Mills eq.

ADHM eq. (≒0dim. ASDYM)

$$F_{z_1\bar{z}_1}+F_{z_2\bar{z}_2}=0$$

$$F_{z_1z_2}=0$$

N × N PDE

1:1

$$[B_1,B_1^+]+[B_2,B_2^+]+I\,I^+-J^+J=0$$

$$[B_1,B_2]+IJ=0$$

k × k matrix eq.

BPST instanton
(G=U(2), C_2 =1)

↔

Sol.=ADHM data
(G=`U(1)’)

$$A_\mu=\frac{i(x-b)^\nu\eta^{ASD}_{\mu\nu}}{(z-\alpha)^2+\rho^2},\quad 2\times 2$$

$$F_{\mu\nu}=\frac{2i\rho^2}{((z-\alpha)^2+\rho^2)^2}\eta^{ASD}_{\mu\nu}$$

$$B_{1,2}=\alpha_{1,2},\quad 1\times 1$$

$$I=(\rho,0),\ J=\begin{pmatrix}0\\ \rho\end{pmatrix}$$

``The first time abstract modern mathematics had been of any use!” **(Polyakov)**

ADHM(Atiyah-Drinfeld-Hitchin-Manin) construction

Ex.) Commutative BPST instanton（N=2, k=1）

4dim. ASDYang-Mills eq.

ADHM eq. (≒0dim. ASDYM)

$$F_{z_1\bar{z}_1}+F_{z_2\bar{z}_2}=0$$

$$F_{z_1 z_2}=0$$

N × N PDE

1:1

$$[B_1,B_1^+]+[B_2,B_2^+]+I\,I^+-J^+J=0$$

$$[B_1,B_2]+IJ=0$$

k × k matrix eq.

BPST instanton
(G=U(2), C_2 =1)

↔

Sol.=ADHM data
(G=`U(1)’)

$$A_\mu=\frac{i(x-b)^\nu\,\eta^{ASD}_{\mu\nu}}{(z-\alpha)^2+\rho^2},\qquad 2\times 2$$

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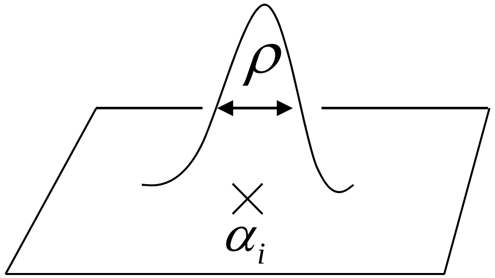


position

$$B_{1,2}=\overset{\updownarrow}{\alpha}_{1,2},\qquad 1\times 1$$

$$I=(\underset{\updownarrow}{\rho},0),\ J=\begin{pmatrix}0\\ \rho\end{pmatrix}$$

size

ADHM(Atiyah-Drinfeld-Hitchin-Manin) construction

Ex.) Commutative BPST instanton（N=2, k=1）

4dim. ASDYang-Mills eq.

$$F_{z_1\bar{z}_1}+F_{z_2\bar{z}_2}=0$$

$$F_{z_1 z_2}=0$$

N × N PDE

ADHM eq. (≒0dim. ASDYM)

$$[B_1,B_1^+]+[B_2,B_2^+]+I\,I^+-J^+J=0$$

$$[B_1,B_2]+IJ=0$$

k × k matrix eq.

1:1

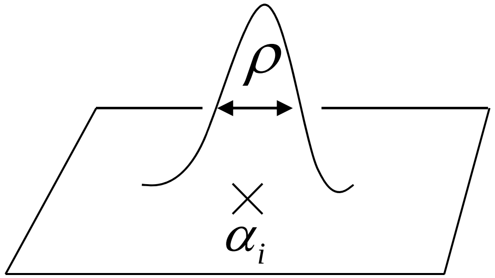
$$\longleftrightarrow$$

BPST instanton
(G=U(2), C2 =1)

Sol.=ADHM data
(G=`U(1)’)

$$A_\mu=\frac{i(x-b)^\nu\eta^{ASD}_{\mu\nu}}{(z-\alpha)^2+\rho^2},\quad 2\times 2$$

$$F_{\mu\nu}=\frac{2i\rho^2}{((z-\alpha)^2+\rho^2)^2}\eta^{ASD}_{\mu\nu}$$



$\rho\to 0$: singular

position

$$B_{1,2}=\alpha_{1,2},\quad 1\times 1$$

$$I=(\rho,0),\ J=\begin{pmatrix}0\\ \rho\end{pmatrix}$$

size

ADHM(Atiyah-Drinfeld-Hitchin-Manin) construction

Ex.) Commutative BPST instanton（N=2, k=1）

4dim. ASDYang-Mills eq.

$$F_{z_1\bar{z}_1} + F_{z_2\bar{z}_2} = 0$$

$$F_{z_1 z_2} = 0$$

N × N PDE

ADHM eq. (≒0dim. ASDYM)

$$\mu_R = [B_1, B_1^+] + [B_2, B_2^+] + I\,I^+ - J^+J = 0$$

$$\mu_C = [B_1, B_2] + IJ = 0$$

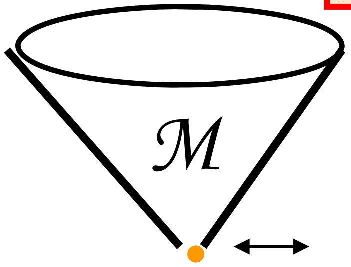
k × k matrix eq.

BPST instanton

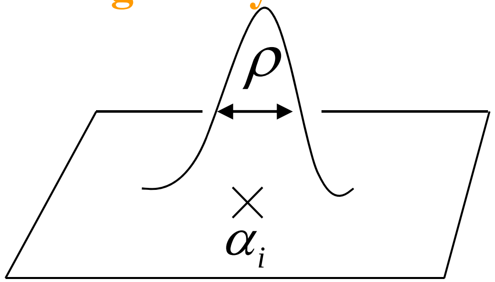
(G=U(2), C$_2$ =1)

$$A_\mu = \frac{i(x-b)^\nu \eta_{\mu\nu}^{ASD}}{(z-\alpha)^2 + \rho^2}, \qquad 2\times 2$$

$$F_{\mu\nu} = \frac{2i\rho^2}{((z-\alpha)^2+\rho^2)^2}\eta_{\mu\nu}^{ASD}$$



singularity $\longleftrightarrow$ $\rho = 0$



$\rho \to 0$ ：singular

Sol.=ADHM data

(G=`U(1)’)

position

$$B_{1,2} = \alpha_{1,2}, \quad 1\times 1$$

$$I = (\rho, 0),\ J = \begin{pmatrix} 0 \\ \rho \end{pmatrix}$$

size

ADHM(Atiyah-Drinfeld-Hitchin-Manin) construction

Ex.) NC BPST instanton（N=2, k=1）

NC ASDYang-Mills eq.

$$F_{z_1\bar{z}_1} + F_{z_2\bar{z}_2} = 0$$

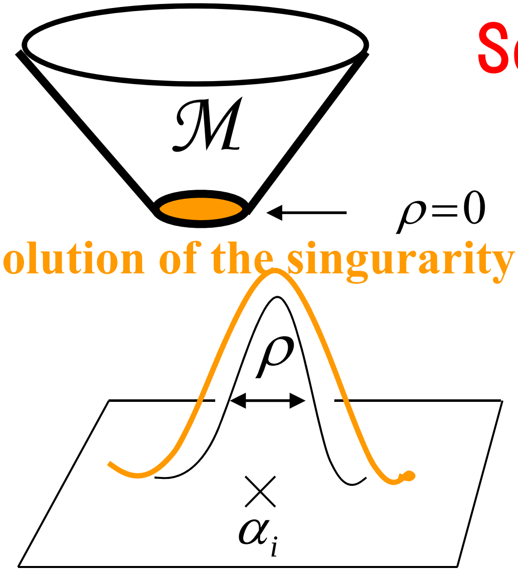
$$F_{z_1 z_2} = 0$$

N × N PDE

NC BPST instanton

(G=U(2), C₂ =1)

$A_\mu, F_{\mu\nu}$ **:exact sol.**



Resolution of the singurarity!

$\rho \to 0$: regular!

NC ADHM eq.

$$\mu_R = [B_1, B_1^+] + [B_2, B_2^+] + I\,I^+ - J^+ J = \zeta$$

$$\mu_C = [B_1, B_2] + IJ = 0$$

k × k marix eq.

Sol.：ADHM data

(G=`U(1)’)

position

↕

$$B_{1,2} = \alpha_{1,2},$$

$$I = (\sqrt{\rho^2 + \zeta}, 0),\ J = \begin{pmatrix} 0 \\ \rho \end{pmatrix}$$

↕

size Fat by ζ !

§3 4dim Wess-Zumino-Witten (WZW) model

[Donaldson '85]
[Losev-Moore-Nekrasov-Shatashvili '96]
[Inami-Kanno-Ueno-Xiong '96] ...

4-dim WZW (4dWZW) model

- analogue of 2-dim WZW model
- EOM = Yang's eq ≡ Anti-Self-Dual Yang-Mills eq.
  (ASD)
- In the split signature (+,+,−,−), SFT action of $N=2$ string theory [Ooguri-Vafa '91]

← Today we focus on

We discuss classical soliton sols. of it

⤴ implication
application

Action: $S_{WZW_4} = S_\sigma + S_{WZ}$

$\sigma(x) \in G$

↑

NOT G_{YM}

$$S_\sigma = \frac{i}{4\pi}\int_{M_4} \omega \wedge \mathrm{Tr}\left[(\partial\sigma)\sigma^{-1} \wedge (\tilde{\partial}\sigma)\sigma^{-1}\right]$$

$$S_{WZ} = -\frac{i}{12\pi}\int_{M_4} A \wedge \mathrm{Tr}\left[(d\sigma)\sigma^{-1}\right]^3$$

$(z, w, \tilde{z}, \tilde{w})$: local coords of M_4

w/ $\omega = dA$: Kähler form of M_4

M_4 : flat 4-dim space-time $\quad \omega = \frac{i}{2}(dz \wedge d\tilde{z} - dw \wedge d\tilde{w})$

$$d = \partial + \tilde{\partial}, \quad \partial = dw\,\partial_w + dz\,\partial_z, \quad \tilde{\partial} = d\tilde{w}\,\partial_{\tilde{w}} + d\tilde{z}\,\partial_{\tilde{z}}$$

EoM: $\tilde{\partial}\left(\omega \wedge (\partial\sigma)\sigma^{-1}\right) = 0 \Leftrightarrow$ Yang's eq.

⇕

ASDYM eq. !

N=2 string theory

# WS SUSY	Name	Target sp.	field contents
$N=0$	Bosonic String	$(1+25)$ dim	$g_{\mu\nu}, B_{\mu\nu}, \phi, \cdots$
$N=1$	Superstring	$(1+9)$ dim	〃 〃
$N=2$	$N=2$ string	$(2+2)$ dim	massless scalar only!

open $N=2$ string

$\sigma = e^{\varphi}$ ← the massless scalar

[Ooguri-Vafa, '91]

$$S_{WZW_4} = \underbrace{\left(\text{in terms of } \varphi\right)}_{\substack{||| \\ S_{N=2\ \text{string (SFT)}}}} \rightsquigarrow \text{n-pt. fcn of } \varphi$$

(coincides with WS calculations)

§4 Soliton Solutions of ASDYM eq.

Yang's eq. (on $\mathbb{C}^4$: complexified space-time)

$$\partial_{\tilde{z}}\left((\partial_z \sigma)\sigma^{-1}\right) - \partial_{\tilde{w}}\left((\partial_w \underset{\sim}{\sigma})\sigma^{-1}\right) = 0$$

$\in G = GL(N;\mathbb{C})$

※ Real slice

$(z, w, \tilde{z}, \tilde{w}) \in \mathbb{C}^4$, $\quad ds^2 = dz\,d\tilde{z} - dw\,d\tilde{w}$

① $\downarrow$ $z = x^1 + x^3,\ w = x^2 + x^4$

$\tilde{z} = x^1 - x^3,\ \tilde{w} = x^4 - x^2$

$\mathbb{R}^4(+,+,-,-)$

Ultrahyperbolic sp. $\mathbb{U}$

② $\downarrow$ $z = x^1 + i x^2,\ w = x^3 + i x^4$

$\tilde{z} = \bar{z},\ \tilde{w} = -\bar{w}$

$\mathbb{R}^4(++++)$

Euclid sp. $\mathbb{E}$

Lax representation :

N×N const matrix

$$(*)\begin{cases} Lf = \sigma\,\partial_w(\sigma^{-1}f) - (\partial_{\tilde z} f)\,\underset{\sim}{\zeta} = 0 & \left(\begin{array}{c}\text{right}\\ \text{action}\end{array}\right) \\ Mf = \sigma\,\partial_z(\sigma^{-1}f) - (\partial_{\tilde w} f)\,\underset{\sim}{\zeta} = 0 \end{cases}$$

compatible condition ⇒ Yang's eq.

$$L(M\phi) - M(L\phi) \doteq 0$$

Darboux trf. [Nimmo-Gilson-Ohta '00] [Gilson-H-Huang-Nimmo '20]

$$(D)\begin{cases} \tilde f = f\zeta - \theta\underset{\sim}{\Lambda}\theta^{-1}f \\ \tilde\sigma = -\theta\underset{\sim}{\Lambda}\theta^{-1}\sigma \end{cases}$$

$\underset{N\times N}{\theta}$: special sol. for $\underset{\sim}{\Lambda}$

special value

Under the Darboux trf. (*) is form invariant $\left(\text{i.e. } \begin{array}{l}\tilde L\tilde f = 0\\ \tilde M\tilde f = 0\end{array}\right.$

n-iterations of (D) from a trivial seed sol.

$(\sigma = 1)$

$$\underset{N\times N}{\sigma_n} = \begin{vmatrix} \overset{N\times N}{\theta_1} & \cdots & \theta_n & \overset{N\times N}{1} \\ \theta_1^{(1)} & \cdots & \theta_n^{(1)} & 0 \\ \vdots & & \vdots & \vdots \\ \theta_1^{(n-1)} & \cdots & \theta_n^{(n-1)} & 0 \\ \theta_1^{(n)} & \cdots & \theta_n^{(n)} & \boxed{0} \end{vmatrix}$$

$$\theta_k^{(\ell)} := \theta_k \Lambda_k^{\ell}$$

$$(\theta_i, \Lambda_i) : \partial_w \theta_i = \partial_{\tilde{z}} \theta_i \Lambda_i$$

$$\partial_z \theta_i = \partial_{\tilde{w}} \theta_i \Lambda_i$$

Wronskian-type!

Quasideterminant

$$\begin{vmatrix} A & B \\ C & \boxed{d} \end{vmatrix} := \underset{N\times N}{\overset{\text{squares}}{d}} - C \overset{\text{squares}}{A}{}^{-1} B \quad \text{(Schur complement)}$$

n-soliton sols. for $G = SL(2,\mathbb{C})$:

[H-Huang. '20]

$$\sigma_n = \begin{vmatrix} \theta_1 & \cdots & \theta_n & 1 \\ \theta_1^{(1)} & \cdots & \theta_n^{(1)} & 0 \\ \vdots & & \vdots & \vdots \\ \theta_1^{(n-1)} & \cdots & \theta_n^{(n-1)} & 0 \\ \theta_1^{(n)} & \cdots & \theta_n^{(n)} & \boxed{0} \end{vmatrix}$$

$$\theta_k = \begin{pmatrix} e^{L_k} & e^{-\bar{L}_k} \\ -e^{-L_k} & e^{\bar{L}_k} \end{pmatrix}, \quad \Lambda_k = \begin{pmatrix} \lambda_k & 0 \\ 0 & \mu_k \end{pmatrix}$$

$$L_k = \lambda_k \alpha_k z + \beta_j \tilde{z} + \lambda_j \beta_j w + \alpha_j \tilde{w}$$

(linear in space-time coord)

Rmk (U) $\mu_k = \bar{\lambda}_k$, $|\mu_k| = 1$

$\Rightarrow G = SU(2)$

(E) $\mu_k = -1/\bar{\lambda}_k$, $|\mu_k| = 1$

$\Rightarrow G = U(2)$

Non-abelian system

&

Calculate the WZW action density of them

One soliton (on $\mathbb{U}$)

$$\sigma = -\theta\wedge\theta^{-1},\quad \theta = \begin{pmatrix} e^{L} & e^{-\bar{L}} \\ -e^{-L} & e^{\bar{L}} \end{pmatrix},\ \Lambda = \underline{\begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix}}$$

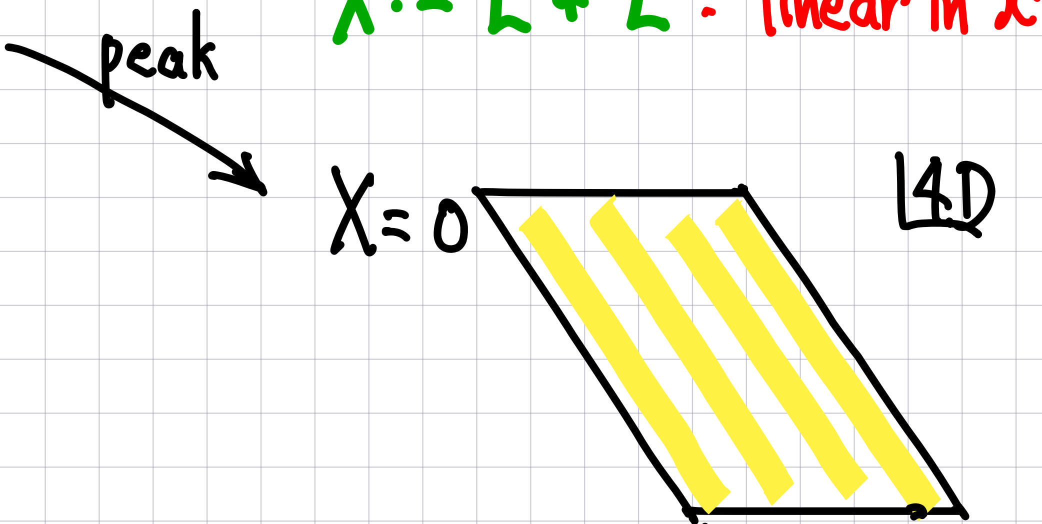
$$\operatorname{sech} x = \frac{1}{\cosh x}$$

$$\Downarrow$$

$$\mathcal{L}_{\sigma} = \frac{1}{8\pi}\underline{d_{11}}\operatorname{sech}^2 X$$

$$X := L + \bar{L}:$$ linear in x^μ

$$\mathcal{L}_{WZ} \equiv 0$$ (identically)



3-dim hyperplane

(codim 1)

not instanton!

One soliton (on Ⅲ)

※ $\lambda = \bar{\lambda} \Rightarrow \mathcal{L}_\sigma \equiv 0$

$$\sigma = -\theta \Lambda \theta^{-1}, \quad \theta = \begin{pmatrix} e^{L} & e^{-\bar{L}} \\ -e^{-L} & e^{\bar{L}} \end{pmatrix}, \quad \Lambda = \underbrace{\begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix}}$$

$$\operatorname{sech} x = \frac{1}{\cosh x}$$

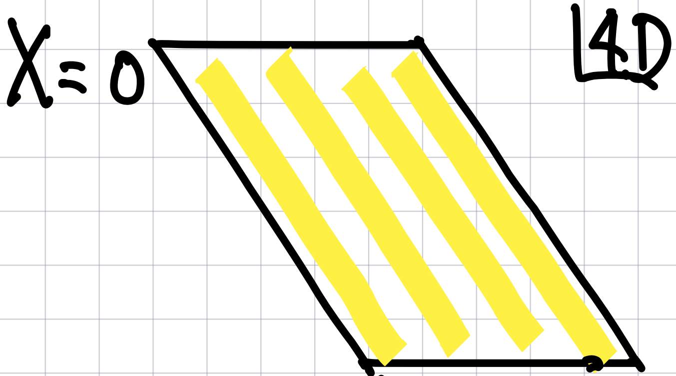
$$\Big\downarrow$$

$$\mathcal{L}_\sigma = \frac{1}{8\pi} \underbrace{d_{11}}_{\propto \, (\lambda - \bar{\lambda})^3} \operatorname{sech}^2 X$$

$X := L + \bar{L}$: linear in x^μ

$\mathcal{L}_{WZ} \equiv 0$ (identically)

peak



3-dim hyperplane
(codim 1)

not instanton!

Similar!

cf. KP soliton

$$u = 2\partial_x^2 \log\left(e^{X} + e^{-X}\right) \propto \operatorname{sech}^2 \underbrace{X}$$

linear in t, x, y

Two Soliton ($\mathcal{L}_\sigma$)

$$X_k = L_k + \bar{L}_k,\quad \Theta_{12} = \Theta_1 - \Theta_2$$
$$i\Theta_k = L_k - \bar{L}_k$$

$$\mathcal{L}_\sigma = \frac{\left[A\cosh^2 X_1 + B\cosh^2 X_2 + C_\pm \cosh^2\left(\frac{X_1 + X_2 \pm i\Theta_{12}}{2}\right) + D_\pm \cosh^2\left(\frac{X_1 - X_2 \pm i\Theta_{12}}{2}\right)\right]}{2\pi\left(a\cosh(X_1+X_2) + b\cosh(X_1-X_2) + c\cos\Theta_{12}\right)^2}$$

non-singular

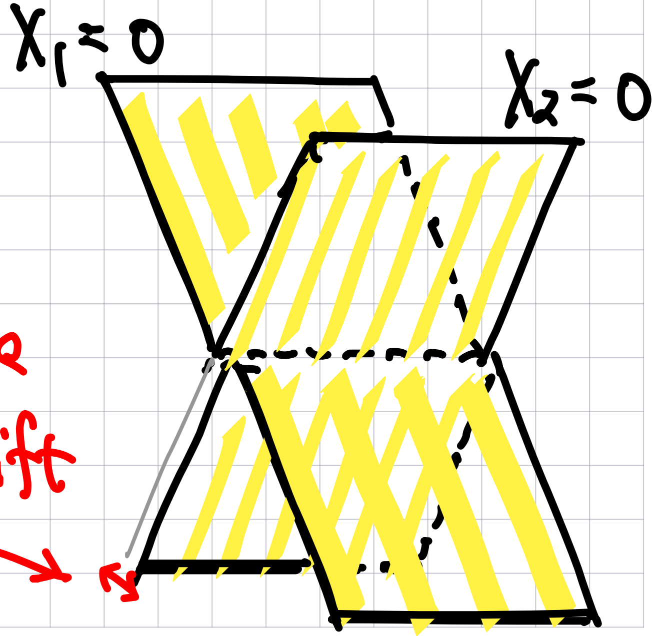
$$\xrightarrow[X_1:\,\mathrm{const}]{r\to\infty} \propto \operatorname{sech}^2(X_1 \pm \underset{\sim}{\delta_1})$$

$$\xrightarrow[X_2:\,\mathrm{const}]{r\to\infty} \propto \operatorname{sech}^2(X_2 \pm \underset{\sim}{\delta_2})$$

$$\xrightarrow[\mathrm{otherwise}]{r\to\infty} \quad 0$$

phase shift

(non-linear effect)



Two Soliton ($\mathcal{L}_{WZ}$)

$\mathcal{L}_{WZ}$ = (very long many terms) non-singular

$\xrightarrow{r\to\infty}$ 0 (in any direction)

$\mathcal{L}_{total} = \mathcal{L}_{\alpha}$ + ("dressing" in the middle region)

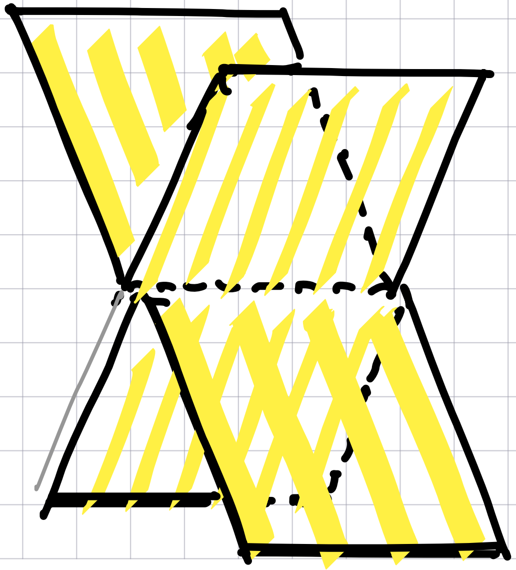
Two Soliton

$\mathcal{L}_{WZ}$ = (very long many terms) non-singular

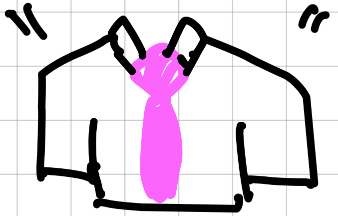
$\xrightarrow{r\to\infty}$ 0 (in any direction)

$\mathcal{L}_{total} = \mathcal{L}_{\alpha}$ + ("dressing" in the middle region)

=



+



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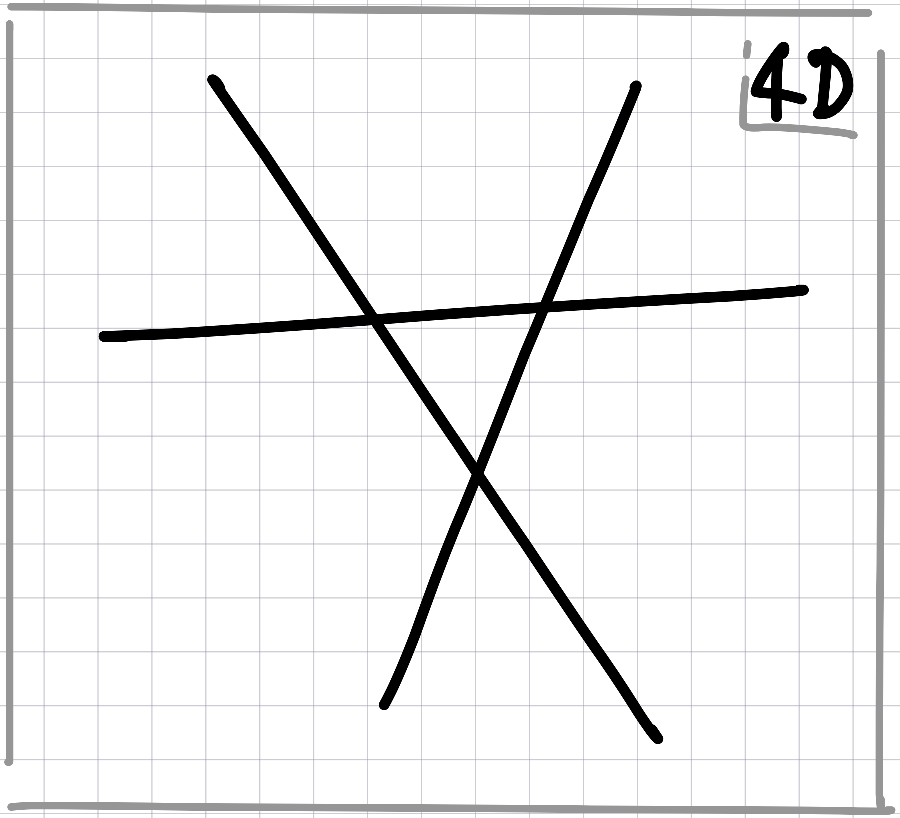
n- soliton sol. = "non-linear superposition of n one solitons [H-Huang '22]



intersecting n hyperplanes (with phase shifts)

Rmk 1 Reduction to (1+2) dim.

Consider $(x^1, x^2, x^3, x^4) \longrightarrow (\overset{\overset{t\,(\text{time})}{\shortparallel}}{x^1}, x^3, x^4)$

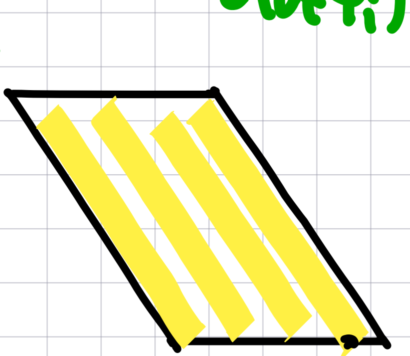
The soliton sol. $\sigma(\alpha_k = \lambda_k \beta_k)$ solves EoM in (1+2)d

$$\Updownarrow$$

∴ $L_k = (\lambda_k \alpha_k + \beta_k) x^1 + \underbrace{(\lambda_k \beta_k - \alpha_k)}_{=0} x^2 + \cdots$ ▨

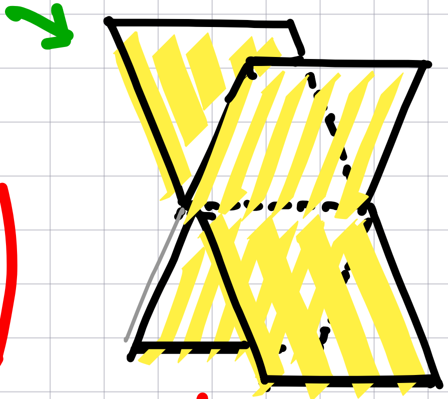
Hamiltonian $\mathcal{H} = \sum_{i=1}^{3} \frac{\partial \mathcal{L}}{\partial(\partial_t \phi_i)} \partial_t \phi_i - \mathcal{L}$ $(\mathcal{H}_{WZ} \equiv 0\,!)$

One soliton



Two soliton

(no dressing)



Energy density has the same peaks as action density.

Rmk 2 Euclidean case $\mathbb{E}$

The soliton sols. : almost the same as in $\mathbb{U}$

Instanton solution (well-known in YM)

(Ex) $G_{YM} = SU(2)$ 't Hooft 1-instanton

$$\mathcal{L}_a \propto \frac{(z\bar{z}+w\bar{w})^3}{\underline{(z\bar{z}w\bar{w})}^2(1+z\bar{z}+w\bar{w})^2}$$

$$\mathcal{L}_{wz} \propto \frac{(z\bar{z}+w\bar{w})(z\bar{z}-w\bar{w})^2}{\underline{(z\bar{z}w\bar{w})^2}(1+z\bar{z}+w\bar{w})^4}$$

↕ localized at the origin (codim 4)

singular ⇝ resolved in NC spaces?

§5 Conclusion and Discussion

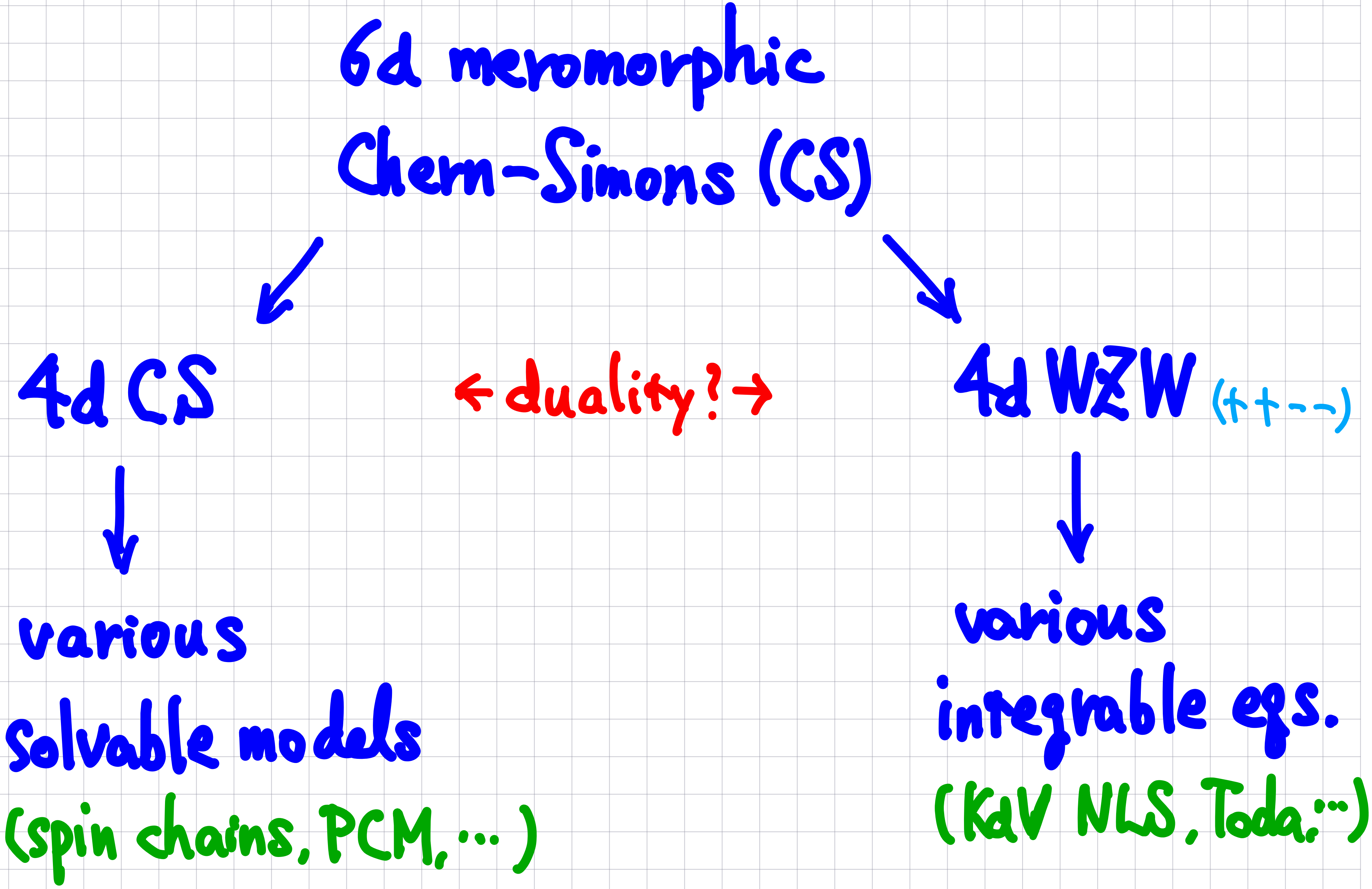
We constructed new-type of codim 1 solitons and calculated action densities of WZW_4 model.

$\rightsquigarrow$ intersecting 3-branes in the $N=2$ string
(new branes)

There are many things to be seen :

- Solitonic properties (charge, mass, moduli, ⋯)
- Classification of the "soliton planes" cf. [Kodama-Williams '14]
- Reduced systems (YMH, Hitchin system, Ernst eq. ⋯)

A Unified theory of integrable systems



6d meromorphic Chern-Simons (CS)

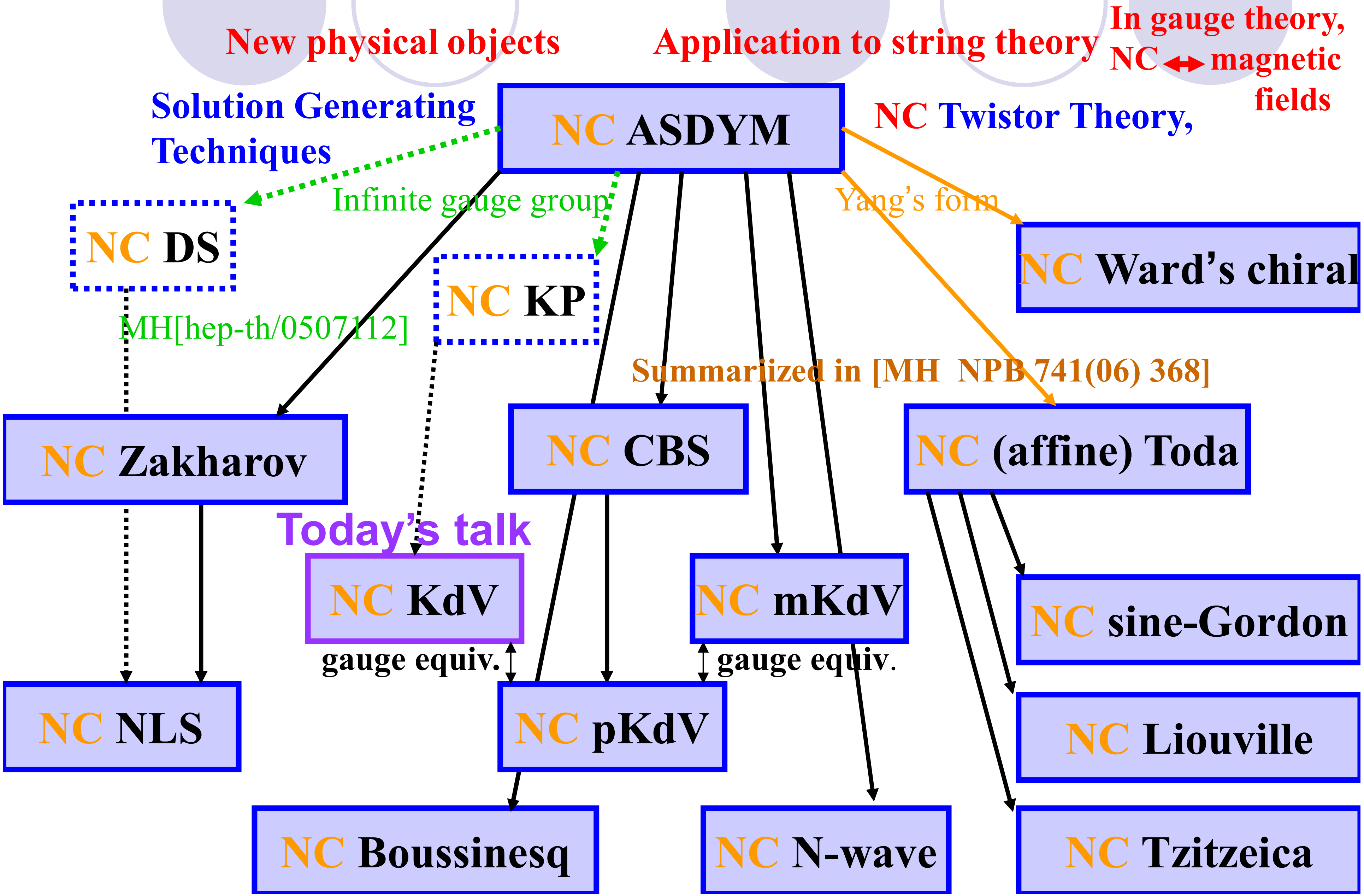
4d CS ← duality? → 4d WZW (++--)

various solvable models (spin chains, PCM, ⋯)

various integrable eqs. (KdV NLS, Toda, ⋯)

NC Ward's conjecture: Many (perhaps all?) NC integrable eqs are reductions of the NC ASDYM eqs.

MH &K.Toda, PLA316 (03)77 [hepth/0211148]



New physical objects

Application to string theory

In gauge theory, NC ↔ magnetic fields

NC Twistor Theory,

Solution Generating Techniques

NC ASDYM

Infinite gauge group

Yang's form

NC DS

NC KP

NC Ward's chiral

MH[hep-th/0507112]

Summariized in [MH NPB 741(06) 368]

NC Zakharov

NC CBS

NC (affine) Toda

Today's talk

NC KdV

NC mKdV

NC sine-Gordon

gauge equiv.

gauge equiv.

NC NLS

NC pKdV

NC Liouville

NC Boussinesq

NC N-wave

NC Tzitzeica

A Unified theory of NC integrable systems

6d meromorphic
NC Chern-Simons (CS)

↙ ↘

NC 4d CS ← duality? → NC 4d WZW (+ + - -)

key: Quasideterminant?

↓ ↓

various NC solvable models
(spin chains, PCM, ⋯)

various NC integrable eqs.
(KdV NLS, Toda,⋯)

Thank You Very Much

Merci!