(Non-)unitarity of massless higher spin fermions in de Sitter space

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Motivation to study field theory in de Sitter spacetime

- D-dimensional de Sitter spacetime (dS_D) $\rightarrow$ maximally symmetric solution of vacuum Einstein equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 0$$

  with positive cosmological constant $\Lambda = \frac{(D-2)(D-1)}{2\mathscr{R}^2}$
- $\mathscr{R}$ is the 'de Sitter radius'
- Relevant to inflationary cosmology and late-time cosmology.

Background material: dS spacetime as a hyperboloid

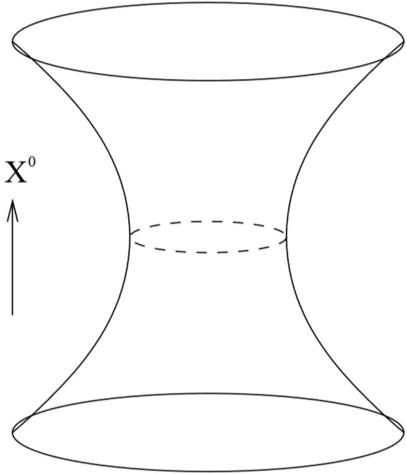
- dS_D can be represented as one-sheeted hyperboloid embedded in $\mathbb{R}^{D,1}$

$$-(X^0)^2 + (X^1)^2 + \ldots + (X^D)^2 = \mathscr{R}^2$$

- Often work in units with $\mathscr{R} = 1$.
- We can draw dS_2 embedded in 3D Minkowski

2D dS manifold embedded in 3D Minkowski

(Figure taken from ‘Les Houches Lectures on de Sitter Space’ by Spradlin, Strominger and Volovich)



- Hyperboloid picture manifests the isometry group $SO(D,1)$ (**dS group**)
- dS group has $D(D+1)/2$ generators (max. number of Killing vectors).
- Let us denote the dS algebra as $spin(D,1)$

- Following Wigner's classification in D-dimensional Minkowski spacetime, we expect: **Elementary particles in dS_D $\rightarrow$ UIR's** of the **de Sitter algebra spin$(D, 1)$**.

Mathematical question I am interested in

- Study realizations of UIRs of spin$(D, 1)$ on the classical solution space? $\sim$(i.e. one-particle Hilbert space from QFT viewpoint)
  Which free theories are unitary?
- Known for the case of bosons with arbitrary spin [Higuchi's PhD Thesis and Basile, Bekaert, Boulanger].
- This question has not yet been fully answered for higher-spin $s \geq 3/2$ fermions on dS_D.

Aim of the talk

- In this talk:
  • introduce useful rep-theoretic machinery that will uncover new features of spin-3/2 fermions on dS_D and
  • Main result:

The representation of $spin(D,1)$ $(D \geq 3)$ corresponding to the massless spin-3/2 field in dS_D is unitary only for $D = 4$.

Background material: massless spin-3/2 field in dS_D

In dS_D, the Rarita-Schwinger equations are:

$$\nabla\!\!\!/ \psi_\mu = -M\psi_\mu, \quad \gamma^\mu \psi_\mu = \nabla^\mu \psi_\mu = 0.$$

Gauge invariance occurs for imaginary M

$$M^2 = -\frac{(D-2)^2}{4}.$$

For this M, the field ψ_μ is a gauge potential, and we call it massless.

- On-shell gauge-invariance manifests itself as:

$$\delta\psi_\mu = \left(\nabla_\mu \pm \frac{i}{2}\gamma_\mu\right)\epsilon$$

where $\nabla\!\!\!/ \epsilon = \mp i\frac{D}{2}\epsilon$. (Below choose $M = +i(D-2)/2$.)

Our method consists of three steps:
i) Write down explicitly the mode solutions ψ_μ. Choose to work in global coordinates

$$ds^2 = -dt^2 + \cosh^2 t \, d\Omega_{D-1}^2, \ (S^{D-1} \text{ spatial slices})$$

ii) Determine the action of spin$(D,1)$ generators (i.e. Killing vectors ξ) on solutions:

- Use spinorial Lie derivative $\mathbb{L}_\xi \psi_\mu$.

Method - three basic steps

iii)Definition of unitarity: Do scalar products with the following properties exist?

- $(\mathbb{L}_\xi \psi, \psi') + (\psi, \mathbb{L}_\xi \psi') = 0$ (dS invariance / anti-hermitian generators)
- positive-definiteness of the norm.

Constructing massless spin-3/2 eigenmodes

- The first step is to construct the eigenmodes in global coordinates.
- $\psi_\mu(t, \boldsymbol{\theta}) = (\psi_t(t, \boldsymbol{\theta}), \psi_{\theta_{D-1}}(t, \boldsymbol{\theta}), ..., \psi_{\theta_1}(t, \boldsymbol{\theta}))$.
- The eigenmodes can be constructed using **separation of variables** $\sim$ (time dependence)$\times$(angular dependence).
- To be specific, express eigenmodes on dS_D in terms of (vector-)spinor spherical harmonics on S^{D-1}.

Constructing the gravitino eigenmodes on dS_D - even $D \geq 4$ - physical modes

Case 1: **even $D \geq 4$.**

Apply separation of variables.

Constructing physical modes: Vanishing t-component $\psi_t(t, \boldsymbol{\theta}) = 0$. Separate variables for non-zero components as:

$$\psi_j^{(phys;-\ell,\underline{m})}(t,\boldsymbol{\theta}) = \begin{pmatrix} \phi_\ell(t)\,\tilde{\psi}^{(\ell,\underline{m})}_{(-)j}(\boldsymbol{\theta}) \\ -i\,\psi_\ell(t)\,\tilde{\psi}^{(\ell,\underline{m})}_{(-)j}(\boldsymbol{\theta}) \end{pmatrix},$$

$$\psi_j^{(phys;+\ell,\underline{m})}(t,\boldsymbol{\theta}) = \begin{pmatrix} i\psi_\ell(t)\,\tilde{\psi}^{(\ell,\underline{m})}_{(+)j}(\boldsymbol{\theta}) \\ -\phi_\ell(t)\tilde{\psi}^{(\ell,\underline{m})}_{(+)j}(\boldsymbol{\theta}) \end{pmatrix},$$

where $\tilde{\psi}^{(\ell,\underline{m})}_{(-)j}, \tilde{\psi}^{(\ell,\underline{m})}_{(+)j}$ are spherical harmonics on S^{D-1}.

Constructing the gravitino eigenmodes on dS_D - even $D \geq 4$ - physical modes

From a representation-theoretic viewpoint: we study spin$(D,1)$ representations in the decomposition spin$(D,1) \supset$ spin(D). The spin(D) content of the representation is understood from the vector-spinor spherical harmonics:

- Two inequivalent vec-spin *spin*(D) irreps for even D
- 1) $\tilde{\nabla}\!\!\!/\,\tilde{\psi}^{(\ell,\underline{m})}_{(+)j}(\boldsymbol{\theta}) = +i\left(\ell + \frac{D-1}{2}\right)\tilde{\psi}^{(\ell,\underline{m})}_{(+)j}$ with highest weight $\vec{f}^{+} = (\ell + \frac{1}{2}, \frac{3}{2}, \frac{1}{2}, ..., \frac{1}{2}, +\frac{1}{2})$
- 2) $\tilde{\nabla}\!\!\!/\,\tilde{\psi}^{(\ell,\underline{m})}_{(-)j}(\boldsymbol{\theta}) = -i\left(\ell + \frac{D-1}{2}\right)\tilde{\psi}^{(\ell,\underline{m})}_{(-)j}$ with highest weight $\vec{f}^{-} = (\ell + \frac{1}{2}, \frac{3}{2}, \frac{1}{2}, ..., \frac{1}{2}, -\frac{1}{2})$, $\ell = 1, 2, ...$

Now, we will study the transformation properties under $spin(D,1)$:

✓i) Write down explicitly the gravitino eigenmodes.

→ ii) Study their transformation properties under $spin(D,1)$ using the spinorial Lie derivative (i.e. action of $spin(D,1)$ generators on eigenmodes)

$$\mathbb{L}_\xi \psi_\mu = \xi^\nu \nabla_\nu \psi_\mu + \xi^\nu \nabla_\mu \psi_\nu + \frac{1}{4} \nabla_\kappa \xi_\lambda \gamma^\kappa \gamma^\lambda \psi_\mu.$$

Transformation properties under spin$(D,1)$ - even $D \geq 4$

- Focus on transformation generated by the boost Killing vector

$$\xi^\mu \partial_\mu = \cos\theta_{D-1} \frac{\partial}{\partial t} - \tanh t \, \sin\theta_{D-1} \frac{\partial}{\partial \theta_{D-1}}.$$

- the dS algebra *spin*$(D,1)$ is generated by ξ^μ and the *spin*(D) Killing vectors $\rightarrow$ [Boost , Rotation] $=$ Boost'.

Transformation properties under spin$(D, 1)$ - even $D \geq 4$

After a (very) long calculation, we find:
dS boost on physical modes:

$$\mathbb{L}_\xi \psi_\mu^{(phys;\boldsymbol{\sigma\ell},\underline{m})} = -\underbrace{i\,M}_{\uparrow real}(D-4)\,c_{\ell,\underline{m}}\,\psi_\mu^{(phys;\boldsymbol{-\sigma\ell},\underline{m})} + \ldots$$

where $i\,M = -(D-2)/2$ and $c_{\ell,\underline{m}} \in \mathbb{R} \setminus \{0\}$.
The terms in '...' contain eigenmodes that must always be orthogonal to $\psi_\mu^{(phys;\pm\boldsymbol{\ell},\underline{m})}$ with resect to any dS invariant scalar product (because of different *spin*(D) content)

Transformation properties under spin$(D,1)$ - even $D \geq 4$

What can we learn from this expression?

For even $D > 4$:

- Under a dS boost, modes with $\sigma = +$ (i.e. *spin*(D) content $\vec{f}^{+}$) always mix with eigenmodes with $\sigma = -$ (i.e. *spin*(D) content $\vec{f}^{-}$).

For $D = 4$: The rep. is reducible

- The modes with $\sigma = +$ (positive helicity) and the ones with $\sigma = -$ (negative helicity) do not mix under the dS boost.
- The two sets $\{\psi_{\mu}^{(phys;-\ell,\underline{m})}\}$ and $\{\psi_{\mu}^{(phys;+\ell,\underline{m})}\}$ separately form irreps. of *spin*$(4,1)$.

Now, we have to check unitarity.

✓i) Write down explicitly the gravitino eigenmodes.

✓ii) Study their transformation properties under *spin*$(D,1)$.

$\rightarrow$ iii)Unitarity: Does a scalar product with the following properties exist?

- $(\mathbb{L}_\xi \psi^{(1)}, \psi^{(2)}) + (\psi^{(1)}, \mathbb{L}_\xi \psi^{(2)}) = 0$ (dS invariance)
- positive-definiteness of the norm.

Check (non-)unitarity - even $D \geq 4$

• Suppose that $(\psi^{(1)}, \psi^{(2)})$ is a dS invariant scalar product for the gravitino eigenmodes.

• dS invariance means $(\mathbb{L}_\xi \psi^{(1)}, \psi^{(2)}) + (\psi^{(1)}, \mathbb{L}_\xi \psi^{(2)}) = 0$ for any two modes and any Killing vector ξ.

• Let us apply our transformation formulae for the dS boost.

Check (non-)unitarity - even $D \geq 4$

- Requiring dS invariance of $(\psi^{(\textbf{phys};+\ell,\underline{m})}, \psi^{(\textbf{phys};-\ell,\underline{m})})$:

$$(\mathbb{L}_\xi \psi^{(\textbf{phys};+\ell,\underline{m})}, \psi^{(\textbf{phys};-\ell,\underline{m})}) + (\psi^{(\textbf{phys};+\ell,\underline{m})}, \mathbb{L}_\xi \psi^{(\textbf{phys};-\ell,\underline{m})}) \stackrel{!}{=} 0$$

Use:

$$\mathbb{L}_\xi \psi_\mu^{(phys;\pm\ell,\underline{m})} = -\, i\, M\, (D-4)\, c_{\ell,\underline{m}}\, \psi_\mu^{(phys;\mp\ell,\underline{m})} + \dots$$

- We find

$$(D-4)\times\left(\left(\psi^{(\textbf{\textit{phys}};-\ell,\underline{m})},\psi^{(\textbf{\textit{phys}};-\ell,\underline{m})}\right)\right.$$
$$\left.+\left(\psi^{(\textbf{\textit{phys}};+\ell,\underline{m})},\psi^{(\textbf{\textit{phys}};+\ell,\underline{m})}\right)\right)\stackrel{!}{=}0$$

$\Rightarrow$ Negative norms are unavoidable unless $D=4$!

Conclusion for even $D \geq 4$

- We conclude that for even $D > 4$ the gravitino field theory is not unitary.
- For $D = 4$, the gravitino field is unitary.

Case 2: D odd

- Let us discuss briefly the case with odd $D \geq 3$.

Case 2: odd $D \geq 3$

- Follow the same steps as in the case with D even:

i) Find gravitino eigenmodes (separation of variables).
ii) Study their transformation properties under $spin(D, 1)$.
iii) Check (non-)unitarity (i.e. existence of a positive-definite and dS invariant scalar product.)

Non-unitarity for odd N

- there is no dS invariant scalar product that is not identically zero for the gravitino modes $\Rightarrow$ non-unitary rep.

Main result

Thus, we arrive at our main result:

The gravitino field theory on dS_D ($D \geq 3$) is not unitary unless $D = 4$.

Final remarks

- The dimensionality $D = 4$ that plays a special role coincides with the dimensionality of our physical Universe.
- This feature of dS field theory (i.e. dimension decides unitarity) does not appear in Minkowski nor in Anti-de Sitter spacetimes.

Final remarks

- The results presented in this talk can be also obtained by studying the classification of the UIR's of $spin(D,1)$ (see my JHEP paper https://doi.org/10.1007/JHEP05(2023)015.)
- The classification of the UIR's of $spin(D,1)$ suggests that the **main result extends to all strictly/partially massless fermionic reps. with arbitrary spin** $s \geq 3/2$.
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Current work

- Canonical quantization of the spin-3/2 theory on dS_4 and study UIRs in the QFT Hilbert space (with Anninos, Fukelman, Silva, and Sempe)

Thank you!

Thank you!