Scattering Theory of Quantum N-particle systems

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N-particle Hamiltonians

- Consider a system of N non-relativistic particles in $\mathbb{R}^\nu$.
- configuration space: $X = X_1 \times \cdots \times X_N$, $X_i = \mathbb{R}^\nu$, with $x = (x_1, \cdots, x_N)$,
- x_i, $D_i = \mathrm{i}^{-1}\partial_{x_i}$ position , momentum of particle i, $D = (D_1, \cdots, D_N)$ $X^\#$-valued selfadjoint operator.
- Hilbert space

$$\mathcal{H} = \otimes_{i=1}^N L^2(X_i) = L^2(X).$$

- statistics of the particles easily incorporated and will be forgotten.

N-particle Hamiltonians

- $N-$particle Hamiltonian:

$$H = \sum_{i=1}^{N} -\frac{1}{2m_i}\Delta_i + \sum_{i<j} v_{ij}(x_i - x_j),$$

m_i mass of particle i, $v_{ij} : \mathbb{R}^\nu \to \mathbb{R}$ interaction potential between particles i and j.

Collision planes

- Inside X subspaces where two or more particles collide are very important.
- If $1 \leq i \neq j \leq N$ we set

$$X_{(ij)} := \{x \in X : x_i = x_j\}.$$

- complete the set of collision subspaces by intersections.
- one obtains a family of subspaces X_a, indexed by the set $\mathcal{A}$ of partitions of $\{1, \dots, N\}$.
- $a = \{C_1, \dots, C_k\}$, the C_i correspond to clusters of a.
- write $(ij) \leq a$ if x_i and x_j are in the same cluster of a and set

$$X_a = \{x \in X : x_i = x_j \text{ if } (ij) \leq a\}.$$

Collision planes

- set $\mathcal{A}$ is equipped with an order relation defined by $a \leq b$ if $X_a \supset X_b$ (a is finer than b).
- minimal partition is $a_{\min} = \{\{1\}, \ldots, \{N\}\}$
- maximal partition is $a_{\max} = \{\{1, \ldots, N\}\}$.
- associated subspaces are

$$X_{a_{\min}} = X, \; X_{a_{\max}} = \{x \in X : x_i = x_j \; \forall i, j\}.$$

Separation of the center of mass

- H commutes with translations:

$$u(x_1, \cdots, x_n) \mapsto u(x_1 - v, \cdots, x_n - v), \ v \in \mathbb{R}^d,$$

- equivalently H commutes with translations $\mathrm{e}^{\mathrm{i}y\cdot D_x}$ for $y \in X_{a_{\max}}$
- we can write $\mathcal{H}$ and H as direct integrals:

$$\mathcal{H} = \int_{\mathbb{R}^{\nu *}}^{\oplus} \mathcal{H}(p)dp, \ H = \int_{\mathbb{R}^{\nu *}} H(p)dp.$$

- explicit version of $\mathcal{H}(p)$ and $H(p)$: (in the old times, done with 'Jacobi coordinates').
- kinetic part in H equal to $\frac{1}{2}D_x^2$ for $\xi \cdot \xi = \sum_{i=1}^{N} \frac{1}{m_i}\xi_i^2, \ \xi \in X^{\#}$.

Separation of the center of mass

- $\xi\cdot\xi$ dual (aka inverse) of $x\cdot x=\sum_{i=1}^{N} m_i x_i^2$.
- Set $X^{a_{\max}}=X_{a_{\max}}^{\perp}$, identify $X\sim X^{a_{\max}}\oplus X_{a_{\max}}$.
- we get

$$V(x)=\sum_{i=1}^{N} v_{ij}(x_i-x_j)=:V(x^{a_{\max}})$$

- $D_x^2=(D_{x^{a_{\max}}})^2+(D_{x_{a_{\max}}})^2$.
- we take $p=\xi_{a_{\max}}$:

$$\mathcal{H}(\xi_{a_{\max}})=L^2(X^{a_{\max}}),$$

$$H(\xi_{a_{\max}})=H^{a_{\max}}+\frac{1}{2}(\xi_{a_{\max}})^2$$

$$H^{a_{\max}}:=\frac{1}{2}(D_{x^{a_{\max}}})^2+V(x^{a_{\max}}).$$

Agmon Hamiltonians

- fix a finite dimensional Euclidean space X (automatically equipped with a Lebesgue measure).
- fix a family $\{X_a : a \in \mathcal{A}\}$ of subspaces of X closed under intersections and containing X.
- set $a \leq b$ if $X_a \supset X_b$, $a, b \in \mathcal{A}$.
  so $X_{a_{\min}} = X$ and $X_{a_{\max}} = \bigcap_{a\in\mathcal{A}} X_a$.
- we can assume that $X_{a_{\max}} = \{0\}$. (If not separate the 'center of mass' as explained above).
- 'number of particles': consider a chain $a_1 < \cdots < a_k$ connecting $a_1 = a_{\min}$ to $a_k = a_{\max}$.
- number of particles is the maximal length of such chains. (reflects the complexity of the lattice of subspaces $\{X_a : a \in \mathcal{A}\}$.

Agmon Hamiltonians

- set

$$X^a := X_a^\perp, \ x = x^a + x_a, \ a \in \mathcal{A}.$$

- by duality we can similarly split

$$X^\# = X^{a\#} \oplus^\perp X_a^\#, \ \xi = \xi^a + \xi_a.$$

- for $a \in \mathcal{A}$ we fix a real function $v^a : X \to \mathbb{R}$ such that

$$v^a(x) = v^a(x + y_a), \ \forall y_a \in \mathcal{X}_a.$$

- a Agmon Hamiltonian is

$$H = \frac{1}{2} D_x^2 + \sum_{a \in \mathcal{A}} v^a, \text{ acting on } \mathcal{H} = L^2(X).$$

Agmon Hamiltonians

- for $a \in \mathcal{A}$ set

$$V^a(x) = \sum_{b \leq a} v^b(x), \text{ function on } X^a,$$

- and

$$H^a = \frac{1}{2}(D_x^a)^2 + V^a(x^a), \text{ acting on } \mathcal{H}^a = L^2(X^a).$$

Since $X = X^{a_{\max}}$ we have $H = H^{a_{\max}}$.

Agmon Hamiltonians

- we have

$$H = H_a + I_a,$$

$$H_a = H^a + \frac{1}{2}D_{x_a}^2, \;\; I_a(x) = \sum_{b \neq a} v_b(x).$$

- H_a describes the non interacting clusters of a, whose centers of masses move freely.
- I_a is the intercluster potential.

Agmon Hamiltonians

Advantages of this framework:

- notational and conceptual simplification.
- easy to incorporate particles of infinite masses (very heavy nuclei):
- add to the above family the subspaces $\{x \in \mathbb{R}^{Nd} : x_j = 0\}$ and their intersections,
- on can consider also multi-particle interactions, for example with potential $v_{ijk}(x_i - x_j, x_j - x_k)$, for $v_{ijk} : \mathbb{R}^{2d} \to \mathbb{R}$, associated to the subspace $X_{(ijk)} = \{x \in X : x_i = x_j = x_k\}$.

Pair potentials

- typical $2-$body potentials decay near infinity and have a Coulomb type singularity at 0.
- a natural assumption: $v^a(x^a)(-\Delta^a+1)^{-1}$ is compact on $L^2(X^a)$.
- note that $v^a(x^a)(-\Delta+1)^{-1}$ not compact on $L^2(X)$ (unless $a=a_{\max}$). By Kato-Rellich we obtain:

Theorem

H with domain $H^2(X)$ is selfadjoint and bounded from below on $L^2(X)$.

- for standard $N-$body Hamiltonians with Coulomb interactions $v_{ij}(x)=\dfrac{q_iq_j}{|x|}$ and $d=3$ first important result of Kato ('stability of matter of the first kind').

The HVZ theorem

- describes the essential spectrum of H (important result from the 60's, nowadays very easy to prove).
- important role played by the thresholds:
- the set of thresholds of a subsystem $a \in \mathcal{A}$ is

$$\mathcal{T}^a := \bigcup_{b<a} \sigma_{\rm pp}(H^b).$$

- $\mathcal{T}^{a_{\max}}$ will be simply denoted by $\mathcal{T}$.
- set also $\Sigma^a := \inf(\mathcal{T}^a)$ and $\Sigma := \Sigma^{a_{\max}} = \inf(\mathcal{T})$.
- easy to show using trial functions (the 'variational argument') that $\inf \sigma(H^a) \leq \mathcal{T}^a$.

- by energy conservation, $\Sigma^a - \Sigma$ is the minimal energy needed to decompose the system into freely moving clusters of a.
- $-\Sigma$ is the minimal energy needed to fully decompose the system.
  Note that $\sigma_{\rm pp}(H^a_{\rm min}) = \{0\}$ hence $\Sigma \leq 0$.

Theorem

Let H be an $N-$particle Hamiltonian with $X_{a_{\rm max}} = \{0\}$. Then the essential spectrum of H is equal to

$$\sigma_{\rm ess}(H) = [\Sigma, \infty[.$$

Partitions of unity

We split the configuration space X into regions describing different cluster decompositions.

- set
$$Z_a = X_a \setminus \cup_{b \not\leq a} X_b.$$
- $\{Z_a\}_{a \in \mathcal{A}}$ is a partition of X.
- thicken the Z_a into
$$Z_a^{\epsilon,\delta} = \{x \in X : |x^a| < \epsilon, \ |x^b| \geq \delta \text{ for } b \not\leq a\},$$
- construct a partition of unity $\{q_a\}_{a \in \mathcal{A}}$ such that
$$\operatorname{supp} q_a \subset Z_a^{\epsilon,\delta}, \qquad \sum_{a \in \mathcal{A}} q_a(x) = 1,$$
$$|\partial_x^\alpha q_a(x)| \leq C_\alpha, \quad 0 \leq q_a(x) \leq 1.$$

Functional calculus formula

recall the celebrated Hellfer-Sjöstrand formula:

- let $\chi \in C_0^\infty(\mathbb{R})$. Then there exist a function $\tilde{\chi} \in C_0^\infty(\mathbb{C})$, called an almost-analytic extension of χ, such that
-
$$\tilde{\chi}\big|_{\mathbb{R}} = \chi, \quad |\frac{\partial \tilde{\chi}}{\partial \overline{z}}(z)| \leq C_N |\mathrm{Im} z|^N, \quad N \in \mathbb{N}.$$
- if H selfadjoint operator on a Hilbert space $\mathcal{H}$ then

$$\chi(H) = \frac{i}{2\pi} \int_{\mathbb{C}} \frac{\partial \tilde{\chi}}{\partial \overline{z}}(z)(z - H)^{-1} \mathrm{d}z \wedge \mathrm{d}\overline{z}.$$

Proof of the HVZ theorem

- prove by induction on k that $\sigma_{\rm ess}(H^a) = [\Sigma^a, +\infty[$ for all a with $\sharp a \leq k$.
- the hard part is $\subset$ ($\supset$ proved with Weyl sequences).
- assume that the theorem holds for all $a < a_{\max}$ and consider $H = H^{a_{\max}}$.
- choose $\chi \in C_0^\infty(\mathbb{R})$ with $\operatorname{supp}\chi \subset]-\infty, \Sigma[$. Then

$$\chi(H) = \sum_{a \in \mathcal{A}} \chi(H) q_a(\frac{x}{R}), \ \ R \gg 1.$$

- on support of q_a, the interaction potential I_a is $o(R^0)$, so we can replace $\chi(H)$ by $\chi(H_a)$ modulo a small error (use HS formula).

Proof of the HVZ theorem

- one gets
$$\chi(H) = \sum_{a\in\mathcal{A}} \chi(H_a) q_a(\frac{x}{R}) + o(R^0).$$
- since $H_a = H^a + \frac{1}{2}D_{x_a}^2$, $\chi(H_a) = 0$ for $a \neq a_{\max}$ because of support of χ,
- $\chi(H) q_{a_{\max}}(\frac{x}{R})$ compact for any χ, because $q_{a_{\max}}$ has compact support.
- Therefore $\chi(H)$ is compact (norm limit of compact operators).

The Mourre estimate

- study of the nature of the essential spectrum of H revolutionized in the 80's by the Mourre method (positivity of a commutator).
- let H, A be two selfadjoint operators on a Hilbert space $\mathcal{H}$.
- one requires that H is of class $C^1(A)$, ie $\mathbb{R} \ni t \mapsto \mathrm{e}^{\mathrm{i}tA}(H+\mathrm{i})^{-1}\mathrm{e}^{-\mathrm{i}tA}$ is strongly C^1.
- the commutator $[H, \mathrm{i}A]$ makes then sense as a bounded hermitian form on $\mathrm{Dom}\, H$.
- the Mourre estimate holds at $\lambda \in \mathbb{R}$ if there exists an open interval $\Delta \ni \lambda$, $c > 0$ and K compact such that:

$$\mathbb{1}_{\Delta}(H)[H, \mathrm{i}A]\mathbb{1}_{\Delta}(H) \geq c\mathbb{1}_{\Delta}(H) + K. \tag{3.1}$$

- the strict Mourre estimate holds at λ if one can take $K = 0$.

The Mourre estimate

- assuming higher regularity of H w.r.t. $\mathrm{e}^{\mathrm{i}tA}$ one deduces from the strict Mourre estimate at λ weighted estimates on $(H-\lambda\mp\mathrm{i}0)^{-1}$.(original motivation of the Mourre method).
- these resolvent estimates are not necessary for the time-dependent scattering theory that we will describe here.

The virial theorem

- the virial theorem states that

$$\mathbb{1}_{\{\lambda\}}(H)[H, \mathrm{i}A]\mathbb{1}_{\{\lambda\}}(H) = 0.$$

- formally obvious by 'undoing' the commutator.
- rigorous proof requires a lot of care, since A is unbounded.
- first consequence: if the Mourre estimate holds at λ then the eigenvalues of H cannot accumulate at λ.
- second consequence: if the Mourre estimate holds at λ and $\lambda \notin \sigma_{\mathrm{pp}}(H)$ then the strict Mourre estimate holds at λ.

Best constant in the Mourre estimate

- nice polishing of various arguments, due to Amrein-Boutet de Monvel-Georgescu .
- set $\rho(\lambda)$= larger c such that

$$\mathbb{1}_{\Delta}(H)[H, \mathrm{i}A]\mathbb{1}_{\Delta}(H) \geq c\mathbb{1}_{\Delta}(H).$$

- $\rho(\lambda)$ best constant in the strict Mourre estimate at λ.
- define $\tilde{\rho}(\lambda)$ similarly, adding a compact error term.

Best constant in the Mourre estimate

virial theorem can be rephrased by stating that

- if $\lambda \in \sigma_{\mathrm{pp}}(H)$ and $\tilde{\rho}(\lambda) > 0$, then $\rho(\lambda) = 0$.
- if $\lambda \notin \sigma_{\mathrm{pp}}(H)$ then $\rho(\lambda) = \tilde{\rho}(\lambda)$.
- if $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, $H = H_1 \otimes 1\!\!1 + 1\!\!1 \otimes H_2$, $A = A_1 \otimes 1\!\!1 + 1\!\!1 \otimes A_2$, and H_i bounded from below then:

$$\rho(\lambda) = \inf_{\lambda_1 + \lambda_2 = \lambda} \rho_1(\lambda_1) + \rho_2(\lambda_2).$$

- (looks easy but rather tricky to prove).

Mourre estimate for N-body Hamiltonians

- set $A = \frac{1}{2}(x \cdot D_x + D_x \cdot x)$ (generator of dilations).
- the expression $x\nabla_x V$ is understood as $[V, \mathrm{i}A]$ (allows to 'undo the commutator' if needed).
- stronger assumption:

$$(1-\Delta^a)^{-1} x^a \nabla_{x^a} v^a (1-\Delta^a)^{-1} \text{compact on } L^2(X^a),\ a \in \mathcal{A}.$$

Theorem

For $\lambda \in [\Sigma, \infty[$, let $d(\lambda) := \inf\{\lambda - \tau \mid \tau \leq \lambda, \tau \in \mathcal{T}\}$. Then for any $\epsilon > 0$, $\lambda \in [\Sigma, \infty[$, there exists an open interval Δ containing λ and a compact operator K such that

$$\mathbb{1}_\Delta(H)[H, \mathrm{i}A]\mathbb{1}_\Delta(H) \geq 2\,(d(\lambda) - \epsilon)\,\mathbb{1}_\Delta(H) + K. \tag{3.2}$$

Mourre estimate for N-body Hamiltonians

- $\mathcal{T}$ is a closed countable set and $\sigma_{\rm pp}(H)$ can accumulate only at $\mathcal{T}$.
- with terminology introduced above (3.2) means that $\tilde{\rho}(\lambda) \geq 2d(\lambda)$.
- Using trial functions one can show that $\tilde{\rho}(\lambda) = 2d(\lambda)$.

Idea of proof

- applying recursively the abstract theory to H^a for $a < a_{\max}$ gives that $\mathcal{T}$ is a closed countable set.
- it suffices prove Mourre estimate by induction on $\sharp a_{\max}$. If $\sharp a_{\max} = 1$ then $H = \frac{1}{2}D_x^2$, $[H, \mathrm{i}A] = D_x^2$.
- assume that Mourre estimate holds for all H^a with $a \neq a_{\max}$.
- use a partition of unity as before but with $\sum_{a\in\mathcal{A}} q_a^2(x) = 1$.
- controlling double commutator terms gives

$$\chi(H)[H, \mathrm{i}A]\chi(H) = \sum_{a\in\mathcal{A}} q_a(\frac{x}{R})\chi(H)[H, \mathrm{i}A]\chi(H)q_a(\frac{x}{R}) + O(R^{-2}).$$

Idea of proof

- on support of q_a, one can replace H by H_a modulo small errors, so

$$\chi(H)[H, \mathrm{i}A]\chi(H) = \sum_{a\in\mathcal{A}} q_a(\frac{x}{R})\chi(H_a)[H_a, \mathrm{i}A]\chi(H_a)q_a(\frac{x}{R}) + o(R^0). \tag{3.3}$$

- write $L^2(X) = L^2(X^a) \otimes L^2(X_a)$ so $H_a = H^a \otimes \mathbb{1} + \mathbb{1} \otimes \frac{1}{2}D^2_{x_a}$, $A = A^a \otimes \mathbb{1} + \mathbb{1} \otimes A_a$.
- for $a \neq a_{\max}$ Mourre estimate for H^a gives $\rho^a(\lambda) \geq 2\inf\{\lambda - \tau^a : \tau^a \in \mathcal{T}^a \cup \sigma_{\rm pp}(H^a)\}$.
- since $[\frac{1}{2}D^2_{x_a}, \mathrm{i}A_a] = D^2_{x_a}$, abstract result for tensor products gives:

$$\rho_a(\lambda) = \inf_{\lambda^a+\lambda_a=\lambda} \rho^a(\lambda^a) + 2|\lambda_a| \geq 2\inf\{\lambda - \tau : \tau \in \mathcal{T}, \tau \leq \lambda\}.$$

Idea of proof

- for $\lambda \in \mathbb{R} \setminus \mathcal{T}$ for all $\epsilon > 0$ there exists $\chi \in C_0^\infty(\mathbb{R})$ with $\chi(\lambda) \neq 0$ such that for all $a < a_{\max}$:

$$\chi(H_a)[H_a, \mathrm{i}A]\chi(H_a) \geq 2\,(d(\lambda) - \epsilon)\,\chi^2(H_a).$$

- winding back the partition of unity gives:

$$\chi(H)[H, \mathrm{i}A]\chi(H) \geq 2\,(d(\lambda) - \epsilon)\,\chi^2(H) + K_1(R) + K_2(R) + o(R^0).$$

- $$K_1(R) = q_{a_{\max}}(\tfrac{x}{R})\chi(H^{a_{\max}})[H^{a_{\max}}, \mathrm{i}A]\chi(H^{a_{\max}})q_{a_{\max}}(\tfrac{x}{R}),$$
$$K_2(R) = q_{a_{\max}}(\tfrac{x}{R})\chi^2(H^{a_{\max}})q_{a_{\max}}(\tfrac{x}{R}),$$

  are compact ($q_{a_{\max}}$ compactly supported).
- pick $R \gg 1$ to obtain Mourre estimate for H.

Wave operators

- consider first 1-particle case:

$$H = \frac{1}{2}D_x^2 + V(x),\ H_0 =: \frac{1}{2}D_x^2,$$

  potential V tends to 0 at infinity.
- describe the asymptotic behavior when $t \to \pm\infty$ of $\mathrm{e}^{-\mathrm{i}tH}u$ for $u \in \mathcal{H}_{\mathrm{c}}(H)$.
- the case of bound states $u \in \mathcal{H}_{\mathrm{pp}}(H)$ is obvious (superposition of oscillations).
- assume that

$$V(x) \in O(\langle x\rangle^{-\mu}),\ \text{for } \mu > 0 \text{ when } x \to \infty.$$

- V is short-range if $\mu > 1$ long-range if $0 < \mu \leq 1$.
- Coulomb potential is long-range.

Wave operators

- time-dependent method starts with wave operators

$$\Omega^{\pm} := \mathrm{s}-\lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH}\mathrm{e}^{-\mathrm{i}tH_0}.$$

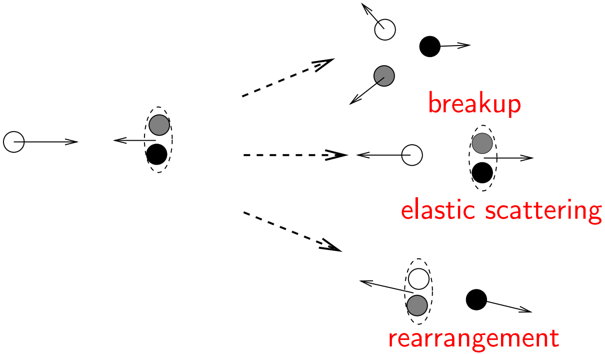
- proof of the existence of $\Omega^{\pm}$ easy in the short-range case for $N=1$. (Cook method).
- asymptotic completeness is the statement that $\mathrm{Ran}\Omega^{\pm} = \mathcal{H}_{\mathrm{c}}(H)$.
- means that for any $u \in \mathcal{H}_{\mathrm{c}}(H)$, there exists $u^{\pm}$ such that

$$\lim_{t\to\pm\infty} \mathrm{e}^{-\mathrm{i}tH}u - \mathrm{e}^{-\mathrm{i}tH_0}u^{\pm} = 0.$$

- if asymptotic completeness holds asymptotic behavior of $\mathrm{e}^{-\mathrm{i}tH}u$ for all $u \in L^2(X)$ is completely understood.

Wave operators in the N-body case

- In the N-particle case other scattering scenarios are possible: freely moving stable clusters of particles can form.



- Several wave operators needed to exhaust all the possibilities.
- $$\Omega_a^{\pm} := \mathrm{s-}\lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH}\mathrm{e}^{-\mathrm{i}tH_a}1\!\!1_{\mathrm{pp}}(H^a).$$
  note that $\Omega_{a_{\max}}^{\pm} = 1\!\!1_{\mathrm{pp}}(H_{a_{\max}})$.
- existence of $\Omega_a^{\pm}$ is easy in the short-range case by the Cook method.

- easy to see that for $a \neq b$ $\mathrm{Ran}\Omega_a^{\pm}$ and $\mathrm{Ran}\Omega_b^{\pm}$ are mutually orthogonal.
- asymptotic completeness is the statement that:

$$\bigoplus_{a\in\mathcal{A}} \mathrm{Ran}\Omega_a^{\pm} = L^2(X).$$

- much more difficult !
- additional difficulty in the long-range case: free motion of the center of masses has to be modified (already present in the 1-particle case). One needs modified wave operators.

Historical sketch of N-particle asymptotic completeness

- brief sketch of the $N-$body asymptotic completeness via time-dependent methods:
- Enss 1978 2-particles, 1986-1989 3-particle short and long range.
- Sigal-Soffer 1987, elegant proof by Graf 1990 N-particle short-range.
- Derezinski 1993, Zielinski 1994, Sigal-Soffer 1994 $N-$particle long-range.
- Gérard 1993, Skibsted 2003 $3-$particle long range decay $\mu > \frac{1}{2}$
- Yafaev 1996 counterexample for 3-particle $0 < \mu < \frac{1}{2}$.

Unitary dynamics

- one needs often to consider also time-dependent Hamiltonians.
- unitary dynamics: strongly continuous map $\mathbb{R}\times\mathbb{R}\ni(t,s)\mapsto U(t,s)\in B(\mathcal{H})$ with

$$U(t,s) \text{ unitary},\ U(s,s)=\mathbb{1}, U(t,u)U(u,s)=U(t,s),\ \forall t,u,s\in\mathbb{R}.$$

- what is the generator $H(t)$ of $U(t,s)$?
- one can require that for B some strictly positive operator

$$\partial_s U(t,s)B^{-1}=\mathrm{i}U(t,s)H(s)B^{-1},\ \text{Dom}\, B\subset \text{Dom}\, H(s),$$

hence

$$\partial_t B^{-1}U(t,s)=-\mathrm{i}B^{-1}H(t)U(t,s).$$

- choose the reference initial time $s=0$ and set $U(t):=U(t,0)$.

Heisenberg derivatives

- fundamental rule: do not consider evolution of states (too complicated) but of observables.
- replace Schroedinger equation by Heisenberg equation.
- if $U_i(t,s)$, $i = 1, 2$ are two unitary dynamics with generators $H_i(t)$, set

$$ {}_2\mathrm{D}_1\Phi(t) = \partial_t\Phi(t) + \mathrm{i}(H_2(t)\Phi(t) - \Phi(t)H_1(t)), $$

for $\Phi : \mathbb{R} \to B(\mathcal{H})$ of class C^1. If $H_1(t) = H_2(t) = H(t)$, denote ${}_2\mathrm{D}_1$ simply by D.

Cook method

- Cook method is the simplest and oldest method to show existence of limits like wave operators.
- based on L^1 in time arguments.
- simplest version: if $U_i(t,s)$ are generated by $H_i(t)$ and $H_2(t) = H_1(t) + V(t)$ with $\|V(t)\|_{B(\mathcal{H})} \in L^1(\mathbb{R})$, then

$$\mathrm{s-}\lim_{t\to\pm\infty} U_2(0,t)U_1(t,0) \text{ exists.}$$

  proof obvious (time derivative is integrable in norm).
- sufficient to show existence of wave operators, not for completeness.
- does not take advantage of the Hilbert space structure, (works on Banach spaces).

Propagation estimates

- better to rely on more symmetric L^2 in time estimates.
- assume $\mathbb{R} \ni t \mapsto \Phi(t) \in B(\mathcal{H})$ uniformly bounded and there exist $C_0 > 0$ and operator valued functions $B(t)$ and $B_i(t)$, $i = 1, \dots, n$, such that

$$\mathrm{D}\Phi(t) \geq C_0 B^*(t)B(t) - \sum_{i=1}^{n} B_i^*(t)B_i(t),$$

$$\int_1^\infty \|B_i(t)U(t)\phi\|^2 \mathrm{d}t \leq C_i \|\phi\|^2, \quad i = 1, \dots, n.$$

Then there exists C such that

$$\int_1^\infty \|B(t)U(t)\phi\|^2 \mathrm{d}t \leq C\|\phi\|^2. \tag{4.1}$$

Existence of limits

- assume that $\mathbb{R} \ni t \mapsto \Phi(t) \in B(\mathcal{H})$ is uniformly bounded and

$$|(\psi_2|_2\mathcal{D}_1\Phi(t)\psi_1)| \leq \sum_{i=1}^{n} \|B_{2i}(t)\psi_2\|\|B_{1i}(t)\psi_1\|, \text{ with}$$
$$\int_1^\infty \|B_{2i}(t)U_2(t)\phi\|^2 \mathrm{d}t \leq C\|\phi\|^2, \quad \phi \in \mathcal{H}, \quad i = 1, \dots, n,$$
$$\int_1^\infty \|B_{1i}(t)U_1(t)\phi\|^2 \mathrm{d}t \leq C\|\phi\|^2, \quad \phi \in \mathcal{H}, \quad i = 1, \dots, n.$$

- then the limit

$$\mathrm{s-}\lim_{t\to+\infty} U_2^*(t)\Phi(t)U_1(t) \text{ exists.}$$

What is a selfadjoint operator ?

- provocative but meaningful definition:
- a (possibly non densely defined) selfadjoint operator on $\mathcal{H}$ is a continuous $*$-morphism $\gamma : C_\infty(\mathbb{R}) \mapsto B(\mathcal{H})$.
- a densely defined selfadjoint operator on $\mathcal{H}$ is a continuous $*$-morphism $\gamma : \mathcal{C}_\infty(\mathbb{R}) \mapsto B(\mathcal{H})$ such that $\mathrm{s-}\lim_{R\to+\infty}\gamma(\chi_R) = \mathbb{1}$ for $\chi_R(\lambda) = \chi(R^{-1}\lambda)$ with $\chi \in C_\infty(\mathbb{R})$ and $\chi(0) = \mathbb{1}$.
- there is a unique selfadjoint operator H such that $\gamma(\chi) = \chi(H)$ for all $\chi \in C_\infty(\mathbb{R})$. γ uniquely extends to $B(\mathbb{R})$ (space of bounded Borel functions), using the monotone class theorem.
- replacing $\mathbb{R}$ by $\mathbb{R}^n$ one obtains the definition of a commuting family $(H_1, \cdots, H_n)$ of selfadjoint operators.

- Jan Derezinski invented the notion of asymptotic velocity.
- gives a bird's eye view of scattering theory and asymptotic completeness.
- a crucial tool in his proof of completeness for long-range potentials.
- assume that $\partial_{x^a}^{\alpha} v^a(x^a) \in O(\langle x^a \rangle^{-\mu}),\ \mu > 0, |\alpha| \leq 1.$
- then

$$\mathrm{s}-\lim_{t \to \pm\infty} \mathrm{e}^{\mathrm{i}tH} \chi(\frac{x}{t}) \mathrm{e}^{-\mathrm{i}tH} =: \gamma^{\pm}(\chi) \text{ exist, } \chi \in C_{\infty}(X).$$

- $\gamma^{\pm}(\chi) = \chi(P^{\pm})$, $P^{\pm}$ future/past asymptotic velocity.
- $P^{\pm}$ commute with H.
  Interpretation: $x(t) = \mathrm{e}^{\mathrm{i}tH} x \mathrm{e}^{-\mathrm{i}tH}$ is the position at time t, $\frac{x(t)}{t}$ the average velocity at time t, $P^{\pm}$ their limits when $t \to \pm\infty$.

Properties of the asymptotic velocity

- Recall that for
$$Z_a = X_a \setminus \cup_{b \not\leq a} X_b,$$
the $\{Z_a\}_{a\in\mathcal{A}}$ are a partition of unity.
- therefore we have
$$\mathbb{1} = \sum_{a\in\mathcal{A}} \mathbb{1}_{Z_a}(P^{\pm}).$$
- $u = \mathbb{1}_{Z_a}(P^{\pm})u$ then $\mathrm{e}^{-\mathrm{i}tH}u$ for $t \to \pm\infty$ is decomposed into independent clusters of a, whose size is $o(t)$.

Properties of the asymptotic velocity

- in particular for $a = a_{\max}$, $Z_{a_{\max}} = \{0\}$ and if $u \in \mathrm{Ran}\mathbb{1}_{\{0\}}(P^{\pm})$ then

$$\mathrm{s-}\lim_{t\to\pm\infty} \mathbb{1}_{[\delta,+\infty[}(\frac{|x|}{t})\mathrm{e}^{-H}u = 0, \ \forall\delta > 0,$$

  ie $x(t)$ is of size $o(t)$, u is an 'almost bound state'.
- the Mourre estimate implies the following fundamental result:

$$\mathbb{1}_{\{0\}}(P^{\pm}) = \mathbb{1}_{\mathrm{pp}}(H),$$

  ie almost bound states are necessarily bound states.
- not true in the classical case !

Joint energy-velocity spectrum

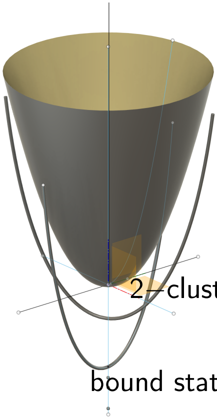
- since $[H, P^{\pm}] = 0$ on can study the joint energy-velocity spectrum.
- it gives a first 'spectral' understanding of scattering theory.

Theorem

The joint energy-velocity spectrum is

$$\sigma(H, P^{\pm}) = \bigcup_{a \in \mathcal{A}} \{(\xi_a, \tau + \frac{1}{2}\xi_a^2) : \xi_a \in X_a, \ \tau \in \sigma_{\rm pp}(H^a)\}.$$

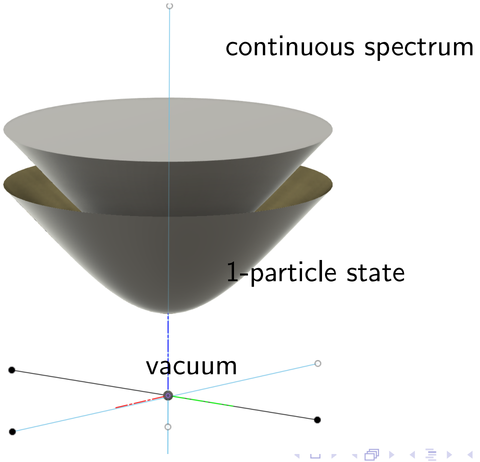
Joint energy-velocity spectrum



Energy-momentum spectrum in relativistic QFT

- similar result in relativistic QFT.
- if the theory is invariant under the Poincaré group, one can study the energy-momentum spectrum (space-time translations).
- typical spectrum is shown in the next slide. Parabolas are replaced by hyperbolas (and Galilei group by Lorentz group).

Energy-momentum spectrum in relativistic QFT



Asymptotic absolute continuity

- one can ask about the nature of the spectral measure of $P^{\pm}$.
- a result in this direction is the following (called asymptotic absolute continuity):
- assume that $\nabla^{\alpha}_{x^a} v^a \in O(\langle x^a \rangle^{-|\alpha|-\mu})$ for $|\alpha| \leq 1$ and $\mu > \frac{1}{2}$.
- then if $a \in \mathcal{A}$ and $\theta \subset Z_a$ is of measure zero on X_a one has

$$\mathbb{1}_{\theta}(P^{\pm}) = 0.$$

Large velocity estimates

- the Heisenberg derivative of x is D_x, controlled by H.
- if total energy is bounded, the position x cannot grow faster than Ct.
- (not true for the N-body problem of Celestial Mechanics !)
- Let $\chi \in C_0^\infty(\mathbb{R})$. Then there exists $\theta > 0$ such that

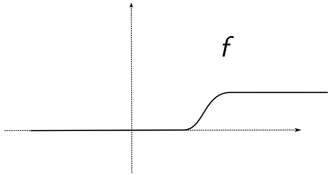
$$1) \int_1^{+\infty} \|\mathbb{1}_{[\theta,\theta']}(\frac{|x|}{t})\chi(H)\mathrm{e}^{-\mathrm{i}tH}u\|^2 \frac{dt}{t} \leq C\|u\|^2.$$

Moreover

$$2) \lim_{t\to\pm\infty} \mathbb{1}_{[\theta,+\infty[}(\frac{|x|}{t})\chi(H)\mathrm{e}^{-\mathrm{i}tH}u = 0.$$

- statement 2) means exactly that $P^{\pm}$ is densely defined.

- proof of 1): take $\Phi(t) = \chi(H)F(\frac{x}{t})\chi(H)$,
- $\mathrm{D}\Phi(t) = \frac{1}{2}\chi(H)(D_x - \frac{x}{t}) \cdot \nabla F(\frac{x}{t}) + \mathrm{h.c.})\chi(H)$.
- take $F(x) = f(|x|)$, with $f' = \mathbb{1}_{[\theta,\theta']}$.



- since $D\chi(H)$ is bounded we get that

$$\mathrm{D}\Phi(t) \leq -\frac{C}{t}\chi(H)\mathbb{1}_{[\theta,\theta']}(\frac{|x|}{t})\chi(H)$$

negative Heisenberg derivative.

- proof of 2): replace $\Phi(t)$ by $\Phi_R(t) = \chi(H)F(\frac{x}{Rt})\chi(H)$, for $R \gg 1$.
- we want to show that $\mathrm{s-}\lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH}\Phi_1(t)\mathrm{e}^{-\mathrm{i}tH} = 0$.
- $\mathrm{D}\Phi_R(t)$ is controlled by terms under the integral in 1) so
- $\mathrm{s-}\lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH}\Phi_R(t)\mathrm{e}^{-\mathrm{i}tH}$ exists.

- keeping track of R in computation of $\mathrm{D}\Phi_R(t)$ one obtains that

$$\mathrm{s-}\lim_{R\to+\infty}\mathrm{s-}\lim_{t\to\pm\infty}\mathrm{e}^{\mathrm{i}tH}\Phi_R(t)\chi(H)\mathrm{e}^{-\mathrm{i}tH}=0.$$

- $\Phi_R(t)-\Phi_1(t)$ supported in $\theta\leq\frac{|x|}{t}\leq\theta'$, so
- $\int_1^{+\infty}\|(\Phi_R(t)-\Phi_1)\mathrm{e}^{-\mathrm{i}tH}\chi(H)u\|^2\frac{dt}{t}<\infty$, hence
- $\mathrm{s-}\lim_{t\to\pm\infty}\mathrm{e}^{\mathrm{i}tH}(\Phi_R(t)-\Phi_1(t))\chi(H)\mathrm{e}^{-\mathrm{i}tH}=0.$
- taking $R\to+\infty$ gives $\mathrm{s-}\lim_{t\to\pm\infty}\mathrm{e}^{\mathrm{i}tH}\Phi_1(t)\chi(H)\mathrm{e}^{-\mathrm{i}tH}=0.$

Phase-space propagation estimates: $1-$particle case

- free case ($V = 0$): let $R(x) = \frac{1}{2}x^2$ and $\Phi(t) = \frac{1}{2}(D_x - \frac{x}{t}) \cdot \nabla R(\frac{x}{t}) + \mathrm{h.c.}) + R(\frac{x}{t})$.
- then

$$\mathrm{D}\Phi(t) = \partial_t \Phi(t) + [\frac{1}{2}D_x^2, \mathrm{i}\Phi(t)] = \frac{1}{t}\| D_x - \frac{x}{t}\|^2 \geq 0.$$

- problem: $\Phi(t)$ is not bounded.
- solution: replace $\Phi(t)$ by $\chi(H)F(\frac{x}{t})\Phi(t)F(\frac{x}{t})\chi(H)$, F supported in $|x| \leq R$, $R \gg 1$.
- extra terms coming from $\mathrm{D}F(\frac{x}{t})$ are controlled by large velocity estimates.

Phase-space propagation estimates: 1−particle case

- assume now that $V(x) \in O(\langle x\rangle^{-\mu})$, $\mu > 0$.
- extra term in $\mathrm{D}\Phi(t)$ is $-\nabla R(\frac{x}{t}) \cdot \nabla_x V(x)$, not controlled.
- solution: modify $R(x)$ such that $\nabla_x R(x) = 0$ in $|x| \leq \epsilon$, keeping $\nabla_x^2 R(x) \geq 0$.
- for example take $R(x) = \max(\frac{1}{2}\epsilon^2, \frac{1}{2}x^2)$ (convex !).
- if necessary smooth out R by convolution w.r.t. ϵ.
- then if $\nabla_x V(x) \in O(\langle x\rangle^{-1-\mu})$, extra term is $O(t^{-1-\mu})$, integrable in norm.
- We obtain $\int_1^{+\infty} \|\mathbb{1}_{[\theta,\omega']}(\frac{x}{t})(D_x - \frac{x}{t})\mathrm{e}^{-\mathrm{i}tH}u\|^2 \frac{dt}{t} \leq C\|u\|^2$.
- example of a phase space propagation estimate.

the Graf function

- Graf ingenious construction: modify $R(x)$ in the N-body case: modify $R(x) = \frac{1}{2}x^2$ so that
- 1) $R(x)$ depends only on x_a near Z_a.
- 2) $\nabla_x^2 R(x) \geq \pi_a$ near Z_a, where $\pi_a : X \to X_a$ orthogonal projection.
- one chooses

$$R^\rho(x) = \frac{1}{2}\max_{a\in\mathcal{A}}\{x_a^2 + \rho_a\}, \ \ \rho = (\rho_a)_{a\in\mathcal{A}}.$$

  R^ρ satisfies 1) and 2) for ρ in some open set.
- smooth out R^ρ w.r.t. ρ:

$$R(x) = \int R^\rho(x) f(\rho) d\rho, \text{ for } \int f(\rho) d\rho = 1,$$

Phase-space propagation estimates: N-particle case

- properties 1) 2) still satisfied.
- recall that $Z_a^{\epsilon,\delta} \subset X$ defined by

$$|x^a| \leq \epsilon, \ |x^b| \geq \delta \ \forall b \not\leq a.$$

- if $\frac{x}{t} \in Z_a^{\epsilon,\delta}$, then clusters of a have distance at least $\delta|t|$ and size $\epsilon|t|$.
- let $\chi \in C_0^\infty(\mathbb{R})$, $F \in C_0^\infty(X)$. Then

$$\int_1^{+\infty} \|\chi(H)F(\frac{x}{t})\mathbb{1}_{Z_a^{\epsilon,\delta}}(\frac{x}{t})(D_{x_a} - \frac{x_a}{t})\mathrm{e}^{-\mathrm{i}tH}u\|^2 \frac{dt}{t} \leq C\|u\|^2.$$

Phase-space propagation estimates: N-particle case

- propagation observable:
  $\Phi(t) = \frac{1}{2}((D_x - \frac{x}{t}) \cdot \nabla R(\frac{x}{t}) + \mathrm{h.c.}) + R(\frac{x}{t})$.
- add energy cutoffs $\chi(H)$ and large distance cutoffs $F(\frac{x}{t})$.
- $\mathrm{D}\Phi(t) = \partial_t \Phi(t) + [\frac{1}{2}D_x^2, \mathrm{i}\Phi(t)] + [V(x), \mathrm{i}\Phi(t)]$.
- first two terms can be computed exactly as

$$\frac{1}{t}(D_x - \frac{x}{t}) \cdot \nabla^2 R(\frac{x}{t})(D_x - \frac{x}{t}) \geq c \sum_{a \in \mathcal{A}} \mathbb{1}_{Z_a}(\frac{x}{t})(D_{x_a} - \frac{x_a}{t})^2.$$

- for $\frac{x}{t} \in Z_a^{\epsilon,\delta}$ split the second term as

$$-\nabla_{x^a} V^a(x^a) \cdot \nabla_{x^a} R(\frac{x}{t}) - \nabla_x I_a(x) \cdot \nabla_x R(\frac{x}{t})$$

- first term is 0, because near Z_a, $\nabla_x R$ depends only on x_a.
- second term is $O(t^{-1-\mu})$ so integrable in norm. $\square$

Asymptotic velocity

- goal: prove that for $F \in C_0^\infty(X)$ (dense in $C_\infty(X)$):

$$\mathrm{s}-\lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH} F(\frac{x}{t}) \mathrm{e}^{-\mathrm{i}tH} \text{ exists}.$$

- we can take F in a C^0 dense subspace of $C_0^\infty(X)$: good choice: $F(x)$ depends only on x_a near X_a.
- set

$$\Phi(t) = F(\frac{x}{t}) + \nabla F(\frac{x}{t}) \cdot (D_x - \frac{x}{t}).$$

$$\mathrm{D}\Phi(t) = (D - \frac{x}{t}) \cdot \nabla^2 F(\frac{x}{t})(D_x - \frac{x}{t}) - \nabla F(\frac{x}{t}) \cdot \nabla V(x).$$

1) first term is integrable along the evolution by phase space propagation estimates.
2) second term is $O(t^{-1-\mu})$ in norm.

Asymptotic velocity

- therefore

$$\mathrm{s-}\lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH}(F(\frac{x}{t}) + \nabla F(\frac{x}{t}) \cdot (D_x - \frac{x}{t}))\mathrm{e}^{-\mathrm{i}tH} \text{ exists.}$$

- next show (by computing its Heisenberg derivative) that

$$\mathrm{s-}\lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH}(\nabla F(\frac{x}{t}) \cdot (D_x - \frac{x}{t})\mathrm{e}^{-\mathrm{i}tH} \text{ exists.}$$

- this observable is integrable along the evolution hence the limit has to be 0.
- therefore $\mathrm{s-}\lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH} F(\frac{x}{t})\mathrm{e}^{-\mathrm{i}tH} = F(P^{\pm})$ exists.
- the fact that $[H, P^{\pm}] = 0$ is a general property (valid for all asymptotic observables).

Minimal velocity estimate

- construction of asymptotic velocity not sufficient to prove asymptotic completeness, even in the short-range case.
- one needs the additional spectral information:
  $\mathbb{1}_{\{0\}}P^{\pm}) = \mathbb{1}_{\rm pp}(H)$.
  (almost bound states are bound states).
- this will follow from a minimal velocity estimate due to Graf: let $\chi \in C_0^\infty(\mathbb{R})$ with $\operatorname{supp}\chi \cap \mathcal{T} \cup \sigma_{\rm pp}(H) = \emptyset$.
- then there exists $\epsilon_0 > 0$ such that

$$\int_1^{+\infty} \|\mathbb{1}_{[0,\epsilon]}(\frac{x}{t})\chi(H)\mathrm{e}^{\mathrm{i}tH}u\|^2 \frac{dt}{t} \leq C\|u\|^2.$$

Minimal velocity estimate

- set $M(t) = J(\frac{x}{t}) + \frac{1}{2}((D_x - \frac{x}{t}) \cdot \nabla J(\frac{x}{t}) + \mathrm{h.c.})$, J depends only on x_a near Z_a, $J(x) = 1$ in $|x| \leq \epsilon$.
- take
$$\Phi(t) = \chi(H)M(t)\chi(H)\frac{A}{t}\chi(H)M(t)\chi(H).$$
- when computing $\mathrm{D}\Phi(t)$, terms coming from $\mathrm{D}M(t)$ will be controlled.
- one has $\mathrm{D}\frac{A}{t} = -\frac{A}{t^2} + \frac{[H,\mathrm{i}A]}{t}$.
- because of the Mourre estimate $\chi(H)[H,\mathrm{i}A]\chi(H) \geq c\chi^2(H)$.
- choosing $\epsilon \ll 1$ we can ensure that $M(t)\chi(H)\frac{A}{t}\chi(H)M(t) \leq \frac{c}{2}M(t)\chi(H)\chi(H)M(t)$.

Minimal velocity estimate

- we obtain $\mathrm{D}\Phi(t) \geq \frac{c}{2}\chi(H)M^2(t)\chi(H)$ modulo already controlled errors. This proves minimal velocity estimate.
- since we know that $\mathrm{s-}\lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH}\chi(H)\mathbb{1}_{[0,\epsilon]}(\frac{x}{t})\chi(H)\mathrm{e}^{-\mathrm{i}tH}$ exists, we obtain that
- $$\mathrm{s-}\lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH}\chi(H)\mathbb{1}_{[0,\epsilon]}(\frac{x}{t})\chi(H)\mathrm{e}^{-\mathrm{i}tH} = 0.$$
- by an easy density argument, this shows that

$$\mathbb{1}_{\{0\}}(P^{\pm}) = \mathbb{1}_{\mathrm{pp}}(H).$$

Asymptotic completeness for short-range N-particle systems

- existence and completeness of short-range wave operators follows very easily from properties of asymptotic velocity.
- it can be neatly formulated as follows: assume that $v^a(x^a) \in O(\langle x^a \rangle)^{-\mu}$ for $\mu > 1$. Then
- the limits

$$1)\ \mathrm{s-}\lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH}\mathrm{e}^{-\mathrm{i}tH_a}1\!\!1_{\mathrm{pp}}(H^a) =: \Omega^{\pm}_{\mathrm{sr},a} \text{ exist.}$$

- the limits

$$2)\ \mathrm{s-}\lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH_a}\mathrm{e}^{-\mathrm{i}tH}1\!\!1_{Z_a}(P^{\pm}) = \Omega^{\pm *}_{\mathrm{sr},a} \text{ exist.}$$

- $\Omega^{\pm}_{\mathrm{sr},a}$ are partial isometries with

$$\mathrm{Dom}\,\Omega^{\pm}_{\mathrm{sr},a} = \mathrm{Ran}1\!\!1_{\mathrm{pp}}(H^a),\ \mathrm{Ran}\Omega^{\pm}_{\mathrm{sr},a} = \mathrm{Ran}1\!\!1_{Z_a}(P^{\pm}).$$

Asymptotic completeness for short-range N-particle systems

- the proofs of 1) and 2) are similar: denoting by $P^{\pm}_{(a)}$, $P^{\pm(a)}$ the asymptotic velocities for H_a, H^a we get

$$P^{\pm}_{(a)} = (D_{x_a}, P^{\pm}(a)).$$

- therefore using Mourre estimate for H^a, we get $\mathbb{1}_{\rm pp}(H^a) = \mathbb{1}_{Z_a}(P^{\pm}_{(a)})$.
- proof of 2): let $u \in {\rm Ran}\mathbb{1}_{Z_a}(P^{\pm})$. By density we can assume that $u = F(P^{\pm})u = \chi(H)u$ for F supported near Z_a, $\chi \in C_0^\infty(\mathbb{R})$.
- so it suffices to prove the existence of

$$\lim_{t\to\pm\infty} \chi(H_a)F(\frac{x}{t})\chi(H){\rm e}^{{\rm i}tH_a}F(\frac{x}{t}){\rm e}^{-{\rm i}tH}u.$$

Asymptotic completeness for short-range N-particle systems

- we compute asymmetric Heisenberg derivative for $H_2 = H_a$, $H_1 = H$:

$$ {}_2\mathrm{D}_1 \chi(H_a) M(t) \chi(H), $$

- for $M(t) = (F(\frac{x}{t}) - \frac{1}{2}((D_x - \frac{x}{t}) \cdot \nabla F(\frac{x}{t}) + \mathrm{h.c.})$.
- it equals

$$ \chi(H_a) \mathrm{D} M(t) \chi(H) + \chi(H_a) \mathrm{i} I_a(x) M(t) \chi(H). $$

- first term is integrable along the evolution (use phase space propagation estimates for H and H_a).
- second term is $O(t^{-\mu})$ in norm so integrable by short-range condition.

Asymptotic completeness for short-range N-particle systems

- To complete the proof of asymptotic completeness, use:
- $$\mathbb{1} = \sum_{a\in\mathcal{A}} \mathbb{1}_{Z_a}(P^{\pm}), \text{(spectral theorem !)}$$
- and $Z_{a_{\max}} = \{0\}$, $\mathbb{1}_{\{0\}}(P^{\pm}) = \mathbb{1}_{\rm pp}(H)$.
- therefore
$$\oplus_{a\in\mathcal{A}} \mathrm{Ran}\Omega^{\pm}_{\mathrm{sr},a} = L^2(X).$$

Modified dynamics 1-particle case

- consider 1-particle Hamiltonian: $H = \frac{1}{2}D_x^2 + V(x)$, $\partial_x^\alpha V(x) \in O(\langle x \rangle)^{-\mu-|\alpha|}$, $\mu > 0$.
- the short-range wave operators

$$\mathrm{s-}\lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH}\mathrm{e}^{-\mathrm{i}t\frac{1}{2}D_x^2} \text{ do not exist !}$$

- the 'long-range tail' of the potential cannot be forgotten.
- purely classical problem, can be completely understood in classical mechanics.
- one needs to modify the free dynamics: various equivalent ways to do it.

Modified dynamics 1-particle case

- time-dependent modifiers: choose a solution of the Hamilton-Jacobi equation:

$$\partial_t S(t,\xi) = \frac{1}{2}\xi^2 + V_t(\nabla_\xi S(t,\xi)),$$

- $V_t(x)$ time-dependent potential, equal to $V(x)$ in $|x| \geq \epsilon|t|$.
- boundary condition for $S(t,\xi)$ is $S(t,\xi) = \frac{1}{2}t\xi^2 + O(t^{1-\mu})$, when $t \to \pm\infty$.
- not unique, no canonical choice.

Modified dynamics 1-particle case

- one introduces the modified wave operators:

$$\Omega_{\rm lr}^{\pm} := \mathrm{s-}\lim_{t\to\pm\infty} \mathrm{e}^{itH}\mathrm{e}^{\mathrm{i}S(t,D_x)}.$$

- Completeness of wave operators is as before statement that:

$$\mathrm{Ran}\Omega_{\rm lr}^{\pm} = \mathbb{1}_{\rm c}(H)L^2(X).$$

- existence easy to prove (stationary phase arguments).
- completeness more difficult: nice time-dependent proof by Sigal

Modified dynamics 1-particle case

- first step: replace $V(x)$ by $V_t(x)$ as above, satisfying $\partial_x^\alpha V_t(x) \in O(t^{-\mu-|\alpha|})$ (use minimal velocity estimates).
- let $U(t,s)$ unitary dynamics generated by $H(t) = \frac{1}{2}D_x^2 + V_t(x)$.
- using asymptotic velocity one shows that

$$\mathrm{s-}\lim_{t\to\pm\infty} U(0,t)\mathrm{e}^{-\mathrm{i}tH}\mathbb{1}_{\mathrm{c}}(H) \text{ exist.}$$

- it remains to show that

$$\mathrm{s-}\lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}S(t,D_x)}U(t,0) \text{ exists.}$$

Modified dynamics 1-particle case

- compute Heisenberg derivative w.r.t. $H(t)$:

$$
\begin{aligned}
& \mathrm{D}(x - \nabla_\xi S(t, D_x)) \\
= \quad & D_x + \nabla_x V_t(x) \nabla_\xi^2 S(t, D_x) - \partial_t \nabla_\xi S(t, D_x) \\
= \quad & (\nabla_x V_t(x) - \nabla_x V_t(\nabla_\xi S(t, D_x))) \nabla_\xi^2 S(t, D_x) \\
= \quad & \nabla_x^2 V_t(x)(x - (\nabla_\xi S(t, D_x))) \nabla_\xi^2 S(t, D_x) \\
= \quad & O(t^{-1-\mu})(x - \nabla_\xi S(t, D_x)).
\end{aligned}
$$

- Gronwall's inequality then gives Sigal's estimate:

$$\|(x - \nabla_\xi S(t, D_x)) U(t, 0) \langle x \rangle^{-1} \| \in O(1).$$

- show that $\mathrm{s-}\lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}S(t,D_x)}U(t,0)$ exists by naive Cook method:
- show that $(\partial_t S(t,D_x) - H(t))U(t,0)u$ integrable in norm.
- we need to show

$$\|(V_t(\nabla_\xi S(t,D_x)) - V_t(x))U(t,0)u\| \in L^1(dt).$$

- pdo calculus gives $V_t(\nabla_\xi S(t,D_x)) - V_t(x)) = O(t^{-1-\mu})(x - \nabla_\xi S(t,D_x)) + O(t^{-1-\mu})$.
- this is in $L^1(dt)$ for $u \in \mathrm{Dom}\langle x\rangle$ by Sigal's estimate.

Long-range N-particle case

- let $u \in \mathrm{Ran} 1\!\!1_{Z_a}(P^{\pm})$. Then:
- size of the clusters of a is $o(t)$, distance between clusters of a greater than $C|t|$.
- We can replace $I_a(x)$ by $I_{a,t}(x)$, with $\partial_x I_{a,t}(x) \in O(t^{-|\alpha|-\mu})$, $I_{a,t}(x) = I_a(x)$ near Z_a.
- the usual argument gives the existence of

$$\Omega^{\pm}_{a,\mathrm{sep}} = \mathrm{s-}\lim_{t\to\pm\infty} U_a(0,t)\mathrm{e}^{-\mathrm{i}tH}\mathrm{Ran} 1\!\!1_{Z_a}(P^{\pm}),$$

- where $U_a(t,x)$ unitary dynamics generated by $H_a(t) = \frac{1}{2}D^2_{x_a} + H^a + I_{a,t}(x)$.
- main problem: $U_a(t,0)$ still couples motion in X^a and in X_a.
- one would like to replace $I_{a,t}(x)$ by $I_{a,t}(0,x_a)$ ie set $x^a = 0$.

Bound on the size of the clusters

- by Taylor's formula $I_{a,t}(x) - I_{a,t}(0, x_a) \in O(t^{-1-\mu})|x^a|$.
- so the key is to estimate the size of the clusters of a when $u \in \mathrm{Ran}\mathbb{1}_{Z_a}(P^{\pm})$, ie replace the $o(t)$ estimate by $O(t^{\delta})$ for some $0 < \delta < 1$.
- Jan Derezinski managed to prove that if $u \in \mathrm{Ran}\mathbb{1}_{Z_a}(P^{\pm})$ then

$$\lim_{t\to\pm\infty} \mathbb{1}_{[\theta,+\infty[}\left(\frac{|x^a|}{t^{\delta}}\right)\mathrm{e}^{-\mathrm{i}tH}u = 0,$$

- for $\delta = 2(2+\mu)^{-1}$. Proof uses the function $r(x) = (2R(x))^{\frac{1}{2}}$ (modification of $|x|$).

Long-range N-particle case

- if $2(2+\mu)^{-1} < \mu$ ie $\mu > \sqrt{3}-1$, then one can replace $I_{a,t}(x)$ by $I_{a,t}(0, x_a)$ on the evolution of such states.
- choose a solution of Hamilton-Jacobi equation:

$$\partial_t S_a(t,\xi_a) = \frac{1}{2}\xi_a^2 + I_{a,t}(0, \nabla_{\xi_a} S(t,\xi_a)),$$

$$S_a(t,\xi_a) = \frac{1}{2}\xi_a^2 + O(t^{1-\mu}).$$

Long-range N-particle case

- one obtains the following theorem
- assume that $\nabla_x^\alpha v^a(x^a) \in O(\langle x^a\rangle)^{-|\alpha|-\mu}$ for $\mu > \sqrt{3}-1$. Then the limits

$$1)\quad \Omega^{\pm}_{\mathrm{lr},a} = \mathrm{s-}\lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}H}\mathrm{e}^{-\mathrm{i}S_a(t,D_{x_a})-\mathrm{i}tH^a}\mathbb{1}_{\mathrm{pp}}(H^a)$$

$$2)\quad \Omega^{\pm *}_{\mathrm{lr},a} = \mathrm{s-}\lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}S_a(t,D_{x_a})+\mathrm{i}tH^a}\mathrm{e}^{-itH}\mathbb{1}_{Z_a}(P^{\pm})$$

  exist.
- $\Omega^{\pm}_{\mathrm{lr},a}$ are partial isometries with

$$\mathrm{Dom}\,\Omega^{\pm}_{\mathrm{lr},a} = \mathrm{Ran}\mathbb{1}_{\mathrm{pp}}(H^a),\ \mathrm{Ran}\Omega^{\pm}_{\mathrm{lr},a} = \mathrm{Ran}\mathbb{1}_{Z_a}(P^{\pm}).$$

Long-range N-particle case

- the wave operators are complete

$$\oplus_{a\in\mathcal{A}}\mathrm{Ran}\Omega^{\pm}_{\mathrm{sr},a} = L^2(X).$$

One has

$$\Omega^{\pm *}_{\mathrm{lr},a} P^{\pm} 1\!\!1_{Z_a}(P^{\pm}) = D_{x_a} \Omega^{\pm *}_{\mathrm{lr},a} 1\!\!1_{Z_a}(P^{\pm}).$$

Thank your for your attention !