SCATTERING THEORY OF CLASSICAL PARTICLES

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Quantum scattering theory is very well developed, both 1-body and N-body. Asymptotic completeness of quantum N-body systems used to be considered an important question of mathematical physics. Classical scattering theory is less famous. It was usually studied as a tool for the quantum case.

In my talk I will describe two topics.

1. Classical scattering theory of a particle in an external potential. Its analysis is later needed in quantum long-range scattering.

2. Classical scattering theory of N-body systems. Can be viewed as a spin-off of the quantum N-body scattering. Quantum results are much more satisfactory than classical ones.

Classical paricle in external potential

Let V be a potential on $\mathbb{R}^d$ decaying at infinity. Consider the classical Hamiltonian

$$H := \frac{1}{2}p^2 + V(x) \tag{1}$$

with the equations of motion

$$\frac{\mathrm{d}x}{\mathrm{d}t} = p, \quad \frac{\mathrm{d}p}{\mathrm{d}t} = -\nabla V(x). \tag{2}$$

Clearly, if $t \mapsto x(t), p(t)$ solves (2), then $H\big(x(t), p(t)\big)$ is constant. It is called the energy of the trajectory $t \to x(t)$.

Theorem 0.1. *Assume that*

$$\nabla V(x)| \leq C\langle x\rangle^{-1-\mu}, \quad \mu > 0. \tag{3}$$

Let $x(t)$ be a trajectory for $t > 0$. Then there are 3 possibilities

1. *Trapped trajectory: $x(t)$ is bounded.*
2. *Almost bounded trajectory: $x(t)$ is un unbounded but $\lim_{t\to\infty} \frac{x(t)}{t} = 0$. This implies that $H = 0$.*
3. *Scattering trajectory: $\lim_{t\to\infty} \frac{x(t)}{t}$ exists and is not zero. This implies that $H > 0$.*

We would like to classify all scattering trajectories.

Theorem 0.2. *Assume the short range condition*

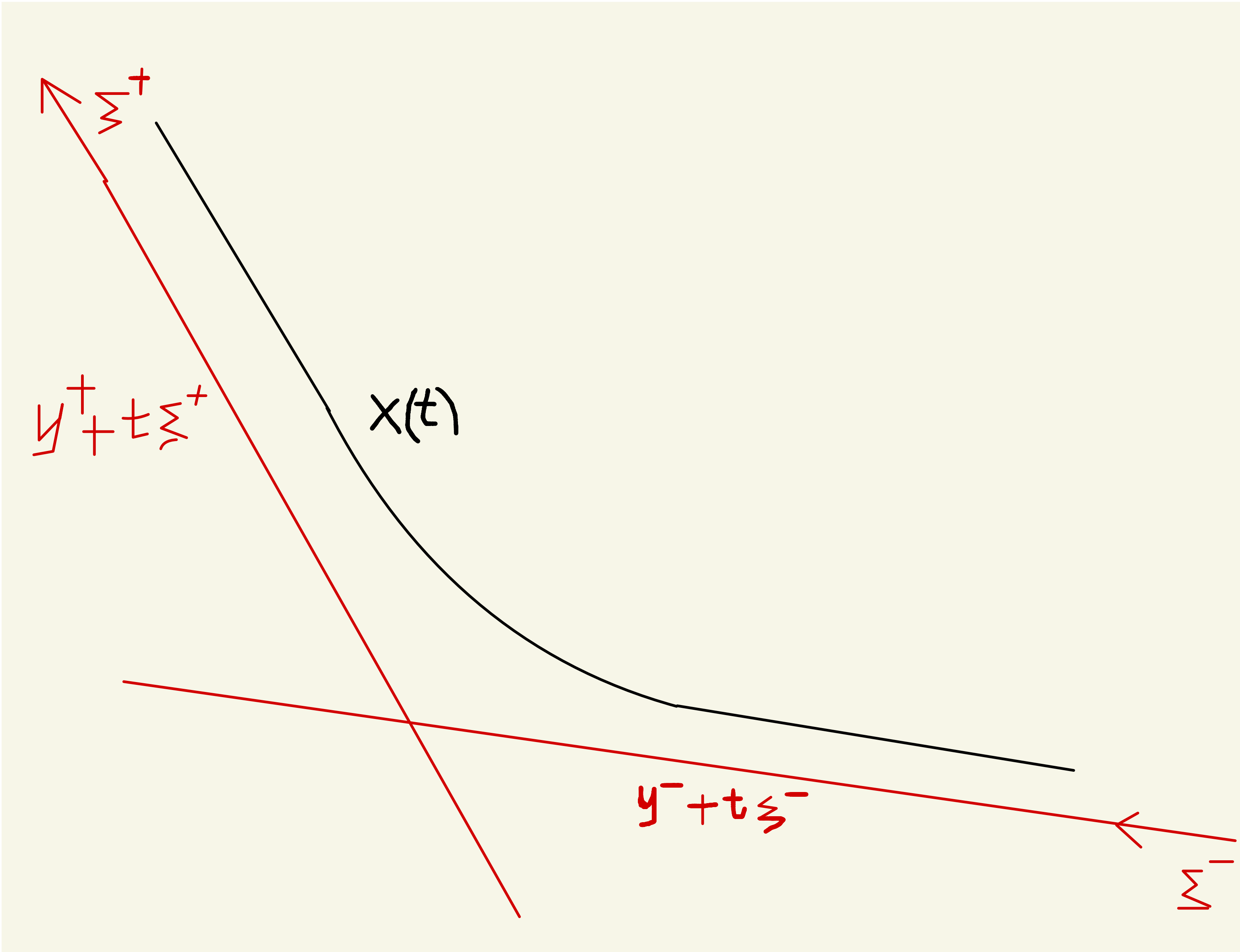
$$|\nabla V(x)|,\ |\nabla^2 V(x)| \leq C\langle x\rangle^{-2-\mu}, \quad \mu > 0. \tag{4}$$

Then for every scattering trajectory $x(t)$ there exist asymptotic position $y^{\pm} \in \mathbb{R}^d$ and asymptotic momentum $\xi^{\pm} \in \mathbb{R}^d \backslash \{0\}$ such that

$$\lim_{t\to\pm\infty} \left(x(t) - t\xi^{\pm} - y^{\pm}\right) = 0. \tag{5}$$

Conversely, for every $y^{\pm} \in \mathbb{R}^d$, $\xi^{\pm} \in \mathbb{R}^d\backslash\{0\}$ there exists a unique scattering trajectory $x^{\pm}(t)$ such that

$$\lim_{t\to\pm\infty} \left(x^{\pm}(t) - t\xi^{\pm} - y^{\pm}\right) = 0. \tag{6}$$



ξ⁺
y⁺+tξ⁺
X(t)
y⁻+tξ⁻
ξ⁻

The above construction does not apply e.g. to Coulomb potentials. Its trajectories do not have the asymptotics $t\xi^{\pm} + y^{\pm}$ because of logarithmic corrections. However, the following fact is almost immediate:

Theorem 0.3. *Assume the long-range condition*

$$\nabla V(x)| \leq C\langle x\rangle^{-1-\mu}, \quad \mu > 0. \tag{7}$$

Then for every scattering trajectory $x(t)$ there exists the asymptotic momentum

$$\lim_{t\to\pm\infty} p(t) \neq 0. \tag{8}$$

Under slightly stronger assumptions one can describe scattering trajectories more precisely also in the long-range case:

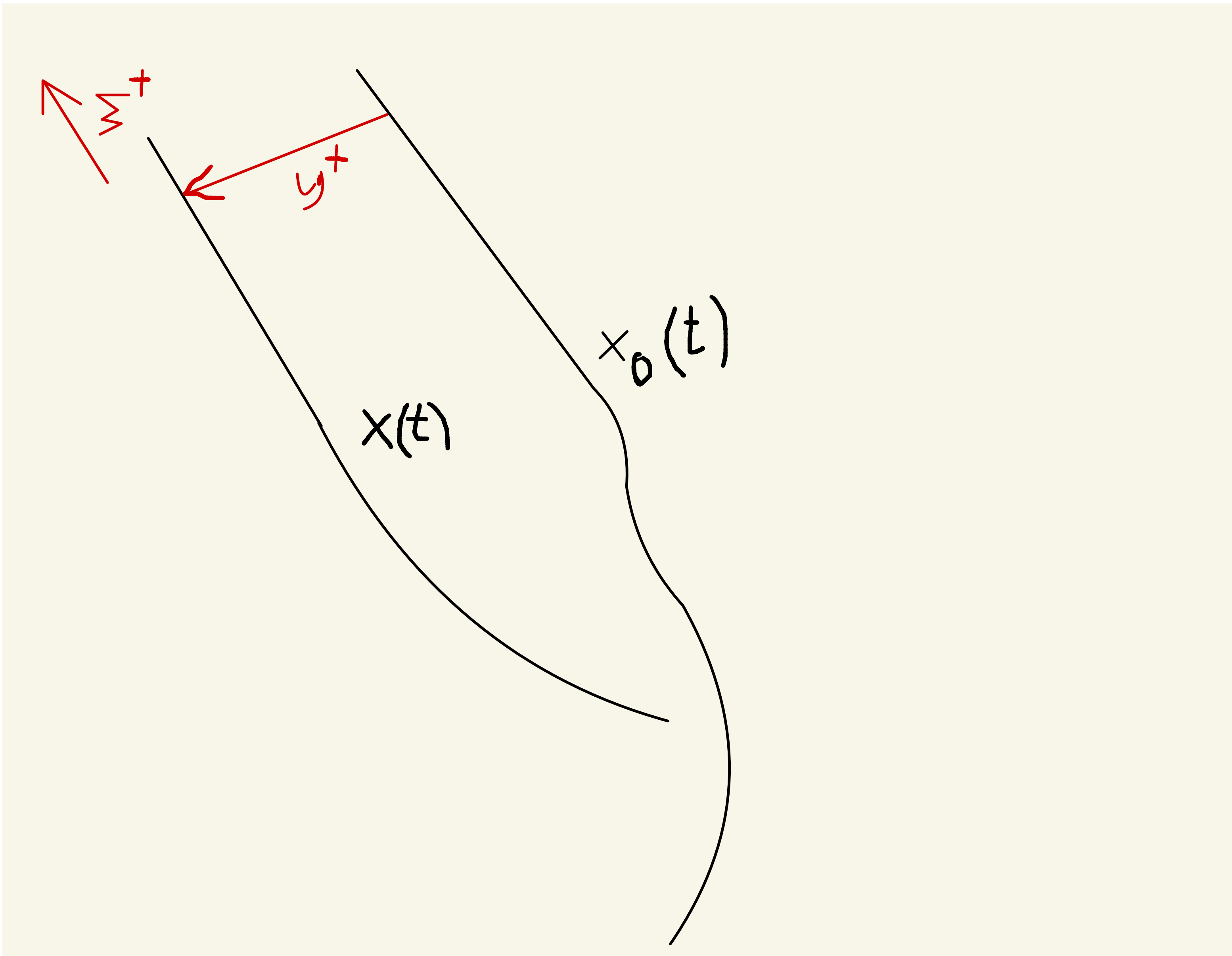
Theorem 0.4. *Assume a stronger long-range condition*

$$\nabla V(x)| \leq C\langle x\rangle^{-1-\mu}, \quad \nabla^2 V(x)| \leq C\langle x\rangle^{-2-\mu}, \quad \mu > 0.$$

1. *Suppose that two scattering trajectories have the same asymptotic momentum:* $\lim_{t\to\pm\infty} p_1(t) = \lim_{t\to\pm\infty} p_2(t)$. *Then there exists the relative asymptotic position* $\lim_{t\to\pm\infty} \big(x_1(t) - x_2(t)\big)$.
2. *Let* $x_0(t)$ *be a scattering trajectory. Let* $y^\pm \in \mathbb{R}^d$. *Then there exist unique trajectories* $x^\pm(t)$ *such that*

$$\lim_{t\pm\infty} p^\pm(t) = \lim_{t\to\pm\infty} p_0(t), \tag{9}$$

$$\lim_{t\pm\infty} \big(x^\pm(t) - x_0(t)\big) = y^\pm. \tag{10}$$



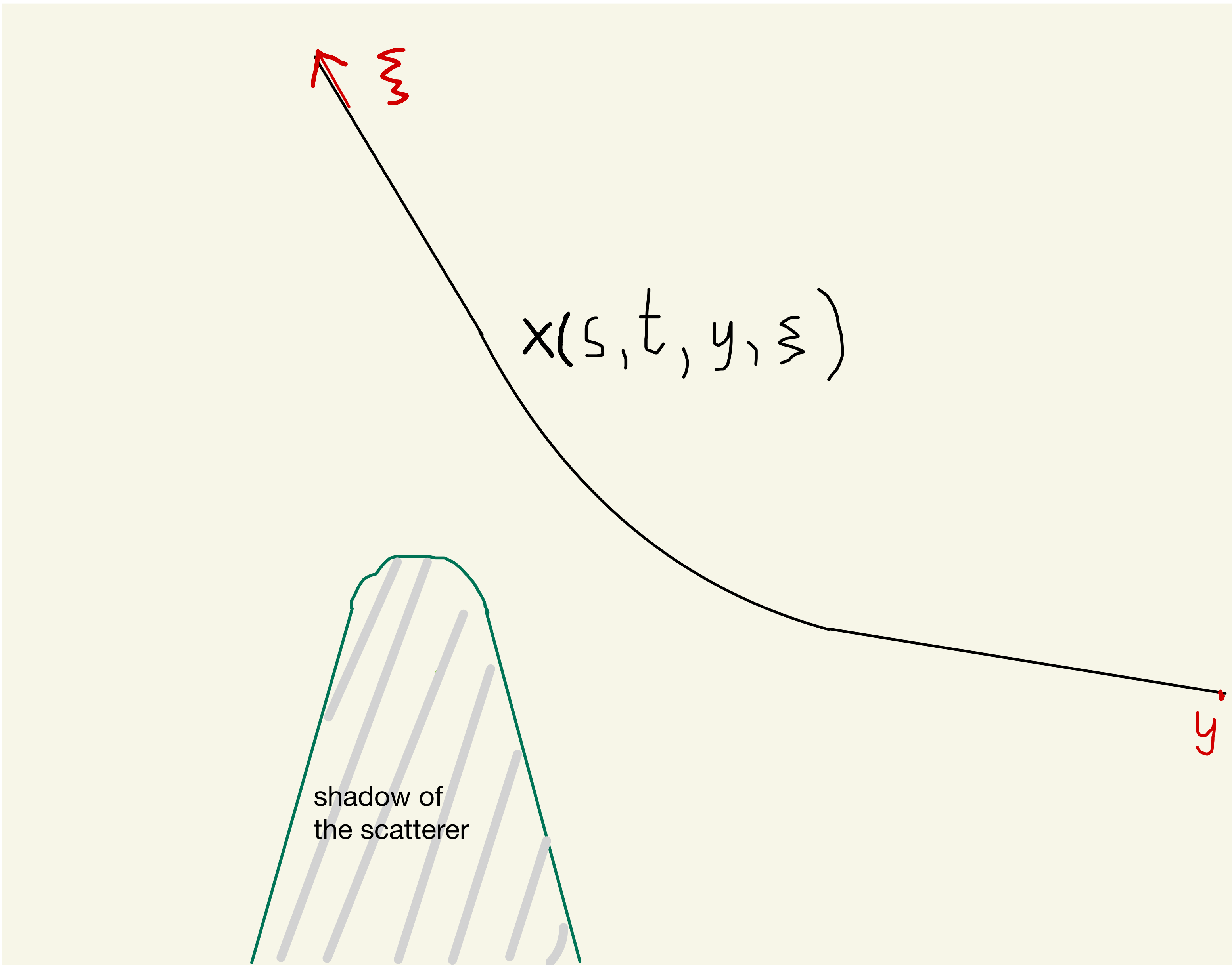
Σ^+
y^+
$X(t)$
$X_0(t)$

Thus, if for every $\xi \neq 0$ we fix reference trajectories $x_0^{\pm}(t,\xi)$, we can classify all scattering trajectories.

For a large class of boundary conditions which are not screened by the potential one can solve the following problem:

Theorem 0.5. *Impose the assumption of the previous theorem. Suppose that $b, \epsilon > 0$. Then there exists $a > 0$ such that if $|y| > a$, $|\xi| > b$, $\frac{y\cdot\xi}{|y||\xi|} > -1 + \epsilon$, then for any t there exists a unique family of trajectories $s \mapsto x(s,t,y,\xi)$ depending continuously on parameters such that*

$$x(0,t,y,\xi) = y, \quad p(t,t,y,\xi) = \xi. \tag{11}$$


ξ
x(s, t, y, ξ)
y
shadow of
the scatterer

The Lagrangian of our particle is $L(x,\dot{x}) := \frac{1}{2}\dot{x}^2 - V(x)$. Define the action along the trajectory $s \mapsto x(s,t,y,\xi)$

$$S(t,y,\xi) := \int_0^t \Big(\frac{1}{2}\dot{x}(s,t,y,\xi)^2 - V\big(x(s,t,y,\xi)\big)\Big)\mathrm{d}s. \qquad (12)$$

$S(t,y,\xi)$ is the generating function of the dynamics:

$$\nabla_y S(t,y,\xi) = p(0,t,y,\xi), \quad \nabla_\xi S(t,y,\xi) = x(t,t,y,\xi), \qquad (13)$$

It also satisfies the Hamilton-Jacobi equation

$$\partial_t S(t,y,\xi) = \frac{1}{2}\xi^2 + V\big(\nabla_\xi S(t,y,\xi)\big) \qquad (14)$$

$$= \frac{1}{2}\big(\nabla_y S(t,y,\xi)\big)^2 + V(y). \qquad (15)$$

Using this construction, with various y in different momentum patches if needed, we can construct a function

$$\mathbb{R} \times \left(\mathbb{R}^d \backslash \{0\}\right) \ni (t, \xi) \mapsto S(t, \xi) \tag{16}$$

such that for any $b > 0$ there exists T such that for $|\xi| > b$, $|t| > T$

$$\partial_t S(t, \xi) = \frac{1}{2}\xi^2 + V\left(\nabla_\xi S(t, \xi)\right). \tag{17}$$

This function provides a choice of reference scattering trajectories and can be used in the construction of quantum modified Møller operators. Note that if $V = 0$, then $S(t, \xi) = \frac{t}{2}\xi^2$.

Recall that almost bounded trajectories statisfy $\lim_{t\to\infty} \frac{x(t)}{t} = 0$. Their energy is always 0.

Here is an example: If $V(x) = -|x|^{-\mu}$, then

$$x(t) = ct^{\frac{2}{2+\mu}}. \tag{18}$$

N-body Schrödinger Hamiltonians

Consider a system of n non-relativistic particles interacting with pair potentials. We suppose that the configuration space of the ith particle is $X_i = \mathbb{R}^d$, $i = 1, \ldots, n$. The Hamiltonian is

$$H = \sum_{i=1}^{n} \frac{1}{2m_i} p_i^2 + \sum_{1 \le i < j \le n} V_{ij}(x_i - x_j). \tag{19}$$

Note that the Hamiltonian is invariant wrt Galileian transformations.

The configuration space $X := X_1 \oplus \cdots \oplus X_n$ is equipped with the scalar product

$$\langle x_1, \dots, x_n | x_1, \dots, x_n \rangle = \sum_{i=1}^{n} m_i x_i^2. \tag{20}$$

The minus Laplacian wrt this product is

$$-\Delta := \sum_{i=1}^{n} \frac{1}{m_i} p_i^2. \tag{21}$$

The kinetic energy is the half of (21).

We will say cluster for a subset of $\{1, \ldots, n\}$. An example of a cluster is a pair $\{i, j\}$.

A cluster decomposition is a partition of $\{1, \ldots, n\}$ into clusters. We denote by $\mathcal{A}$ the set of cluster decompositions.

Let a, b be cluster decompositions. We say that $b \leq a$ if b is finer than a. In particular, $\{1\} \ldots \{n\}$ is the minimal and $\{1, \ldots, n\}$ is the maximal element of $\mathcal{A}$.

For any cluster decomposition $a \in \mathcal{A}$ we define the corresponding collision plane

$$X_a := \{(x_1, \ldots, x_n) \in X \mid x_i = x_j, \ (i, j) \leq a\}. \qquad (22)$$

We set $X^a := X_a^\perp$. Clearly, $X = X_a \oplus X^a$, For every $a \in \mathcal{A}$ we have the corresponding factorization of the configuration space into internal and external degrees of freedom, the cluster Hamiltonian H_a, the internal Hamiltonian H^a and the external interaction:

$$X = X_a \oplus X^a, \tag{23}$$

$$\begin{aligned} H_a &:= \frac{1}{2}p^2 + \sum_{(i,j) \leq a} V_{ij}(x_i - x_j) \\ &= \frac{1}{2}p_a^2 + \frac{1}{2}(p^a)^2 + V_a(x^a) \ = \ \frac{1}{2}p_a^2 + H^a, \end{aligned} \tag{24}$$

$$I_a := \sum_{(ij) \not\leq a} V_{ij}(x_i - x_j), \tag{25}$$

$$H = H_a + I_a(x). \tag{26}$$

For instance, here is the separation of the center-of-mass motion:

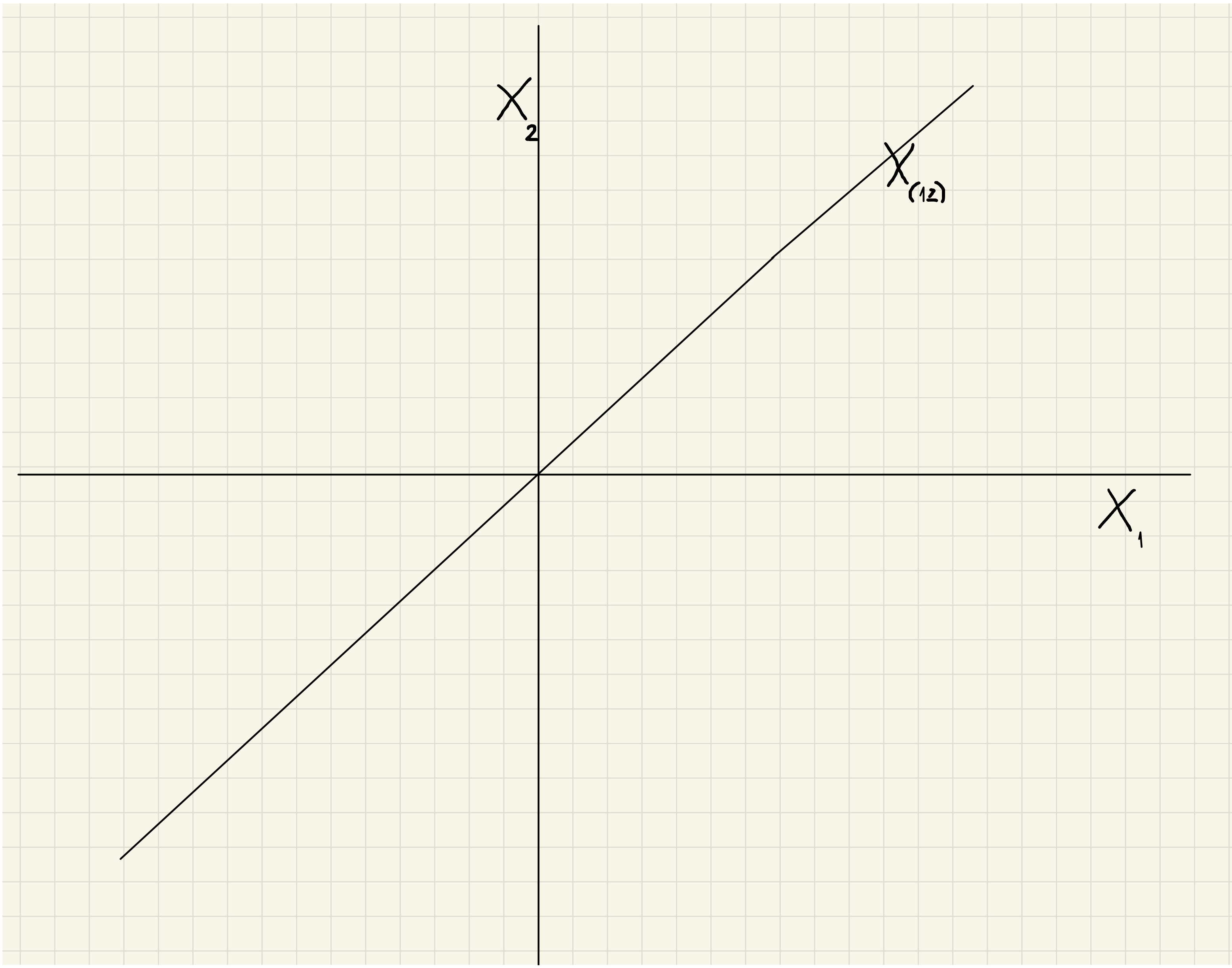
$$H = H_{\{1,\dots,n\}} = -\frac{1}{2}p^2_{\{1,\dots,n\}} + H^{\{1,\dots,n\}}. \tag{27}$$

The free Hamiltonian is $$H_{\{1\}\dots\{n\}} = \frac{1}{2}p^2. \tag{28}$$

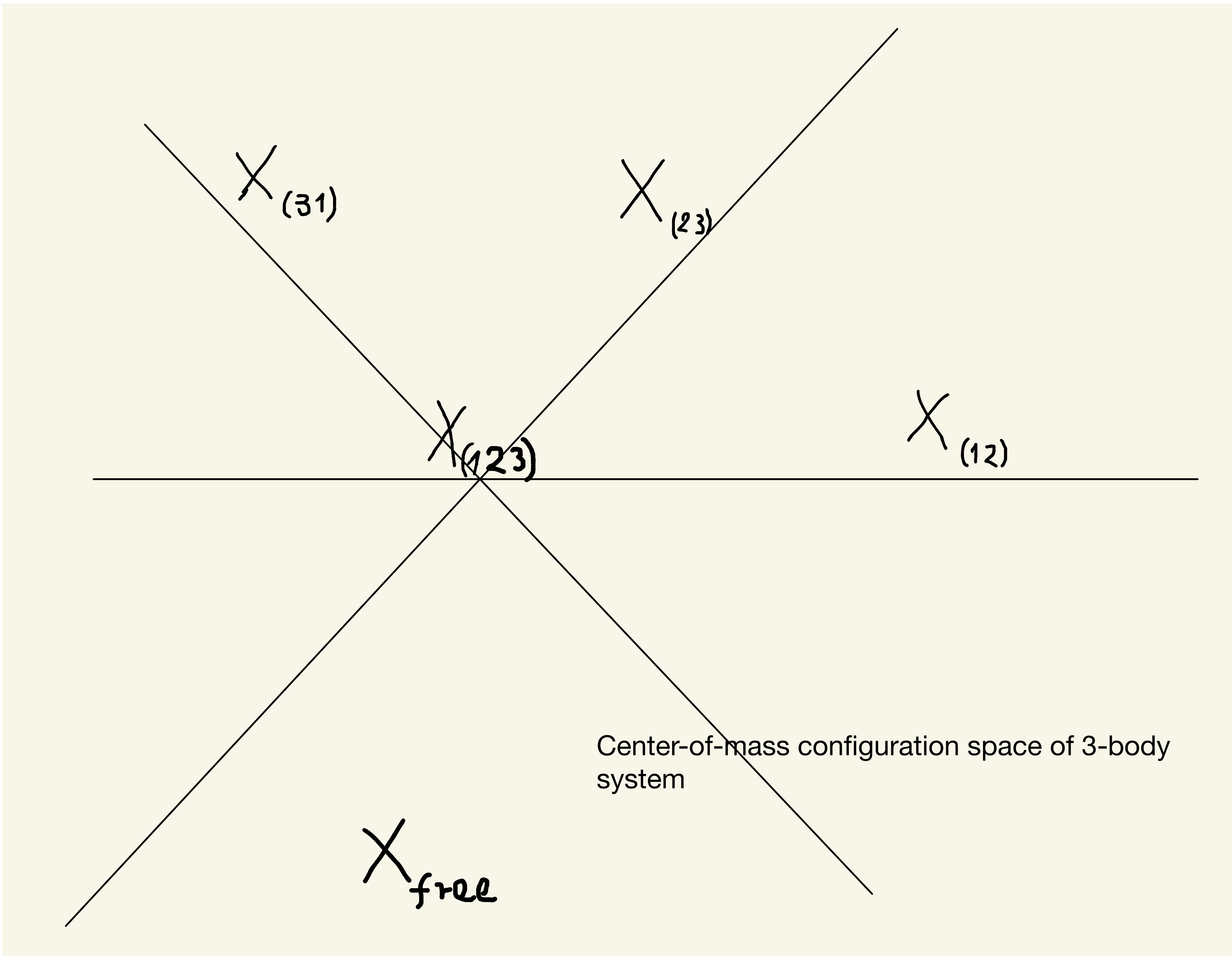
Note that $H^{\{1\}\dots\{n\}} = 0$.

If $a = \{c_1, \dots, c_k\}$ (a cluster decomposition a consists of clusters $c_1, \dots, c_k$) , then

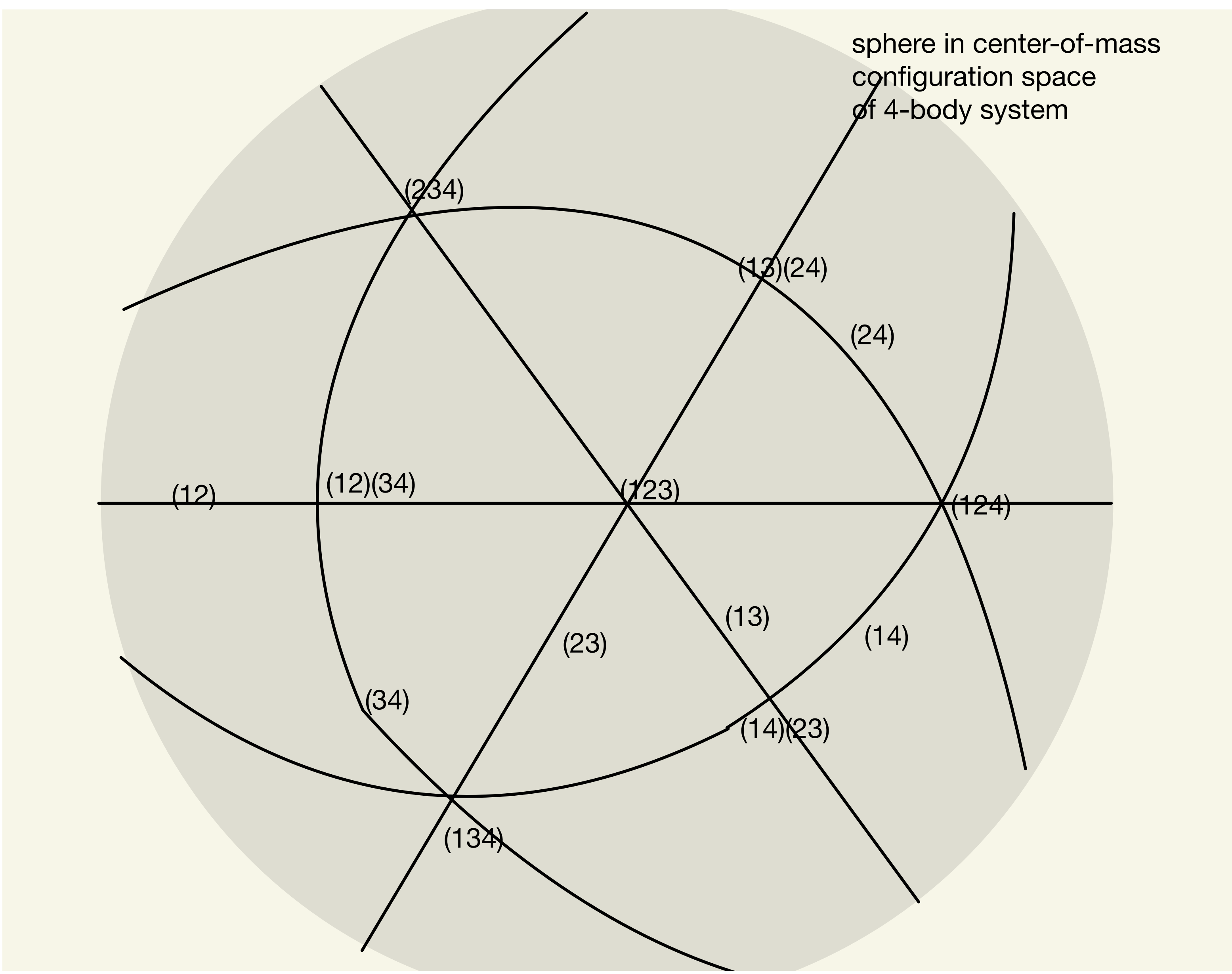
$$X_a = \bigcap_{i=1}^{k} X_{c_i}, \qquad X^a = X^{c_1} \oplus \cdots \oplus X^{c_k}. \tag{29}$$



X_2
$X_{(12)}$
X_1



Center-of-mass configuration space of 3-body system


sphere in center-of-mass
configuration space
of 4-body system
(234)
(13)(24)
(24)
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(13)
(14)
(23)
(34)
(14)(23)
(134)

It is probably impossible to classify all scattering trajectories of classical N-body systems in a similar way as in the quantum case. Thus classical N-body scattering theory is much less satisfactory than the quantum one.

However, there are some results. They can be viewed as a spin-off of the theory developed for the quantum case.

Theorem 0.6. *Suppose that*

$$|\nabla V_{ij}| \leq C\langle x_{ij}\rangle^{-1-\mu}, \quad \mu > 0. \tag{30}$$

Then for every trajectory $x(t) = (x_1(t), \ldots, x_n(t))$ there exists asymptotic velocity

$$\lim_{t\to+\infty} \frac{x(t)}{t}. \tag{31}$$

If the asymptotic velocity is zero, then

$$|x(t)| \leq C\langle t\rangle^{\frac{\mu}{2+\mu}}. \tag{32}$$

Theorem 0.7. *Suppose that the asymptotic velocity* p_a^+ *is contained in* $X_a \setminus \bigcup_{b \not\subset a} X_b$.

(1) **Short range case**. *If* $|\nabla V_{ij}| \leq C\langle x_{ij}\rangle^{-2-\mu}, \quad \mu > 0.$ *then there exists*

$$\lim_{t\to+\infty} \big(x_a(t) - tp_a^+\big). \tag{33}$$

(2) **Long range case**. *Suppose that* $\mu > \sqrt{3} - 1$ *and*

$$|\nabla V_{ij}| \leq C\langle x_{ij}\rangle^{-1-\mu}, \quad |\nabla^2 V_{ij}| \leq C\langle x_{ij}\rangle^{-2-\mu}, \tag{34}$$

Then there exists a unique trajectory $\tilde{x}_a(t)$ *of* $\frac{1}{2}p_a^2 + I_a(x_a)$ *such that*

$$\lim_{t\to+\infty} \big(x_a(t) - \tilde{x}_a(t)\big) \quad \textit{exists}. \tag{35}$$

Sketch of proof.

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\big(x_a(t) - \tilde{x}_a(t)\big) = \nabla_a I_a\big(x(t)\big) - \nabla_a I_a\big(x_a(t)\big) \tag{36}$$

$$= O\big(\nabla^2 I_a\big) O\big(x^a(t)\big) = \langle t \rangle^{-2-\mu} \langle t \rangle^{\frac{2}{2+\mu}}. \tag{37}$$

(37) is twice integrable if

$$\frac{2}{2+\mu} < \mu. \tag{38}$$

Thus we need to solve the quadratic equation $\mu^2 + 2\mu - 2 = 0$, which yields $\mu = \sqrt{3} - 1 \approx 0.73$.

In the quantum case, for potentials decaying as $x^{-\mu}$ with $\mu > \sqrt{3} - 1$ a much more satisfactory result can be shown: asymptotic completeness.

THANK YOU

FOR YOUR ATTENTION!