Introduction to data assimilation
Mini project 3

Miniproject 3: Adjoint code for logistic growth model w/ piecewise constant α.
We aim to explore the logistic growth model, where now the growth rate $\alpha = \alpha(t)$ is piecewise constant on (known) subintervals. In this context, the number of parameters can be large (it is equal to the number of subintervals for $\alpha(t)$, plus the capacity K and the initial condition y_0). Therefore we propose to implement an adjoint model to estimate the parameters.

We consider the following ODE that models the logistic growth of a population:

$$\begin{cases} y'(t) = \alpha(t)y(t)(1 - y(t)/K), & t \in [0, T], \\ y(t = 0) = y_0, & t = 0. \end{cases} \tag{1}$$

for a constant $K > 0$. The initial population is $y_0 > 0$. The growth rate $\alpha(t)$ is piecewise constant and defined as follows. Let $0 = u_0 < u_1 < \ldots < u_N = T$, then

$$\alpha(t) = \alpha_i \qquad t \in (u_i, u_{i+1}).$$

In this exercise we fix the time interval [0,T] = [0,200]. The values of y are computed on 1000 points equally spaced in this interval (option `t_eval` of `solve_ivp`).

a) Modify the function from (practical day 2) to implement a function that
   - takes as parameters $\theta = (K, y_0, \mathbf{u}, \boldsymbol{\alpha})$ where $\mathbf{u} = (u_1, \ldots, u_{N-1})$ and $\boldsymbol{\alpha} = (\alpha_0, \ldots, \alpha_{N-1})$
   - returns the solution of problem (1) on the interval $[0, T]$.

   Display different solutions with different values of $\alpha(t)$. (take $N = 5$ or $N = 10$).

b) The solution is observed at the time points $t_i, 0 \leq i \leq n - 1$. We assume that $(K, y_0, \mathbf{u})$ are known. The following cost function is to be minimized:

$$j(\boldsymbol{\alpha}) = \frac{1}{2} \sum_i (y(\boldsymbol{\alpha})(t_i) - y_{obs}(t_i))^2.$$

   Calculate the gradient of $j(\theta)$ using the adjoint method.

c) Implement the calculation of the gradient obtained at the previous question. Check the validity using finite differences.

d) *(optional)* Fix some values $\theta^* = (K^*, y_0^*, \mathbf{u}^*, \boldsymbol{\alpha}^*)$ where $K^* = 10^4, y_0^* = 1$ and the α_i are between 0.05 and 0.2 (take a subdivision into $N = 10$ subintervals). Compute the solution $y(\theta^*)$ and add gaussian noise to generate the observations.
   Perform a gradient descent to estimate the true parameters $\boldsymbol{\alpha}^*$ (starting from arbitrary values).