Introduction to data assimilation
Practical session Day 2: tangent and adjoint models

Exercice D2.1. **Tangent model and parameter estimation (logistic model)**
We consider the following ODE that models the logistic growth of a population:

$$\begin{cases} y'(t) = \alpha y(t)(1 - y(t)/K), & t \in [0, T], \\ y(t = 0) = y_0, & t = 0. \end{cases}$$

for some given constants $\alpha > 0$ and $K > 0$. The initial population is $y_0 > 0$. In this exercise we fix the time interval [0,T] = [0,200]. The values of y are computed on 1000 points equally spaced in this interval (option `t_eval` of `solve_ivp`).

a) Compute the derivatives wrt the parameters α, K, y_0. Implement a function Dy that takes as input α, K, y_0 and as output the Jacobian matrix of y wrt these 3 parameters. Check the validity of this function using finite differences.

b) Fix some values $\theta^* = (\alpha^*, K^*, y_0^*)$ close to $(0.1, 10^4, 1)$ up to a factor of e.g. 10. Compute the solution $y(\theta^*)$ and add gaussian noise to generate the observations. (Note: here the observation operator L is the identity operator).
Implement the Gauss-Newton algorithm to estimate the parameters with an initial guess that is not close from θ^*.

c) Perform the same study with a reduced number of observations points (100, then 10, then smaller values)

Exercice D2.2. **Tangent model and parameter estimation (SIR epidemiologic model)**
We consider the following ODE system (SIR model) that models the spread of an epidemics in a population:

$$\begin{cases} S'(t) = -\tau S(t) I(t), \\ I'(t) = \tau S(t) I(t) - \nu I(t) \\ R'(t) = \nu I(t), & t \in [0, T], \end{cases}$$

together with the initial condition: $S(t = 0) = N - I0, I(t = 0) = I_0, R(t = 0) = 0$.

In this exercise we fix the time interval [0,T] = [0,250], and the following values: $N = 6e7, I_0 = 1$. The values of y are computed on every integer point in this interval (observations available every day).

a) Compute the derivatives wrt the parameters τ, ν.
Implement a function Dy that takes as input τ, ν and as output the Jacobian matrix of y wrt these 2 parameters. Check the validity of this function using finite differences.

b) Fix some values $\theta^* = (\tau^*, \nu^*)$ close to $(3.10^{-9}, 10^{-1})$ up to a factor of e.g. 10. Compute the solution $y(\theta^*) = (S, I, R)$ and add gaussian noise to I to generate the observations. (Note: here the observation operator L consists of measuring the quantity $I(t)$).
Implement the Gauss-Newton algorithm to estimate the parameters with an initial guess that is not close from θ^*.

Exercice D2.3. **Adjoint logistic model**
We consider the logistic model:

$$\begin{cases} y'(t) = \alpha y(t)(1 - y(t)/K), & t \in [0, T], \\ y(t = 0) = y_0, & t = 0. \end{cases}$$

for some given constants $\alpha > 0$ and $K > 0$. The initial population is $y_0 > 0$. In this exercise we fix the time interval [0,T] = [0,200]. The values of y are computed on 1000 points equally spaced in this interval (option `t_eval` of `solve_ivp`). The parameters are $\theta = (\alpha, K)$.
We consider observations of the population

$$y_{obs}(t_i) = y(\theta^*)(t_i) + n,$$

where n is the realization of gaussian random noise with variance σ^2, and θ^* is the (true) value of the parameters. The following cost function is to be minimized:

$$j(\theta) = \frac{1}{2} \sum_i (y(\theta)(t_i) - y_{obs}(t_i))^2.$$

a) Calculate the gradient of j using the adjoint method.

b) Compute the adjoint state. Note: since it is a backwards ODE the solution using `solve_ivp` has to be carefully written (indicate the integration bounds $[T, 0]$ in this order).

c) Make a numerical check of the validity of the gradient thus obtained.