Sur les représentations linéaires d'ordre deux de la sixième équation de Painlevé

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Institut Henri Poincaré, Paris, 9 février 2024

Outline

Warning

Nothing new in this talk (last results date back to 2016, 2017).
The goal is twofold:
(i) to be aware of the existence of Lax pairs of P_{VI} other than Fuchs 1905, Jimbo Miwa 1981;
(ii) to present several open problems.

$\mathrm{P_{VI}}$, the unique "special" function beyond the elliptic fn.

L. Fuchs, H. Poincaré, É. Picard, R. Fuchs, P. Painlevé, B. Gambier

The so-called "special functions" belong to 3 disjoint subsets:
($\exists$ LODE) e^x, x^n, ${}_1F_2(a,b,c;x)$, $J_\nu(x)$, $\mathrm{Ai}(x)$, ...
($\exists$ NLODE) elliptic function $\wp(x, g_2, g_3)$ (Weierstrass, Jacobi),
($\nexists$ ODE) Euler $\Gamma(x)$ (Hölder 1886), Riemann $\zeta(x)$, ...
Can this set be extended in some systematic way?

Problem (L. Fuchs, H. Poincaré). To define (new) functions by algebraic ordinary differential equations.

Solution (Painlevé 1900, Gambier 1910):
First order ODEs. Except linearizable ODEs, only **one new**: the elliptic function.
Second order ODEs. Except linearizable and elliptic ODEs, only **six new**: the master function $\mathrm{P_{VI}}$ and its 5 degeneracies $\mathrm{P_V}$–$\mathrm{P_I}$.

$\mathrm{P_{VI}}$ of Picard. "Une équation différentielle curieuse"

Consider the elliptic integral

$$U = \frac{1}{2\omega} \int_\infty^u \frac{\mathrm{d}u}{\sqrt{u(u-1)(u-x)}}, \qquad (1)$$

equivalent to,

$$u = \frac{\wp(2\omega U, g_2, g_3) - e_1}{e_2 - e_1}, (e_1, e_2, e_3) = (0, 1, x) - \frac{x+1}{3} \cdot (2)$$

(Picard 1889) The functions $u(x)$ and $U(X)$, with $X = i\pi\omega'/\omega$, obey ODEs without movable critical singularities, namely

$$\frac{\mathrm{d}^2 U}{\mathrm{d}X^2} = 0, \text{(Legendre) } x(x-1)\frac{\mathrm{d}^2\omega}{\mathrm{d}x^2} + (2x-1)\frac{\mathrm{d}\omega}{\mathrm{d}x} + \frac{\omega}{4} = 0, (3)$$

and "une équation différentielle curieuse"

$$u'' = \frac{1}{2}\left[\frac{1}{u} + \frac{1}{u-1} + \frac{1}{u-x}\right] u'^2 - \left[\frac{1}{x} + \frac{1}{x-1} + \frac{1}{u-x}\right] u' + \frac{u(u-1)(u-x)}{2x^2(x-1)^2} \frac{x(x-1)}{(u-x)^2}.$$

The search for complementary terms to Picard

"Complementary" := which do not create movable critical singularities.

Huge efforts (see Appendix B.1 of *The Painlevé handbook*).

1889. For no reason, Roger Liouville discarded the case of all $\mathrm{P_n}$.

1895. Leçons de Stockholm, number 21, see Eq (B) page 508.

1898. Painlevé did enumerate the good class (CRAS 126 p 1185)

$$u'' = \left(\frac{1}{u} + \frac{1}{u-1} + \frac{1}{u-x}\right)\frac{u'^2}{2} + \frac{P_4(u,x)u' + P_6(u,x)}{u(u-1)(u-x)},$$

but argued (CRAS 126 p 1329) that it could only yield $\mathrm{d}^2 U/\mathrm{d}X^2 = 0$ and did not process this class.

1905. Richard Fuchs added four complementary terms.

1906. Painlevé "J'ai laissé échapper un des sous-cas les plus importants".

1910. Gambier proved that the four terms of Fuchs are the only admissible ones.

The complete $\mathrm{P_{VI}}$

In elliptic coordinates (Painlevé 1906)

$$\mathrm{P_{VI}} \; : \qquad \frac{\mathrm{d}^2 U}{\mathrm{d}X^2} = \frac{(2\omega)^3}{\pi^2} \sum_{j=\infty,0,1,x} \theta_j^2 \wp'(2\omega U + \omega_j, g_2, g_3). \quad (4)$$

In rational coordinates

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = \frac{1}{2}\left[\frac{1}{u} + \frac{1}{u-1} + \frac{1}{u-x}\right]\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 - \left[\frac{1}{x} + \frac{1}{x-1} + \frac{1}{u-x}\right]\frac{\mathrm{d}u}{\mathrm{d}x}$$
$$+ \frac{u(u-1)(u-x)}{2x^2(x-1)^2}\left[\theta_\infty^2 - \theta_0^2 \frac{x}{u^2} + \theta_1^2 \frac{x-1}{(u-1)^2} + (1-\theta_x^2)\frac{x(x-1)}{(u-x)^2}\right]$$

Invertible point transformation

$$U = \frac{1}{2\omega}\int_\infty^u \frac{\mathrm{d}u}{\sqrt{u(u-1)(u-x)}}, u = \frac{\wp(2\omega U, g_2, g_3) - e_1}{e_2 - e_1}, \quad (5)$$

$$X = \Omega = i\pi\frac{\omega'}{\omega}, x = \frac{e_3 - e_1}{e_2 - e_1}. \quad (6)$$

$\mathrm{P_{VI}}(U, X)$ is obviously Hamiltonian, therefore so is $\mathrm{P_{VI}}(u, x)$ (Malmquist 1922).

Outline

Invariances of $\mathrm{P_{VI}}$ and τ_{VI}

Invariances of (unique) $\mathrm{P_{VI}}$: 4 involutions $\theta_j \to -\theta_j$ (a tautology for the moment), 4! permutations of the singular points $\infty, 0, 1, x$.
Invariance of (four one-zero) τ_{VI} (Chazy 1911, Okamoto 1979) $\Rightarrow$ birational transformation (an involution) between rational $\mathrm{P_{VI}}(u, x, \theta_j)$ and rational $\mathrm{P_{VI}}(U, X, \Theta_j)$,

$$\mathrm{T_{VI}}: \begin{pmatrix}\theta_\infty \\ \theta_0 \\ \theta_1 \\ \theta_x\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1\end{pmatrix}\begin{pmatrix}\Theta_\infty \\ \Theta_0 \\ \Theta_1 \\ \Theta_x\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1 \\ 1 \\ 1 \\ 1\end{pmatrix} \tag{7}$$

$$\frac{N}{u-U} = \frac{x(x-1)U'}{U(U-1)(U-x)} + \frac{\Theta_0}{U} + \frac{\Theta_1}{U-1} + \frac{\Theta_x - 1}{U-x} \tag{8}$$

$$= \frac{x(x-1)u'}{u(u-1)(u-x)} + \frac{\theta_0}{u} + \frac{\theta_1}{u-1} + \frac{\theta_x - 1}{u-x}, \tag{9}$$

$$N = 1 - \Theta_\infty - \Theta_0 - \Theta_1 - \Theta_x = (1/2)\sum(\theta_j - \Theta_j). \tag{10}$$

Linear representations **cannot** preserve both, they must choose.

Outline

Second order linear representations

Second order $=$ the minimal order.
Notation: indep var $= x, X$; spectral parameter $= t, T$.

Four kinds: scalar or matrix, elliptic (U, X, T) or rational (u, x, t) coordinates.

Second order scalar Lax pair. The theory

Poincaré, Acta math. 1884 (before P_{VI} was known)

$$\left[\partial_t^n + \dots f_{n-1}(t)\partial_t^{n-1} + \dots + f_0(t)\right]\psi(*,t) = 0,\ f_j(t) \text{ rational.}$$

How many Fuchsian singularities (apparent + nonapparent) to put in $f_j(t)$'s for the monodromy to generate something interesting?

Very general counting by Poincaré 1884. Result $n = 2$ is:

3 nonapp. + 0 app. $\Rightarrow$ hypergeometric ODE (linear)

4 nonapp. + 0 app. $\Rightarrow$ Darboux ODE 1882 ($\equiv$ Heun 1889) (linear)

4 nonapp. + 1 app. $\Rightarrow$ nonlinear differential relation between apparent and crossratio(nonapparent).

Poincaré did not perform the computation and stopped there (a frequent behaviour, see talk by Chenciner 2012).

Second order matrix Lax pair. The theory

Schlesinger 1912 (after $\mathrm{P_{VI}}$ was known)

$\mathrm{d}\psi = (L\mathrm{d}x + M\mathrm{d}t)\psi,\ L_t - M_x + [L, M] = 0,\ t$=spectral.

Fuchsian singularities: **4 nonapparent + 0 apparent**

$$M = \frac{M_0}{t} + \frac{M_1}{t-1} + \frac{M_x}{t-x},\ L = -\frac{M_x}{t-x} + L_\infty,\ M_\infty = -M_0 - M_1 - M_x,$$
$$\frac{\mathrm{d}M_0}{\mathrm{d}x} = -\frac{1}{x}[M_0, M_x] - [M_0, L_\infty], \frac{\mathrm{d}M_1}{\mathrm{d}x} = -\frac{1}{x-1}[M_1, M_x] - [M_1, L_\infty],$$
$$\frac{\mathrm{d}M_x}{\mathrm{d}x} = \frac{1}{x}[M_0, M_x] + \frac{1}{x-1}[M_1, M_x] - [M_x, L_\infty], \frac{\mathrm{d}M_\infty}{\mathrm{d}x} = -[M_\infty, L_\infty].$$

- L and M can be chosen traceless. $(M_j, L_\infty)(x, u, u', \theta_j)$.
- $\forall j, \det(M_j) =$ constant ($\Leftrightarrow$ the four parameters of $\mathrm{P_{VI}}$).
- (M_∞ const.) $\Leftrightarrow$ ($L_\infty / M_\infty =$ scalar $f(x)$). This $\not\Rightarrow$ M_∞ const
- If M_∞ is invertible, there exists a change of basis allowing one to set L_∞ to zero and therefore to make the Lax pair unique.

Like Poincaré, Schlesinger did not compute and stopped there.

Outline

Linear representations. Overview of results

Table: R=rational, E=elliptic, S=scalar, M=matrix

type	name	sing.	dependence (θ_j)	invariance
R S	1905 Fuchs	4+1	affine (θ_j^2)	4!
R S	1912 Garnier	4+1	affine (θ_j^2)	4!
R S	1994 Suleimanov	4+0	affine (θ_j^2)	heat PDE, no u
E M	2004 Zotov	4+0	affine (θ_j)	4!
E S	2012 Zabrodin Zotov	4+0	affine (θ_j^2)	heat PDE, no U
R M	1981 Jimbo Miwa	4+0	aff($3\theta_j^2$), mero(θ_∞)	3!
R M	2016 Loray	4+0	quad($4\theta_j$)	3!
R M	2017 Bonnet (RC)	4+0	aff($3\theta_j^2$), quad(θ_x)	3!

Outline

The symmetry $u \leftrightarrow t$ (Garnier)

Fuchs 1905; Garnier 1912

Define the potential function V_{VI}

$$4V_{\mathrm{VI}}(z,s) = (\theta_\infty^2 - s)(z-x) + (\theta_0^2 - s)\left(\frac{x}{z} - 1\right) + (\theta_1^2 - s)\left(-\frac{x-1}{z-1} + 1\right) + (\theta_x^2 - s)\left(\frac{x(x-1)}{z-x} + 2x - 1\right) - 3z.$$

Then the Lax pair of Fuchs 1905

$$\left(\partial_t^2 + \frac{S}{2}\right)\psi_{\mathrm{d}} = 0,\ \left(\partial_x + C\partial_t - \frac{C_t}{2}\right)\psi_{\mathrm{d}} = 0,\ C = -\frac{t(t-1)(u-x)}{x(x-1)(t-u)},$$

displays a nice symmetry $u \leftrightarrow t$,

$$\frac{S}{2} + \frac{3/4}{(t-u)^2} = -\frac{\beta_1 u' + \beta_0}{t(t-1)(t-u)} - \frac{[(\beta_1 u')^2 - \beta_0^2](u-x)}{u(u-1)t(t-1)(t-x)} + \frac{1}{t(t-1)(t-x)}\left[V_{\mathrm{VI}}(u,1) - V_{\mathrm{VI}}(t,1)\right],$$

$$\beta_1 = -\frac{x(x-1)}{2(u-x)},\ \beta_0 = -\frac{u(u-1)}{2}\left(\frac{1}{u} + \frac{1}{u-1}\right). \qquad (11)$$

Outline

Matrix pair of Zotov, elliptic coordinates 1/2

Zotov 2004; Levin, Olshanetsky, Zotov, Russian Math Surveys 2014

Recall the Lamé equation

$$[\mathrm{d}^2/\mathrm{d}t^2 - 2(\wp(t) + \wp(a))]\, \varphi(t,a) = 0,$$

and its two solutions ("élément simple" of Halphen),

$$\varphi(t,a) = \frac{\sigma(t+a)}{\sigma(t)\sigma(a)} e^{-\zeta(a)t} \text{ (one pole } t=0, \text{residue } 1).$$

By the duality $U \leftrightarrow T$, two Lax pairs of the type

$$L = \sum_{j=\infty,0,1,x} \theta_j \partial_1 M_j, M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{\mathrm{d}U}{\mathrm{d}X} + \sum_{j=\infty,0,1,x} \theta_j M_j,$$

$$[\partial_X - L, \partial_T - M] \equiv -\frac{\mathrm{d}^2 U}{\mathrm{d}X^2} + \sum \theta_j^2 \wp'(2\omega U + \omega_j) = 0,$$

$\partial_1 =$ partial derivative with respect to the 1st argument of φ.

Matrix pair of Zotov, elliptic coordinates 2/2

Zotov 2004; Levin, Olshanetsky, Zotov Russian Math Surveys, 2014

First pair, 4 poles U and 1 pole T (Zotov 2004),

$$\left\{ \begin{array}{l} M_j = \begin{pmatrix} 0 & \varphi(U + \omega_j/(2\omega), T) \\ \varphi(-U + \omega_j/(2\omega), T) & 0 \end{pmatrix}, \\ \text{parameters} = (\theta_j). \end{array} \right.$$

Second pair, 4 poles T and 1 pole U (LOZ 2014),

$$\left\{ \begin{array}{l} M_j = \begin{pmatrix} 0 & \varphi(U, T + \omega_j/(2\omega)) \\ \varphi(-U, T + \omega_j/(2\omega)) & 0 \end{pmatrix}, \\ \text{parameters} = (1/2 - \text{elementary birational of } \theta_j). \end{array} \right.$$

Transition matrix between the two is (LOZ (8.32)),

$$P = \begin{pmatrix} +\sigma(2\omega(T - U) + \omega_1) & -\sigma(2\omega(T + U) + \omega_1) \\ -\sigma(2\omega(T - U) + \omega_2) & +\sigma(2\omega(T + U) + \omega_2) \end{pmatrix}.$$

Outline

Matrix pair of Jimbo and Miwa, assumptions

Jimbo and Miwa, Physica D 1981; traceless form, Mahoux, Cargèse 1999

$$M_\infty = \frac{1}{2}\begin{pmatrix} a_\infty & 0 \\ 0 & -a_\infty \end{pmatrix}, L_\infty = 0,$$

$$M_j = \frac{1}{2}\begin{pmatrix} z_j & (\theta_j - z_j)u_j \\ (\theta_j + z_j)/u_j & -z_j \end{pmatrix}, j = 0, 1, x,$$

constant $a_\infty \neq 0$ (otherwise only 3 singularities, $\mathrm{P_{VI}}$ impossible), $z_0, z_1, u_0, u_1 =$ four unknown functions of three variables x, u, u', u of $\mathrm{P_{VI}} :=$ the unique zero $t = u$ of M_{12}.

Matrix pair of Jimbo and Miwa, result

Jimbo and Miwa 1981

$a_\infty = \theta_\infty - 1 \Longrightarrow$ Lax pair does not exist for $\theta_\infty^2 = 1$ (denominator a_∞ as expected in matrix elements).

Off-diagonal elements M_j are not algebraic, only their log deriv is (however not a big problem).
The spectral curve is OK (genus one).

Second order matrix Lax pair of Loray, assumptions

Frank Loray, Izvestiya: Mathematics 2016

Motivation: remove the restriction $\theta_\infty^2 \neq 1$ of JM 1981.
Assumptions:

$$M_\infty = \text{Jordan} = \begin{pmatrix} a_\infty/2 & \neq \text{constant} \\ 0 & -a_\infty/2 \end{pmatrix}, L_\infty = \begin{pmatrix} f_{11}(x) & f_{12}(x) \\ f_{21}(x) & -f_{11}(x) \end{pmatrix},$$

$$M_j = \begin{pmatrix} a_j & b_j \\ (\theta_j^2/4 - a_j^2)/b_j & -a_j \end{pmatrix}, j = 0, 1, x \text{ (invariance 3!)},$$

$$(b_0, b_1, b_x) = \left(-\frac{u}{x}, \frac{u-1}{x-1}, \frac{u-x}{x(1-x)}\right) \text{ so that } M_{12} = \frac{t-u}{t(t-1)(t-x)},$$

$$(a_0, a_1, a_x) = \text{Riccati-like.}$$

(Zero-curvature condition) $\Leftrightarrow$ (values of $f_{11}(x), f_{12}(x), f_{21}(x)$).

Second order matrix Lax pair of Loray, result

Frank Loray, 2016

Sucess to remove the restriction of JM.

$a_\infty = \theta_\infty - 1$.

Quadratic dependence on all θ_j's.

Spectral curve OK (genus one).

Second order matrix Lax pair of Bonnet, assumptions

Bonnet 1867; Weingarten 1897; Bobenko 1994; Bobenko and Eitner 1998; RC 2017

No assumption at all. Three pieces of information glued together (RC 2017):
(i) The mean curvature of Bonnet surfaces is identical to (BE 1998) the log deriv of the one-zero tau fn of Chazy for $\mathrm{P_{VI}}$ (=a Hamiltonian);
(ii) The moving frame of any surface in $\mathbb{R}^3$ (Weingarten) can be described (Bobenko 1994) by a traceless second order matrix pair.
(iii) Existence of a birational transformation (Chazy 1911, Okamoto 1980) between $\mathrm{P_{VI}}$ and τ_{VI}.

Remark. BE 1998 did not find this Lax pair because they took JM pair as an input and forced the moving frame to match JM.

Second order matrix Lax pair of Bonnet, result

RC CRAS, JMP 2017

$$M_\infty = \text{Jordan} = \frac{1}{4}\begin{pmatrix} 2a & -4 \\ a^2 - \theta_\infty^2 & -2a \end{pmatrix},\ a = \text{any constant},$$

$$(M_x)_{12} = 0,$$

$$L_\infty = -\frac{u-x}{x(x-1)} M_\infty,$$

$$M_{12} = \frac{t-u}{t(t-1)},$$

$$(L, M) = \text{affine}(\theta_\infty^2, \theta_0^2, \theta_1^2), \text{quadratic}(\theta_x).$$

The matrix Lax pair of $\mathrm{P_{VI}}$, rational(u,u'), polynomial(θ_j)

RC CRAS, JMP 2017

$$
\begin{cases}
L = -\dfrac{M_x}{t-x} - \dfrac{u-x}{x(x-1)} M_\infty,\ M_\infty = \dfrac{1}{4}\begin{pmatrix} 2a & -4 \\ a^2-\theta_\infty^2 & -2a \end{pmatrix}, \\
M_0 = -\dfrac{1}{2(u-x)} \begin{pmatrix} e_0 & -2u(u-x) \\ \dfrac{e_0^2-\theta_0^2(u-x)^2}{2u(u-x)} & -e_0 \end{pmatrix}, \\
M_1 = \dfrac{1}{2(u-x)} \begin{pmatrix} e_1 & -2(u-1)(u-x) \\ \dfrac{e_1^2-\theta_1^2(u-x)^2}{2(u-1)(u-x)} & -e_1 \end{pmatrix}, \\
M_x = \dfrac{1}{2}\begin{pmatrix} -\Theta_x & 0 \\ 2M_{x,21} & \Theta_x \end{pmatrix}, \\
e = x(x-1)u' + \Theta_x u(u-1), \\
e_0 = e - (\Theta_x - a)u(u-x), \\
e_1 = e - (\Theta_x - a)(u-1)(u-x), \\
-4\det M_j = \theta_j^2,\ j=\infty,0,1;\ -4\det M_x = \Theta_x^2,\ \Theta_x^2 = (\theta_x-1)^2,
\end{cases}
$$

$a =$ irrelevent arb. constant, e.g. $a = \Theta_x$. No residue ever vanishes.

The matrix Lax pair of $\mathrm{P_{VI}}$, normalized by $L_\infty = 0$

$$
\left\{
\begin{array}{l}
\theta_\infty \ne 0: \ \ L = -\frac{M_x}{t-x},\ \ M = \frac{M_0}{t} + \frac{M_1}{t-1} + \frac{M_x}{t-x}, \\
M_\infty = \frac{1}{2}\begin{pmatrix} \theta_\infty & 0 \\ 0 & -\theta_\infty \end{pmatrix},\ \mathrm{tr}\, M_j = 0, M_{j,21}(\theta_\infty) = M_{j,12}(-\theta_\infty), \\
M_{0,11} = \ \ \frac{u-1}{N}\left[(e - \Theta_x u(u-x))^2 - (u-x)^2(\theta_0^2 + \theta_\infty^2 u^2)\right], \\
M_{0,12} = \ \ \frac{u-1}{N}\left[(e - (\Theta_x + \theta_\infty) u(u-x))^2 - (u-x)^2\theta_0^2\right] g, \\
M_{1,11} = -\frac{u}{N}\left[(e - \Theta_x (u-1)(u-x))^2 - (u-x)^2(\theta_1^2 + \theta_\infty^2 (u-1)^2)\right], \\
M_{1,12} = -\frac{u}{N}\left[(e - (\Theta_x + \theta_\infty)(u-1)(u-x))^2 - (u-x)^2\theta_1^2\right] g, \\
M_{x,11} = \ \ \frac{1}{N}\left[e^2 - (u-x)^2\left[(\theta_\infty^2 + \Theta_x^2)u(u-1) - \theta_0^2(u-1) + \theta_1^2 u)\right]\right], \\
M_{x,12} = \ \ \frac{1}{N}\left[e^2 - (u-x)^2\left[((\Theta_x + \theta_\infty)^2)u(u-1) - \theta_0^2(u-1) + \theta_1^2 u)\right]\right] g, \\
-4\,\det M_j = \theta_j^2,\ j = \infty, 0, 1;\ -4\,\det M_x = \Theta_x^2,\ \Theta_x^2 = (\theta_x - 1)^2,
\end{array}
\right. \tag{12}
$$

$$
(\log g)' = \theta_\infty \frac{u-x}{x(x-1)},\ \ N = 4\theta_\infty u(u-1)(u-x)^2.
$$

Outline

A few open problems in rational coordinates

1. Find a discrete Lax pair of some discrete P_{VI} (e-, *q*- of d-) **holomorphic** in all parameters.
Indeed, Jimbo and Sakai 2006 of q-P_{VI} is meromorphic, for the same reason than JM 1981 (diagonal assumption creates a denominator).

2. Discretize the Lax pair of either Loray or Bonnet into a discrete Lax pair holomorphic in the four θ_j.

3. Discretize the Gauss-Codazzi (nonlinear) and/or the Gauss-Weingarten (linear) equations, deduce a "discrete Bonnet surface" i.e. a natural (because of geometric origin) discrete P_{VI}.

A few open problems in elliptic coordinates

Discretize the Lax pair of Zotov (either one of the two dual Lax pairs).

Outline

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Outline

The two fundamental quadratic forms

Gauss 1827

$\mathbf{F}(x_1, x_2)$:=point on the surface, d$\mathbf{F}$=vector in the tangent plane,
$\mathbf{N}$:=any unit vector normal to the tangent plane.
(Surfaces in $\mathbb{R}^3$) $\Leftrightarrow$ (two "fundamental" quadratic forms):
$\mathrm{I} =< \mathrm{d}\mathbf{F}, \mathrm{d}\mathbf{F} >$, $\mathrm{II} = - < \mathrm{d}\mathbf{F}, \mathrm{d}\mathbf{N} >$,
In "conformal coordinates", these quadratic forms

$$\mathrm{I} =< \mathrm{d}\mathbf{F}, \mathrm{d}\mathbf{F} >= e^{\upsilon} \mathrm{d}z \; \mathrm{d}\bar{z},$$
$$\mathrm{II} = - < \mathrm{d}\mathbf{F}, \mathrm{d}\mathbf{N} >= Q \; \mathrm{d}z^2 + e^{\upsilon} H \mathrm{d}z \; \mathrm{d}\bar{z} + \overline{Q} \; \mathrm{d}\bar{z}^2,$$

define four real fields: υ real, Q complex, H real.
Link with the two principal curvatures $1/R_1$ and $1/R_2$:

$$\frac{1}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = \text{mean curvature} = H,$$
$$\frac{1}{R_1 R_2} = \text{total(=Gaussian) curvature} = H^2 - 4e^{-2\upsilon}|Q|^2 = -2e^{-\upsilon}\upsilon_{z\bar{z}}.$$

The moving frame equations, a linear system

Weingarten, Acta mathematica 1897; Bobenko 1994; Bobenko and Eitner 2000

The moving frame $\sigma = {}^{\mathrm{t}}(\mathbf{F}_z, \mathbf{F}_{\bar{z}}, \mathbf{N})$ evolves linearly as
$\sigma_z = \mathbb{U}\sigma, \ \sigma_{\bar{z}} = \mathbb{V}\sigma$,

$$\begin{pmatrix} \upsilon_z & 0 & Q \\ 0 & 0 & (H-c)e^{\upsilon}/2 \\ -(H-c) & -2e^{-\upsilon}Q & 0 \end{pmatrix}, \mathbb{V} = \begin{pmatrix} 0 & 0 & (H+c) \\ 0 & \upsilon_{\bar{z}} & \overline{Q} \\ -2e^{-\upsilon}\overline{Q} & -(H+c) & 0 \end{pmatrix}$$

equivalent to (Bobenko and Eitner 2000)

$$\mathbb{U} = \begin{pmatrix} \upsilon_z/4 & -Qe^{-\upsilon/2} \\ (H+c)e^{\upsilon/2}/2 & -\upsilon_z/4 \end{pmatrix}, \mathbb{V} = \begin{pmatrix} -\upsilon_{\bar{z}}/4 & -(H-c)e^{\upsilon/2}/2 \\ \overline{Q}e^{-\upsilon/2} & \upsilon_{\bar{z}}/4 \end{pmatrix}.$$

c=additional parameter when replacing $\mathbb{R}^3$ by ($\mathbb{S}^3$, $\mathbb{R}^3$ or $\mathbb{H}^3$) $\subset$ $\mathbb{R}^4$.

The three nonlinear equations

Gauss 1827; Peterson 1853 = Mainardi 1856 = Codazzi 1868. Also Bour, Bonnet

The condition $(\sigma_z)_{\bar{z}} = (\sigma_{\bar{z}})_z$ generates three nonlinear PDEs,

$$\begin{cases} \upsilon_{z\bar{z}} + \frac{1}{2}(H^2 - c^2)e^{\upsilon} - 2|Q|^2 e^{-\upsilon} = 0, \text{ (Gauss)} \\ Q_{\bar{z}} - \frac{1}{2}H_z e^{\upsilon} = 0, \ \overline{Q}_z - \frac{1}{2}H_{\bar{z}} e^{\upsilon} = 0. \text{ (Codazzi)} \end{cases}$$

This system (**Gauss-Codazzi** equations) is **underdetermined**, i.e. one additional condition can be enforced.
Thm of Gauss and Bonnet: (Any solution $(\upsilon, H, Q, \overline{Q})$) $\Longrightarrow$ (unique surface up to rigid motion (déplacement)).

Outline

The problem of Bonnet

Pierre-Ossian Bonnet, J. École polytechnique 42 (1867) 1–151, gallica.bnf.fr

Stated as an application of the newly discovered Codazzi equations.

Problem (Bonnet 1867 §11 pp. 72–73). Given a surface in $\mathbb{R}^3$, to find all surfaces which are applicable on that surface and possess the same two principal radii of curvature.

"Applicable" $\Leftrightarrow$ (same first fundamental form) $\Leftrightarrow$ (same υ).
"Same R_1, R_2" $\Leftrightarrow$ (same H and $|Q|^2$).

Therefore, the problem is: given υ, H, $|Q|^2$, to find all $\omega := \arg Q$.

The solution of Bonnet

Pierre-Ossian Bonnet (1867) §11–12 pp 72–92

By elimination of $(\omega_z, \omega_{\bar{z}})$, the variable $e^{i\omega}$ obeys a second degree algebraic equation whose coefficients only depend on υ, H, $|Q|$,

$$H_{\bar{z}}(\log\alpha)_{\bar{z}} e^{i\omega} + H_z(\log\beta)_z e^{-i\omega} - 4e^{-\upsilon}|Q|\upsilon_{z\bar{z}} + e^{\upsilon}|Q|^{-2}H_z H_{\bar{z}} = 0,$$
$$\alpha = e^{\upsilon}|Q|^{-2}H_{\bar{z}},\ \beta = e^{\upsilon}|Q|^{-2}H_z.$$

Two cases $(\alpha_{\bar{z}}\beta_z \neq 0)$, $(\alpha_{\bar{z}} = \beta_z = 0)$, and the second case, which then admits the two first integrals

$$\alpha = g_1(z),\ \beta = g_2(\bar{z}),$$

splits into three subcases $(H_{\bar{z}}, H_z) = (0,0)$, $(0, \neq 0)$,$(\neq 0, \neq 0)$. Finally the subcase $H_{\bar{z}}H_z \neq 0$ splits into three different subsubcases, yielding a total of five types of analytic surfaces.

Solution of Bonnet, the five types of analytic surfaces

Pierre-Ossian Bonnet (1867) §11–12 pp 72–92

Table: The five types of analytic surfaces which solve the Bonnet problem.
The page numbers refer to Bonnet 1867.
CMC:=constant mean curvature surfaces.

	Condition on H	Analytic surfaces	Comment	Pages
1	$H_{\bar{z}} = 0, H_z = 0$	CMC (two arb f + one PDE)	s-Gordon, Liouv.	76–78
2	$H_{\bar{z}} = 0, H_z \neq 0$	One cone (two arb f)	not real (Bonnet)	78–81
3	$H_{\bar{z}} H_z \neq 0$ $h' + h^2 - c^2 = 0$	Dual to minimal (two arb f)	not real (Cartan)	81–84
4	$H_{\bar{z}} H_z \neq 0$ $h' + h^2 - c^2 \neq 0$	Bonnet surfaces (6-param)	(52)=H_{VI}	84–85
5	$\alpha_{\bar{z}} \beta_z \neq 0$	Bonnet pairs	3 PDEs p 90	85–92

The fourth type: Bonnet surfaces

Bonnet (1867) pp 84–85

Definition:

$$\begin{cases} e^{\upsilon}|Q|^{-2}H_{\bar z} = g_1(z) \neq 0,\ \ e^{\upsilon}|Q|^{-2}H_z = g_2(\bar z) \neq 0, \\ \dfrac{\mathrm{d}H}{(1/2)(g_1\mathrm{d}z + g_2\mathrm{d}\bar z)} + H^2 - c^2 \neq 0, \end{cases}$$

Integration: after a conformal transformation,

$$\begin{cases} \xi \text{ defined by } \mathrm{d}\xi = (1/2)(g_1\mathrm{d}z + g_2\mathrm{d}\bar z), \\ |Q| = \dfrac{k}{\sinh k(\xi - \xi_0)},\ \tanh\dfrac{i\omega}{2} = \tanh k\dfrac{\Re(z-z_0)}{2}\coth k\dfrac{\Im(z-z_0)}{2i}, \end{cases}$$

$H = h(\xi)$ obeys the third order ODE (Bonnet Eq. (52) p. 84)

$$-\frac{1}{8}(\log h')'' - \frac{1}{2}h' + \frac{k^2}{\sinh^2 k(\xi-\xi_0)}\left(\frac{1}{8} + \frac{h^2 - c^2}{8h'}\right) = 0.$$

Total six parameters: $z_0, \bar z_0, k$ and three integration constants.

Link to the geometric description page 85=+91

Bonnet surfaces. Integration with $\mathrm{P_{VI}}$, 130 years later

Hazzidakis 1897; Bobenko, Eitner 1998

First integral (Hazzidakis 1897 for $c = 0$)

$$K = \left(\frac{h''}{h'} + 2k \coth k(\xi - \xi_0)\right)^2 + 8\left[\left(\frac{k}{\sinh k(\xi - \xi_0)}\right)^2 \frac{h^2 - c^2}{h'} + h' + 2k \coth k(\xi - \xi_0) h\right] \quad (13)$$

É. Cartan (1942) : "Une étude des singularités de (13) paraît du reste difficile."

BE 1998: the general solution is $h = (\log \tau(\xi))'$, with $\tau =$ the tau-function of either a codimension-two $\mathrm{P_{VI}}$ ($k \neq 0$) or a codimension-three $\mathrm{P_V}$ ($k = 0$). Like entire functions, $\tau(\xi)$ has no movable singularities, it has only one movable simple zero (Chazy 1911).

$\mathrm{P_{VI}}$ and its τ-functions

Fuchs 1905; Painlevé 1906; Chazy 1911 p 341; Malmquist 1922

$\mathrm{P_{VI}}$ is the *unique* second order ODE able to define a function, and irreducible to order one or to linear.

$$\mathrm{P_{VI}}: \ \frac{d^2u}{dx^2} = \frac{1}{2}\left[\frac{1}{u}+\frac{1}{u-1}+\frac{1}{u-x}\right]u'^2 - \left[\frac{1}{x}+\frac{1}{x-1}+\frac{1}{u-x}\right]u' + \frac{u(u-1)(u-x)}{2x^2(x-1)^2}\left[\theta_\infty^2 - \theta_0^2\frac{x}{u^2} + \theta_1^2\frac{x-1}{(u-1)^2} + (1-\theta_x^2)\frac{x(x-1)}{(u-x)^2}\right],$$

and there exist 2x4=8 τ-functions (without movable singularities), e.g. (Painlevé 1906 Eq. (3), only two simple poles of residue one),

$$\frac{d}{x(x-1)dx}\log\tau = \frac{x(x-1)u'^2}{4u(u-1)(u-x)} + \frac{1}{4x(x-1)}\left[\theta_\infty^2\left(\frac{1}{2}-u\right) + \theta_0^2\left(\frac{1}{2}-\frac{x}{u}\right) + \theta_1^2\left(\frac{1}{2}-\frac{x-1}{u-1}\right) + (\theta_x-1)^2\left(\frac{1}{2}-\frac{x(x-1)}{u-x}-x\right)\right] =: \mathrm{H_{VI}}.$$

Comparison with the Lax pair of Jimbo and Miwa

Jimbo and Miwa II, 1981; Lin, RC, Musette, JNMP 2003; RC, RIMS 2007

Jimbo and Miwa made three assumptions:

1. The residue M_∞ is constant and diagonalizable;
2. (following Schlesinger) $L_\infty = 0$, since M_∞ is diagonalizable;
3. representation of det M_j=constant as

$$M_\infty = \frac{1}{2}\begin{pmatrix}\Theta_\infty & 0\\ 0 & -\Theta_\infty\end{pmatrix}, M_j = \frac{1}{2}\begin{pmatrix} z_j & (\theta_j - z_j)u_j\\ (\theta_j + z_j)/u_j & -z_j\end{pmatrix},\ j \neq \infty, \quad (33)$$

$z_0, z_1, u_0, u_1 =$ four functions of three variables x, u, u'.

This resulted in quite intricate expressions, **meromorphic** in Θ_∞.

Thanks to the geometric origin of the present results, none of these assumptions is required.

A posteriori, the structure of the residues is not (33) but

$$M_\infty = \frac{1}{2}\begin{pmatrix}\theta_\infty & 0\\ 0 & -\theta_\infty\end{pmatrix},\ M_j = \frac{f_j}{\theta_\infty}\begin{pmatrix} P_{j,11} & P_{j,12}g\\ -P_{j,21}/g & -P_{j,11}\end{pmatrix},\ j \neq \infty, \quad (34)$$

$P_{j,kl} :=$ second degree, monic polynomials of u'.