Axiomatisation et analyse conceptuelle chez Gödel

Fabien Carbo-Gil (Doctorant CGGG)
fabien.carbo@protonmail.com

Direction : Gabriella Crocco (CGGG)
Co-Direction : Lionel Nguyen Van Thé (I2M)

Incomplétude des systèmes axiomatiques formels

Théorèmes d'incomplétude :

Soit T une théorie du premier ordre récursivement axiomatisable, cohérente et capable de « formaliser l'arithmétique » :

1_On peut construire un énoncé arithmétique qui est indécidable dans T.

2_La cohérence de T est indécidable dans T.

Axiomatisation et Analyse conceptuelle

- Saisir (percevoir) un concept, déterminer son sens, signifie comprendre ses relations avec les autres concepts ET ses conditions d'applicabilité
- On peut axiomatiser un concept même si on ne le perçoit pas de manière claire et complète. Dans ce cas, les axiomes ne sont pas des définitions implicites du concept, mais ils rendent explicites certaines de ses propriétés.
- Le sens de certains concepts mathématiques ne peut être capturé par aucun système formel d’axiomes
- Cela laisse la possibilité d’une axiomatisation informelle du type ZFC+Axiomes de Grands cardinaux

Aix*Marseille université

Progressivité de la perception des concepts

7.3.8 *There are cases where we mix two or more exact concepts in one intuitive concept and then we seem to arrive at paradoxical results. One example is the concept of continuity. Our prior intuition contains an ambiguity between smooth curves and continuous movements. We are not committed to the one or the other in our prior intuition. In the sense of continuous movements a curve remains continuous when it includes vibrations in every interval of time, however small, provided only that their amplitudes tend toward 0 if the time interval does. But such a curve is no longer smooth. The concept of smooth curves is seen sharply through the exact concept of differentiability. We find the example of space-filling continuous curves disturbing because we feel intuitively that a continuous curve, in the sense of being a smooth one, cannot fill the space. When we realize that there are two different sharp concepts mixed together in the intuitive concept, the paradox disappears. Here the analogy with sense perception is close. We cannot distinguish two neighboring stars a long distance away. But by using a telescope we can see that there are indeed two stars.*

7.3.10 *We begin with vague perceptions of a concept, as we see an animal far away or take two stars for one before using the telescope. For example, we had the precise concept of mechanical procedure in mind, but had not perceived it clearly before we knew of Turing's work. »*

Wang 1995, A Logical Journey

Le concept cumulatif d'ensemble et ZFC

« *We understand this concept of set sufficiently well to see, after some deliberation, and in some cases even a great deal of deliberation, that the ordinary axioms of set theory are true for (or with respect to) this concept and to be able to extend these axioms by proposing additional axiom, and recognizing some of them to be true for (or with respect to) it.* »

Wang 1974 From mathematics to philosophy

L’article de 1947

« *It is to be noted, however, that, even if one should succeed in proving its undemonstrability as well, this would […] by no means settle the question definitively. Only someone who (like the intuitionist) denies that the concepts and axioms of classical set theory have any meaning (or any well-defined meaning) could be satisfied with such a solution, not someone who believes them to describe some well-determined reality. For in this reality Cantor's conjecture must be either true or false, and its undecidability from the axioms as known today can only mean that these axioms do not contain a complete description of this reality*; »

« *a complete solution of these problems can be obtained only by a more profound analysis (than mathematics is accustomed to give) of the* ***meanings of the terms occurring in them (such as "set"****, "one-to-one correspondence", etc.) and of* ***the axioms underlying their use.*** »

[Gödel 1947] What is Cantor’s Continuum Problem

L'article de 1947 : Le Programme de Gödel

_**Programme de Gödel** : Recherche de nouveau axiomes visant à compléter ZFC et décidant des énoncés indépendants de ZFC tels que (mais pas exclusivement) l'Hypothèse du Continu.

L’article de 1947 : Justifications intrinsèques et extrinsèques

_Justifications intrinsèques et extrinsèques :

« ***even disregarding*** *the intrinsic necessity of some new axiom, and even in case it has no intrinsic necessity at all, a probable decision about its truth is possible also in another way, namely, inductively by studying its "success".* » [Gödel 1947]

L’article de 1947 : Justifications intrinsèques et extrinsèques

« *even disregarding the intrinsic necessity of some new axiom, and even in case it has no intrinsic necessity at all, a probable decision about its truth is possible also in another way, namely, inductively by studying its "success".* »

[Gödel 1947]

Les débats récents à propos de la distinction intrinsèque/extrinsèque

Maddy envisage justifications intrinsèques et extrinsèques comme distinctes et en compétition :

« *intrinsic considerations are valuable, but only insofar as they correlate with these extrinsic payoffs. This suggests that the importance of intrinsic considerations is merely instrumental, that the fundamental justificatory force is all extrinsic. This casts serious doubt on the common opinion that intrinsic justifications are the grand aristocracy and extrinsic justifications the poor cousins. The truth may well be the reverse* »

[Maddy 2011] Defending the Axioms.

Les débats récents à propos de la distinction intrinsèque/extrinsèque

1.) To show that the distinction between intrinsic and extrinsic justification, andthe notion that there might be a preferable kind, is fraught with problems.

(2.) To propose arguments in favour of a conception of justification as multifaceted but fundamentally indivisible and linked to the notion of explanation.'Intrinsic' and 'extrinsic' justifications should (on our view) be understood as manifestations of explanatory considerations.

Barton, N . ; Ternullo, C. ; Venturi, G. (2020) On Forms of Justification in Set Theory

L'article de 1947 : Justifications intrinsèques et extrinsèques

« ***even disregarding*** *the intrinsic necessity of some new axiom, and even in case it has no intrinsic necessity at all, a probable decision about its truth is possible also in another way, namely, inductively by studying its "success".* »

[Gödel 1947]

Justifications intrinsèques

« **Reflection principle.** The universe of all sets is structurally indefinable. One possible way to make this statement precise is the following : The universe of sets cannot be uniquely characterized (i.e., distinguished from all its initial segments) by any internal structural property of the membership relation in it which is expressible in any logic of finite or transfinite type, including infnitary logics of any cardinal number. [...] The totality of all sets is, in some sense, indescribable. When you have any structural property that is supposed to apply to all sets, you know you have not got all sets. There must be some sets that contain as members all sets that have that property. »

Wang 1974 From Mathematics to Philosophy

Les exemples les plus courant d'axiomes intrinsèquement justifiés sont les axiomes de grands cardinaux.

Implémentation par l’histoire des axiomes de détermination

_La propriété de détermination est issue de la théorie des jeux, elle affirme l’existence d’une stratégie gagnante pour certains jeux définis sur des ensembles de réels.

Définition : AD est l’axiome « Tout ensemble de réels est déterminé »

Théorème : **AD** implique que tout ensemble de réels est Lebesgue-mesurable, a la propriété de Baire et la propriété d’ensemble parfait.

Théorème (Steinhaus 1965) : **AD** contredit l’axiome du Choix

_On formule des restrictions de **AD** (en particulier $\mathbf{AD}^{\mathbf{L(R)}}$ et **PD**)

Théorème (Woodin 1985) : L’existence d’un cardinal mesurable plus grand que tous les cardinaux de Woodin implique $\mathbf{AD}^{\mathbf{L(R)}}$

MERCI

Aix*Marseille université

Bibliographie

[Gödel, 1946] Gödel, K. (1946). Remarks before the Princeton Bicentennial Conference on Problems in Mathematics. In [Gödel,1990] pp. 150--153.

[Gödel, 1947] Gödel, K. (1947). What is Cantor's continuum problem? In [Gödel,1990], pages 176–187. Oxford University Press

[Gödel,1990] Gödel, K. (1990). Collected Works, Volume II: Publications 1938-1974. Oxford University Press. Edited by: Solomon Feferman (Editor-in-chief), John W. Dawson, Jr., Stephen C. Kleene, Gregory H. Moore, Robert M. Solovay, Jean van Heijenoort.

[Kanamori, 2009] Kanamori, A. (2009). The Higher Infinite: Large Cardinals in Set Theory from Their Beginnings. Springer, 2nd edition.

[Maddy, 2011] Maddy, P. (2011). Defending the Axioms. Oxford University Press.

[Dehornoy 2017] Dehornoy, P. (2017) La théorie des ensembles : introduction à une théorie de l'infini. Éd. CALVAGE ET MOUNET

[Barton, Ternullo, Venturi 2020] Barton, N . ; Ternullo, C. ; Venturi, G. (2020) On Forms of Justification in Set Theory. Australasian Journal of Logic 17 (4):158-200.