Table ronde : Elements of purity

Andrew Arana

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25 juin 2024

Cambridge Elements

The Philosophy of Mathematics

Penelope Rush
University of Tasmania
From the time Penny Rush completed her thesis in the philosophy of mathematics (2005), she has worked continuously on themes around the realism/anti-realism divide and the nature of mathematics. Her edited collection *The Metaphysics of Logic* (Cambridge University Press, 2014), and forthcoming essay 'Metaphysical Optimism' (*Philosophy Supplement*), highlight a particular interest in the idea of reality itself and curiosity and respect as important philosophical methodologies.

Stewart Shapiro
The Ohio State University
Stewart Shapiro is the O'Donnell Professor of Philosophy at The Ohio State University, a Distinguished Visiting Professor at the University of Connecticut, and a Professorial Fellow at the University of Oslo. His major works include *Foundations without Foundationalism* (1991), *Philosophy of Mathematics: Structure and Ontology* (1997), *Vagueness in Context* (2006), and *Varieties of Logic* (2014). He has taught courses in logic, philosophy of mathematics, metaphysics, epistemology, philosophy of religion, Jewish philosophy, social and political philosophy, and medical ethics.

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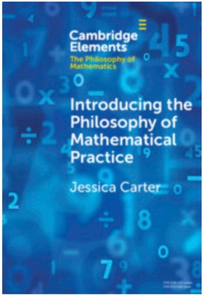

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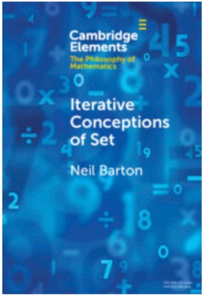
Jessica Carter



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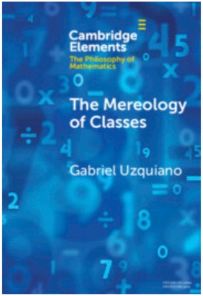

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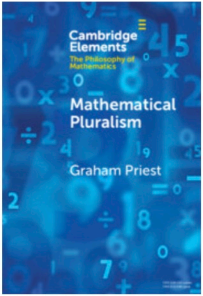

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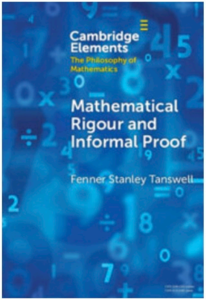

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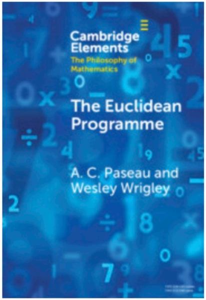

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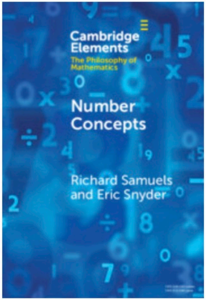

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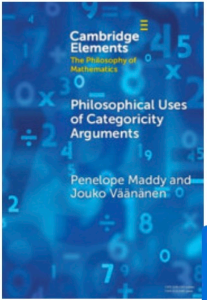

Element

Philosophical Uses of Categoricity Arguments

Penelope Maddy, Jouko Väänänen

Cambridge Elements

The Philosophy of Mathematics

Lakatos and the Historical Approach to Philosophy of Mathematics

Donald Gillies

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2 Lakatos' Contribution to the Philosophy of Mathematics

My main thesis is that Lakatos' very important contribution consisted in his introduction in his 1963–4 paper 'Proofs and Refutations' of the *historical approach* to the philosophy of mathematics. The striking nature of this paper and its interesting results led other researchers in philosophy of mathematics to take up the historical approach and it was in subsequent years strongly developed, although it had never been used by philosophers of mathematics before Lakatos. In some ways, it is strange and surprising that the historical approach to philosophy of mathematics was introduced as late as 1963–4, because the historical approach to philosophy of science had been introduced in 1840, 123 years earlier, and by 1963–4 had become a very well-established approach to philosophy of science. I will next give a brief sketch of the development of the

Elements of purity

Elements in the Philosophy of Mathematics

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Abstract: A proof of a theorem can be said to be pure if it drawn only on what is “close” or “intrinsic” to that theorem. In this Element we will investigate the apparent preference for pure proofs that has persisted in mathematics since antiquity, alongside a competing preference for impurity. In Section 1, we present two examples of purity, from geometry and number theory. In Section 2, we give a brief history of purity in mathematics. In Section 3, we discuss several different types of purity, based on different measures of distance between theorem and proof. In Section 4 we discuss reasons for preferring pure proofs, for the varieties of purity constraints presented in Section 3. In Section 5 we conclude by reflecting briefly on purity as a localism and how issues of translation intersect with the considerations we have raised throughout this work.

Keywords: purity ; philosophy of mathematics ; mathematical practice ; epistemology of mathematics ; normativity in mathematics

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Friedrich Engel, *Der Geschmack in der neueren Mathematik*, 1890

Number theory deals with the properties of integers, so one should actually demand that it prove all of its theorems without leaving the realm of integers. But there is still a long way to go before she can do that. A good number of apparently extremely simple theorems have hitherto only been able to be proved with the use of an enormous apparatus of transcendent means, of theorems from the theory of elliptic functions, and the like.

In 1621 Claude Gaspar Bachet de Méziriac published a Latin edition of Diophantus' *Arithmetic*.

In Book IV, Problem 31, he comments that every positive integer can be written as the sum of four squares, verifying the result explicitly up to 120 and saying that he verified it up to 325. He adds that it "can easily be extended to any number of squares".

In 1638 Descartes wrote of the conjecture to Mersenne, saying that it was "doubtlessly one of the most beautiful that one could find concerning numbers" but that he knew no proof and that he judged it so difficult that he did not dare to start looking for one.

Noting Descartes' difficulties with the problem in a letter to Carcavi in 1659, Fermat claimed to have found a proof by his method of infinite descent, but did not give the proof.

Euler took up the problem but did not solve it.

Building on Euler's work, Lagrange published the first proof in 1770.

After seeing that every positive integer can be written as the sum of four squares, we can ask in how many different ways it can be done.

For instance, Bachet observed that 39 can be written as

$$1+1+1+36=1^2+1^2+1^2+6^2$$

and as

$$1+4+9+25=1+2^2+3^2+5^2.$$

Are there any other combinations that work?

Jacobi solved this problem in 1829, proving what is now known as Jacobi's four squares theorem : the number of representations of n as a sum of four squares is 8 times the sum of the positive divisors of n that are not divisible by 4.

It follows that 39 can be represented in $8 \cdot 56 = 448$ ways, since the positive divisors of 39 are $1, 3, 13, 39$ and their sum is 56.

While this purely arithmetic result is simple to state, Jacobi's first proof, in *Fundamenta nova theoriae functionum ellipticarum* (1829), was transcendental.

He defines an elliptic function known today as a theta function, and shows that it is periodic.

As a periodic function he can represent it by a Fourier series.

From this he can read off a power series whose coefficients, by Euler, have a combinatorial interpretation : they give the number of ways a positive integer can be represented as the sum of four squares.

Announcing the result in 1828, Jacobi writes that it "seems to be difficult to prove by the known methods of number theory" and that his proof, "by the theory of elliptic functions is entirely analytic".

Jacobi, "Über unendliche Reihen...", 1848

Between analysis and number theory, which were long thought to be completely separate disciplines, more and more frequent and often unexpected connections and transitions have recently been discovered. A rich source of mutual relationships between the two, which will remain unexhausted for a long time, is the analysis of elliptic functions.

Jacobi, “De compositione numerorum e quatuor quadratis”, 1834

This theorem is clear even at first glance by comparing the formulas that I have shown in *Fundamenta nova theoriae functionum ellipticarum*. But for the sake of the men of arithmetic, not advocating analytic developments, I will show the matter here, in place of the above-mentioned propositions, starting solely from the theorems that concern the composition of numbers into two squares. You can extract such a demonstration without much trouble from the analysis that we have used.... The less it is concealed, the more likely it can provide a handle for others to further refine the method ....

Jacobi, “Über unendliche Reihen…”, 1848

The derivation of these arithmetic propositions from the analytic developments not only increases the supply of arithmetic proofs, but also the propositions themselves are found in a new, remarkable form. In an earlier case, in which a fundamentally arithmetic theorem resulted as the corollary of an elliptic formula, this theorem received an essentially different version, which gave it a more general character and increased importance.

Jacobi, “Über unendliche Reihen…”, 1848

In the following I have tried to derive the properties of the numbers resulting from analytic developments also from well-known arithmetic theorems, which every time gives a purely arithmetic proof for the analytic formula. Although these arithmetic proofs of results obtained by analytical means do not present any essential difficulties, they are sometimes of a complicated nature…

Jacobi's work raises a number of questions.

What is it for a statement to be proved by means "close" or "intrinsic" to it, or by avoiding what is "extrinsic", "extraneous", "distant", "remote", "alien" or "foreign" to it?

Can such proofs always be found?

Are such proofs simpler or more complex than other proofs?

Are there other reasons to prefer such proofs, or to avoid them?

Contents

In Section 2 I emphasize that attention to purity continues from antiquity to today, despite profound changes in how we think about mathematics and in how we practice it.

Aristotle's theory of knowledge supported the idea that individual mathematical disciplines were autonomous, with their own distinct first principles ; as a consequence, "we cannot, for instance, prove geometrical truths by arithmetic." (*An. Post.*)

The development of algebra at the birth of the modern era brought a shock to the Aristotelian theory of knowledge, but purity took on new forms.

In Section 3, I distinguish between five types of purity :

- Geographical purity
- Topical purity
- Syntactic purity
- Logical purity
- Elemental purity

A proof of a statement is geographically pure if it draws only on what belongs to the branch of mathematics to which the statement belongs.

Harold Davenport, *The Higher Arithmetic*, 1952

We have already said that the proof of Dirichlet's Theorem on primes in arithmetical progressions and the proof of the prime number theorem were analytical, and made use of methods which cannot be said to belong properly to the theory of numbers. The propositions themselves relate entirely to the natural numbers, and it seems reasonable that they should be provable without the intervention of such foreign ideas.

But geographically purity depends on the disciplinary structure of mathematics, a structure that is always shifting.

It also struggles with results that belong to several different branches, like Poincaré's uniformization theorem.

It also does not account for purity projects that only aim to use part of one branch to prove a result : proving the infinitude of primes without using addition which might be judged extraneous to both primality and infinitude.

A proof is topically pure if it draws only on what belongs to the content of the theorem it is proving, i.e. on what must be grasped and accepted in order to comprehend that theorem.

Hilbert, "Lectures on Euclidean Geometry", 1898–1899

Therefore we are for the first time in a position to put into practice *a critique of means of proof*. In modern mathematics such criticism is raised very often, where the aim is to preserve *the purity of method* [*die Reinheit der Methode*], i.e. to prove theorems if possible using means that are suggested by [*nahe gelegt*] the content [*Inhalt*] of the theorem.

Topical purity applies better to the cases that are problematic for geographical purity.

But what belongs to the content of mathematical statements is difficult to determine.

Foundationalists will say that every statement is "ultimately" given its meaning in terms of some foundational theory like set or category theory.

Then every theorem will have a topically pure proof.

Foundationalism makes topical purity trivial.

This is a reason to reject foundationalism.

Barry Mazur, "Number theory as gadfly", 1991

One of the mysteries of the Shimura-Taniyama-Weil conjecture, and its constellation of equivalent paraphrases, is that although it is undeniably a conjecture "about arithmetic," it can be phrased variously, so that : in one of its guises, one thinks of it as being also deeply "about" integral transforms in the theory of one complex variable ; in another as being also "about" geometry.

I develop a distinction between the basic content and deep content of statements, and argue that topical purity is concerned with the former.

To avoid these semantic issues, we can try to construe what belongs to a statement syntactically.

A proof of a statement is Gentzenian pure if it consists only of subformulas of that statement.

The infinitude of primes can be formalized by :

$$\forall a \exists b[b > a \wedge \forall x[\exists y(x \cdot y = b) \rightarrow (x = 1 \vee x = b)]]. \quad (1)$$

Since $x \cdot y$ is a subformula of (1), we can conclude that multiplication can be used in a syntactically pure proof of (1).

Gentzenian syntactic purity is appealing because it is sharply determinate when a proof is pure ; indeed, it is decidable in the sense of the theory of computation.

But it is sensitive to formulations.

(1) has no subformula with successor, so a proof using successor is Gentzenian impure.

But surely the natural numbers are what they are in virtue of being a discrete series.

Gentzen showed that every provable statement in the sequent calculus has a proof such that all of the proof's formulas are subformulas of the statement proved.

So every provable statement in this setting has a Gentzenian pure proof.

But this isn't true outside of this quite restricted, purely logical setting.

A proof of a statement is logically pure if it draws only on what is logically necessary for proving it.

Anand Pillay, "Remarks on Purity of Methods", 2021

There is a context consisting of points and lines in $\mathbb{R}^2$ and a statement about such points and lines. What do we have to know (in terms of assumptions) about this context to prove the statement, and is there a minimum natural collection of such assumptions, or axioms, (other than the statement itself) needed ? Of course, some properties of the basic notions of points and lines (and incidence) will be needed, but maybe not everything about the real field $\mathbb{R}$.

Lipman & Teissier, "Pseudo-rational local rings and a theorem of Briançon-Skoda about integral closures of ideals", *Michigan Mathematical Journal* (1981)

The proof given by Briançon and Skoda of this completely algebraic statement is based on a quite transcendental deep result of Skoda.... The absence of an algebraic proof has been for algebraists something of a scandal—perhaps even an insult—and certainly a challenge.

We can see this “scandal” of impurity as geographical—transcendental results do not belong to algebra—or we can see it as a matter of elementarity : that transcendental results are harder to understand than the algebraic result in question.

A proof of a statement that only draws on what is more elementary than the statement is *elementally pure*, where elementarity is an epistemic notion that can be measured comprehensionally (or computationally).

In Section 4 I turn to what makes purity, and impurity, valuable.

I discuss the relationship between purity and rigour, and between purity and explanation.

I discuss geographical purity as a kind of epistemic localism, a preference for what is local to a particular mathematical culture in proving results deemed to belong to that culture, or what Gaston Bachelard called a "rationalisme régional".

I then discuss how topically pure proofs give a particularly stable kind of knowledge of their conclusions.

Finally, I examine the evidence for whether impure proofs are systematically simpler than pure proofs.

In Section 5 I compare the value of understanding, say, a purely geometrical proof of a geometrical theorem to the value of reading a work of literature in its original language rather than in translation, linking to work of Barbara Cassin.

I conclude that the pursuit of multiple epistemic values, like purity and impurity, contributes to a fuller understanding of mathematics.

We can give many different proofs of a single theorem, with each proof valuable for some different reason.

It is important to cultivate a plurality of epistemic values in order to succeed as a mathematical knower, because to know in the fullest sense requires knowing in as many different ways as we can.