Topological Sum Rules and Spectral Flows of Chiral and Gravitational Axion-like Interactions

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Based on
Work with Stefano Lionetti, Dario Melle (to appear)
and previous work with , M. Creti', S. Lionetti, M. Maglio, R. Tommasi

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PRD
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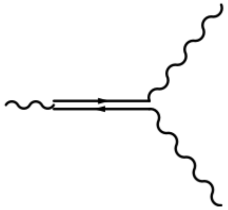
Axion-like interactions are characterised by an off-shell effective action manifesting the exchange of anomaly poles in chiral and gravitational correlators.

We examine **sum rules** in

JJJA (axial-vector/vector-vectorector)
and
JATT (axial-vector/stress-energy tensor) correlators**,**



highlighting the transition of anomaly poles to branch cuts beyond the conformal limit.

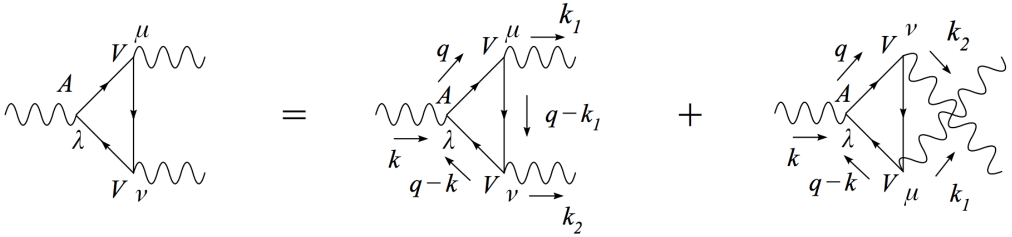
Conformal Ward identities constrain longitudinal and transverse sectors, with spectral density flows shifting the continuum to the massless anomaly pole.

For massless fermions**, anomaly and particle poles align only on-shell, with no particle pole for off-shell vector lines or massive fermions.**

Signatures of Chiral and Conformal anomalies

$$\Delta_{\alpha\mu\nu}(p_1,p_2) = \int \frac{d^4k}{(2\pi)^4}\mathrm{Tr}\left(\frac{1}{\not k - \not p_1 - m}\gamma_\mu \frac{1}{\not k - m}\gamma_\nu \frac{1}{\not k + \not p_2 - m}\gamma_\alpha\gamma_5\right) + [(p_1,\mu)\leftrightarrow(p_2,\nu)]$$

AVV diagram



$$\begin{aligned}
\Delta_0^{\lambda\mu\nu} &= A_1(k_1,k_2)\varepsilon[k_1,\mu,\nu,\lambda] + A_2(k_1,k_2)\varepsilon[k_2,\mu,\nu,\lambda] + A_3(k_1,k_2)\varepsilon[k_1,k_2,\mu,\lambda]k_1{}^{\nu} \\
&+ A_4(k_1,k_2)\varepsilon[k_1,k_2,\mu,\lambda]k_2^{\nu} + A_5(k_1,k_2)\varepsilon[k_1,k_2,\nu,\lambda]k_1^{\mu} + A_6(k_1,k_2)\varepsilon[k_1,k_2,\nu,\lambda]k_2^{\mu}.
\end{aligned}$$

If we change the parameterization of the loop momentum, A1 and A2 change.

$$\begin{aligned}
A_3(k_1,k_2) &= -A_6(k_2,k_1) = -16\pi^2 I_{11}(k_1,k_2), \\
A_4(k_1,k_2) &= -A_5(k_2,k_1) = 16\pi^2\left[I_{20}(k_1,k_2) - I_{10}(k_1,k_2)\right],
\end{aligned}$$

Some are finite by power counting.
A1 and A2 are not

where the general massive I_{st} integral is defined by

$$I_{st}(k_1,k_2)=\int_0^1 dw\int_0^{1-w} dz w^s z^t\left[z(1-z)k_1^2+w(1-w)k_2^2+2wz(k_1k_2)-m^2\right]^{-1},$$

Impose vector Ward identities

Then A1 ans A2 are fixed
without any renormalization

$$\begin{aligned} A_1(k_1,k_2) &= k_1\cdot k_2\, A_3(k_1,k_2)+k_2^2\, A_4(k_1,k_2),\\ A_2(k_1,k_2) &= k_1^2\, A_5(k_1,k_2)+k_1\cdot k_2\, A_6(k_1,k_2), \end{aligned}$$

No renormalization: Chiral anomalies are topological, similarly to the Euler density in the conformal anomaly

This is not the only parameterization. A second one is the **longitudinal/transverse (LT) decomposition**

$$W^{\lambda\mu\nu} = \frac{1}{8\pi^2}\left[W^{L\,\lambda\mu\nu} - W^{T\,\lambda\mu\nu}\right],$$

$$W^{L\,\lambda\mu\nu} = w_L\, k^\lambda \varepsilon[\mu,\nu,k_1,k_2]$$

De Rafael et al

developed in the study of g-2 of the muon

It corrects an erro r in the book by Kerson Huang on particle theory

Only the L part contributes to the Ward Identity

$$\begin{aligned} {W^T}_{\lambda\mu\nu}(k_1,k_2) \;&=\; w_T^{(+)}\left(k^2,k_1^2,k_2^2\right)\, t^{(+)}_{\lambda\mu\nu}(k_1,k_2) + w_T^{(-)}\left(k^2,k_1^2,k_2^2\right)\, t^{(-)}_{\lambda\mu\nu}(k_1,k_2) \\ &+ \widetilde{w}_T^{(-)}\left(k^2,k_1^2,k_2^2\right)\, \widetilde{t}^{(-)}_{\lambda\mu\nu}(k_1,k_2), \end{aligned}$$

$$\begin{aligned} t^{(+)}_{\lambda\mu\nu}(k_1,k_2) \;&=\; k_{1\nu}\,\varepsilon[\mu,\lambda,k_1,k_2] - k_{2\mu}\,\varepsilon[\nu,\lambda,k_1,k_2] - (k_1\cdot k_2)\,\varepsilon[\mu,\nu,\lambda,(k_1-k_2)] \\ &\quad + \frac{k_1^2+k_2^2-k^2}{k^2}\; k_\lambda\;\varepsilon[\mu,\nu,k_1,k_2]\;, \\ t^{(-)}_{\lambda\mu\nu}(k_1,k_2) \;&=\; \left[(k_1-k_2)_\lambda - \frac{k_1^2-k_2^2}{k^2}\;k_\lambda\right]\varepsilon[\mu,\nu,k_1,k_2] \\ \widetilde{t}^{(-)}_{\lambda\mu\nu}(k_1,k_2) \;&=\; k_{1\nu}\,\varepsilon[\mu,\lambda,k_1,k_2] + k_{2\mu}\,\varepsilon[\nu,\lambda,k_1,k_2] - (k_1\cdot k_2)\,\varepsilon[\mu,\nu,\lambda,k]. \end{aligned}$$

Tensor structures involved
In the LT parameterization

Topology comes through the Schouten relations

Armlllis, Delle Rose, CC

$$
\begin{aligned}
A_3(k_1,k_2) &= \frac{1}{8\pi^2}\left[w_L - \tilde{w}_T^{(-)} - \frac{k^2}{(k_1+k_2)^2} w_T^{(+)} - 2\,\frac{k_1\cdot k_2 - k_2^2}{k^2} w_T^{(-)}\right], \\
A_4(k_1,k_2) &= \frac{1}{8\pi^2}\left[w_L + 2\,\frac{k_1\cdot k_2}{k^2} w_T^{(+)} - 2\,\frac{k_1\cdot k_2 + k_2^2}{k^2} w_T^{(-)}\right], \\
A_5(k_1,k_2) &= -A_4(k_2,k_1), \qquad\qquad A_6(k_1,k_2) = -A_3(k_2,k_1),
\end{aligned}
$$

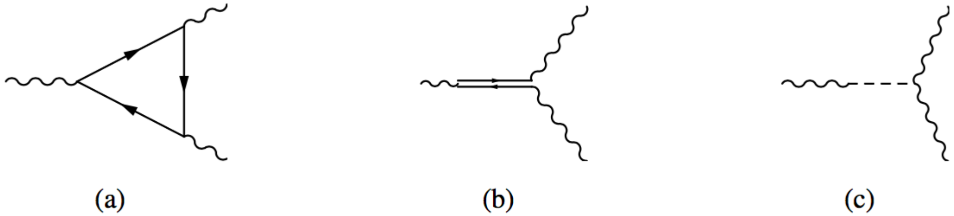
$$
w_L(k^2,\, k_1^2,\, k_2^2) \;=\; \frac{8\pi^2}{k^2}\left[A_1 - A_2\right],
$$

$$
\begin{aligned}
w_L(k^2,\, k_1^2,\, k_2^2) &= \frac{8\pi^2}{k^2}\left[(A_3 - A_6)k_1\cdot k_2 + A_4\, k_2^2 - A_5\, k_1^2\right], \\
w_T^{(+)}(k^2,\, k_1^2,\, k_2^2) &= -4\pi^2\,(A_3 - A_4 + A_5 - A_6)\,, \\
w_T^{(-)}(k^2,\, k_1^2,\, k_2^2) &= 4\pi^2\,(A_4 + A_5)\,, \\
\tilde{w}_T^{(-)}(k^2,\, k_1^2,\, k_2^2) &= -4\pi^2\,(A_3 + A_4 + A_5 + A_6)\,,
\end{aligned}
$$

Notice that if you change the parameterization of The momentum in the loop, A1 and A2 will shift by the same amount, but WL will not change.

Notice the presence of a single pole in the Longitudinal component of the AVV diagram.

$$
\begin{aligned}
w_L(s_1, s_2, s) &= -\frac{4i}{s} \\
w_T^{(+)}(s_1, s_2, s) &= i\frac{s}{\sigma} + \frac{i}{2\sigma^2}\Big[(s_{12}+s_2)(3s_1^2 + s_1(6s_{12}+s_2) + 2s_{12}^2)\log\frac{s_1}{s} \\
&+ (s_{12}+s_1)(3s_2^2 + s_2(6s_{12}+s_1) + 2s_{12}^2)\log\frac{s_2}{s} \\
&+ s(2s_{12}(s_1+s_2) + s_1 s_2(s_1+s_2+6s_{12}))\Phi(s_1, s_2)\Big]
\end{aligned}
$$



The triangle diagram in the fermion case (a), the collinear fermion configuration responsible for the anomaly (b) and a diagrammatic representation of the exchange via an intermediate state (dashed line) (c).

The signature of the chiral anomaly is in the the generation of 1 pole in the axial vector channel

CONFORMAL POLES

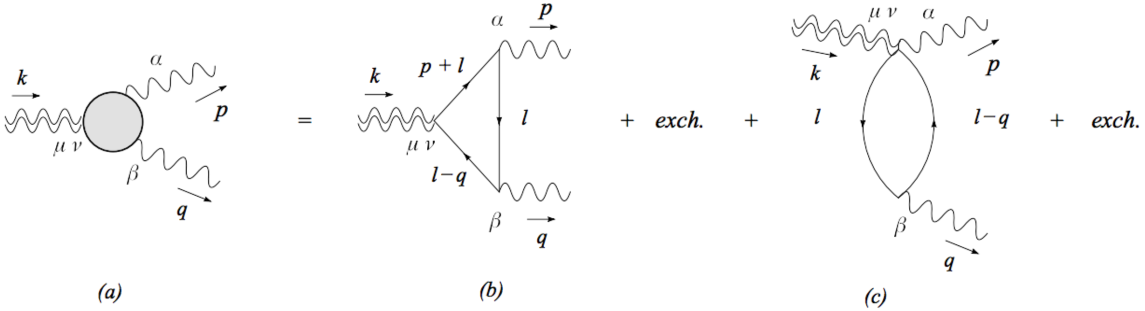
QED

$$T_f^{\mu\nu} = -i\bar{\psi}\gamma^{(\mu}\overleftrightarrow{\partial}^{\nu)}\psi + g^{\mu\nu}(i\bar{\psi}\gamma^{\lambda}\overleftrightarrow{\partial}_{\lambda}\psi - m\bar{\psi}\psi),$$

$$T_{fp}^{\mu\nu} = -\,eJ^{(\mu}A^{\nu)} + eg^{\mu\nu}J^{\lambda}A_{\lambda}\,,$$

$$T_{ph}^{\mu\nu} = F^{\mu\lambda}F^{\nu}{}_{\lambda} - \frac{1}{4}g^{\mu\nu}F^{\lambda\rho}F_{\lambda\rho},$$

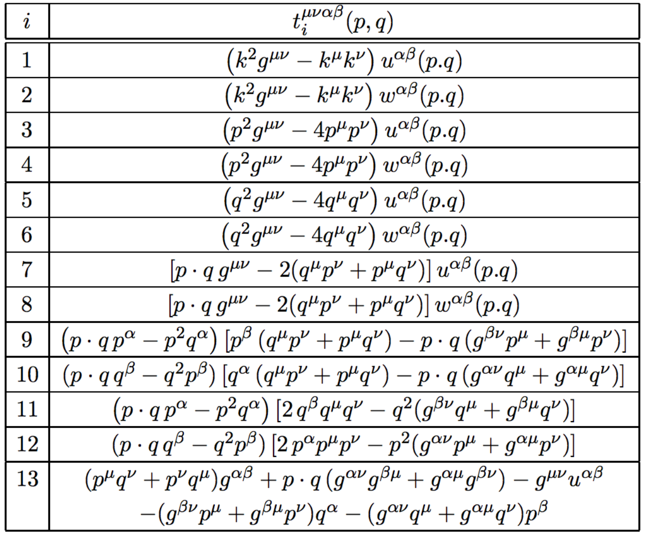
$$T^{\mu\nu} \equiv T_f^{\mu\nu} + T_{fp}^{\mu\nu} + T_{ph}^{\mu\nu}$$



$$\begin{aligned}\langle T_p^{\mu\nu}(z)\rangle_A \;&\equiv\; \int D\psi D\bar{\psi}\; T_p^{\mu\nu}(z)\; e^{i\int d^4x\mathcal{L}+\int J\cdot A(x)d^4x}\\ &=\; \langle T_p^{\mu\nu}\, e^{i\int d^4x\, J\cdot A(x)}\rangle\end{aligned}$$

Armillis, Delle Rose, CC

Giannotti and Mottola

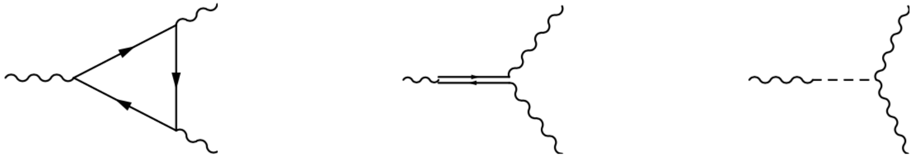


i	$t_i^{\mu\nu\alpha\beta}(p,q)$
1	$\left(k^2 g^{\mu\nu} - k^{\mu}k^{\nu}\right) u^{\alpha\beta}(p.q)$
2	$\left(k^2 g^{\mu\nu} - k^{\mu}k^{\nu}\right) w^{\alpha\beta}(p.q)$
3	$\left(p^2 g^{\mu\nu} - 4p^{\mu}p^{\nu}\right) u^{\alpha\beta}(p.q)$
4	$\left(p^2 g^{\mu\nu} - 4p^{\mu}p^{\nu}\right) w^{\alpha\beta}(p.q)$
5	$\left(q^2 g^{\mu\nu} - 4q^{\mu}q^{\nu}\right) u^{\alpha\beta}(p.q)$
6	$\left(q^2 g^{\mu\nu} - 4q^{\mu}q^{\nu}\right) w^{\alpha\beta}(p.q)$
7	$[p\cdot q\, g^{\mu\nu} - 2(q^{\mu}p^{\nu} + p^{\mu}q^{\nu})]\, u^{\alpha\beta}(p.q)$
8	$[p\cdot q\, g^{\mu\nu} - 2(q^{\mu}p^{\nu} + p^{\mu}q^{\nu})]\, w^{\alpha\beta}(p.q)$
9	$\left(p\cdot q\, p^{\alpha} - p^2 q^{\alpha}\right) [p^{\beta}\left(q^{\mu}p^{\nu} + p^{\mu}q^{\nu}\right) - p\cdot q\left(g^{\beta\nu}p^{\mu} + g^{\beta\mu}p^{\nu}\right)]$
10	$\left(p\cdot q\, q^{\beta} - q^2 p^{\beta}\right) [q^{\alpha}\left(q^{\mu}p^{\nu} + p^{\mu}q^{\nu}\right) - p\cdot q\left(g^{\alpha\nu}q^{\mu} + g^{\alpha\mu}q^{\nu}\right)]$
11	$\left(p\cdot q\, p^{\alpha} - p^2 q^{\alpha}\right) [2\, q^{\beta}q^{\mu}q^{\nu} - q^2(g^{\beta\nu}q^{\mu} + g^{\beta\mu}q^{\nu})]$
12	$\left(p\cdot q\, q^{\beta} - q^2 p^{\beta}\right) [2\, p^{\alpha}p^{\mu}p^{\nu} - p^2(g^{\alpha\nu}p^{\mu} + g^{\alpha\mu}p^{\nu})]$
13	$(p^{\mu}q^{\nu} + p^{\nu}q^{\mu})g^{\alpha\beta} + p\cdot q\left(g^{\alpha\nu}g^{\beta\mu} + g^{\alpha\mu}g^{\beta\nu}\right) - g^{\mu\nu}u^{\alpha\beta}$ $-(g^{\beta\nu}p^{\mu} + g^{\beta\mu}p^{\nu})q^{\alpha} - (g^{\alpha\nu}q^{\mu} + g^{\alpha\mu}q^{\nu})p^{\beta}$

$$\underline{\underline{\mathbf{F_1(s;\, s_1,\, s_2,\, 0)}}} \quad = \quad -\frac{e^2}{18\pi^2 s},$$

Armillis, Delle Rose,CC. PRD

$$\Gamma^{\mu\nu\alpha\beta}(p,q) = \sum_{i-1}^{13} F_i(s; s_1, s_2, m^2)\; t_i^{\mu\nu\alpha\beta}(p,q)\,,$$



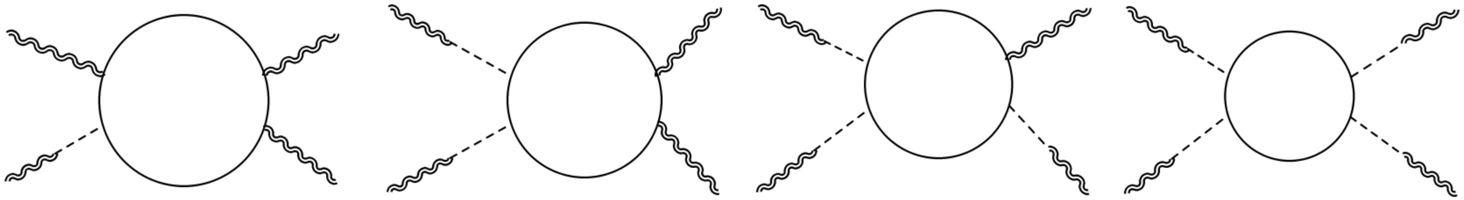
(a) (b) (c)

Bilinear Mixings are associated to projectors, ubiquitous in the conformal anomaly effective action

$$
\begin{aligned}
&\langle T^{\mu_1\nu_1}(p_1)T^{\mu_2\nu_2}(p_2)T^{\mu_3\nu_3}(p_3)T^{\mu_4\nu_4}(\bar{p}_4)\rangle_{poles} = \\
&= \frac{\pi^{\mu_1\nu_1}(p_1)}{3}\,\langle T(p_1)T^{\mu_2\nu_2}(p_2)T^{\mu_3\nu_3}(p_3)T^{\mu_4\nu_4}(\bar{p}_4)\rangle_{anomaly} + (perm.) \\
&- \frac{\pi^{\mu_1\nu_1}(p_1)}{3}\,\frac{\pi^{\mu_2\nu_2}(p_2)}{3}\,\langle T(p_1)T(p_2)T^{\mu_3\nu_3}(p_3)T^{\mu_4\nu_4}(\bar{p}_4)\rangle_{anomaly} + (perm.) \\
&+ \frac{\pi^{\mu_1\nu_1}(p_1)}{3}\,\frac{\pi^{\mu_2\nu_2}(p_2)}{3}\,\frac{\pi^{\mu_3\nu_3}(p_2)}{3}\,\langle T(p_1)T(p_2)T(p_3)T^{\mu_4\nu_4}(\bar{p}_4)\rangle_{anomaly} + (perm.) \\
&- \frac{\pi^{\mu_1\nu_1}(p_1)}{3}\,\frac{\pi^{\mu_2\nu_2}(p_2)}{3}\,\frac{\pi^{\mu_3\nu_3}(p_3)}{3}\,\frac{\pi^{\mu_4\nu_4}(p_4)}{3}\,\langle T(p_1)T(p_2)T(p_3)T(\bar{p}_4)\rangle_{anomaly}\,.
\end{aligned}
$$

For 4-point functions we need to proceed through perturbative analysis of the conformal ward identities.

Maglio, Theofilopoulos, CC, 2021



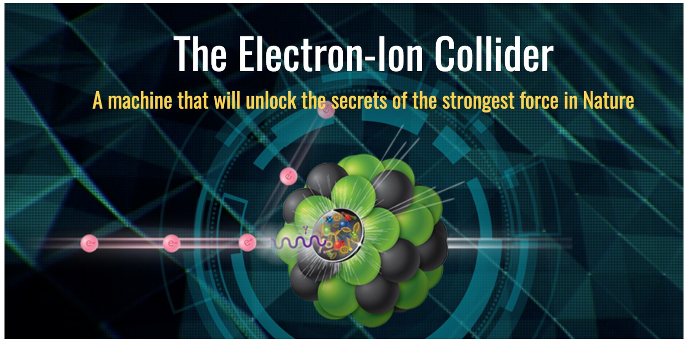
BILINEAR MIXINGS.

They are crucial for anomalies and can be studied
At colliders
In analogue realizations (example topological materials)

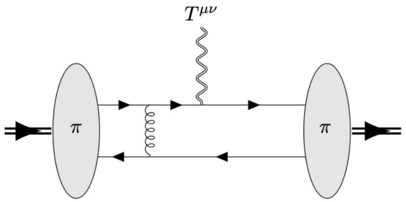
2409.05609 [hep-ph]
Lionetti, Melle, Tommasi, CC

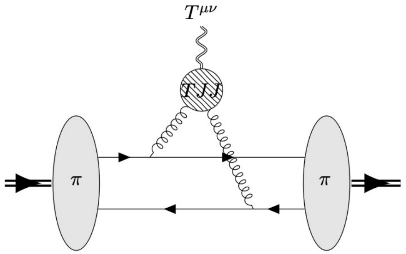
Brookhaven National Laboratory
Jefferson Lab

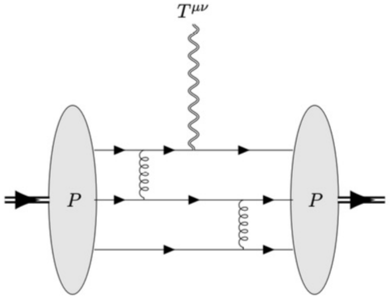
The Electron-Ion Collider (EIC) at Brookhaven National Laboratory is designed to have a highly flexible energy range, with the capability to collide electrons with protons and nuclei at center-of-mass energies ranging from approximately **20 GeV** to **140 GeV (e-p and e-Ions)**

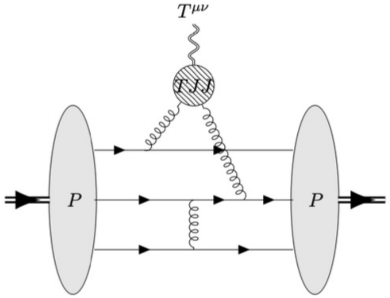


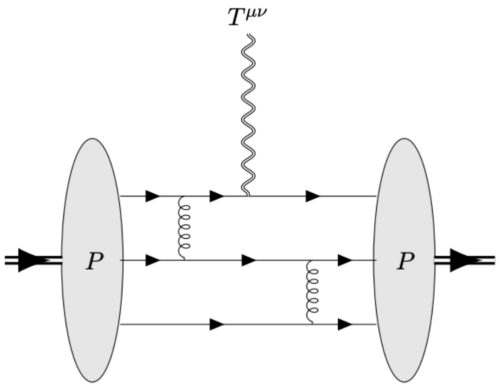
In the factorization picture: the TJJ correlator



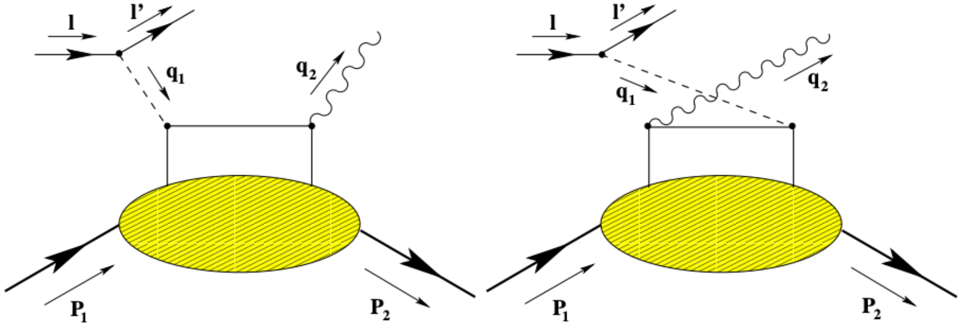








GFF of the proton

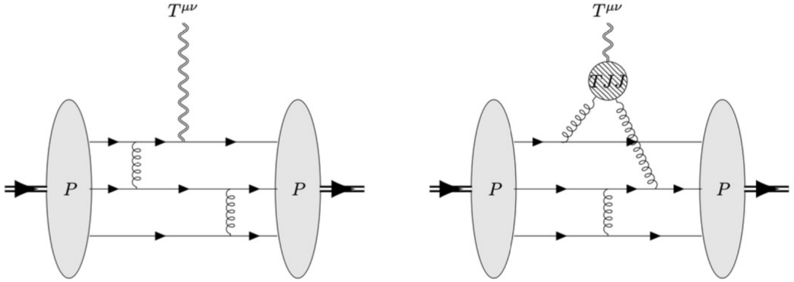


DVCS

Ji’s Sum rule

$$\mathcal{L}_{grav}(x) = -\frac{\kappa}{2} T^{\mu\nu}(x) h_{\mu\nu}(x),$$

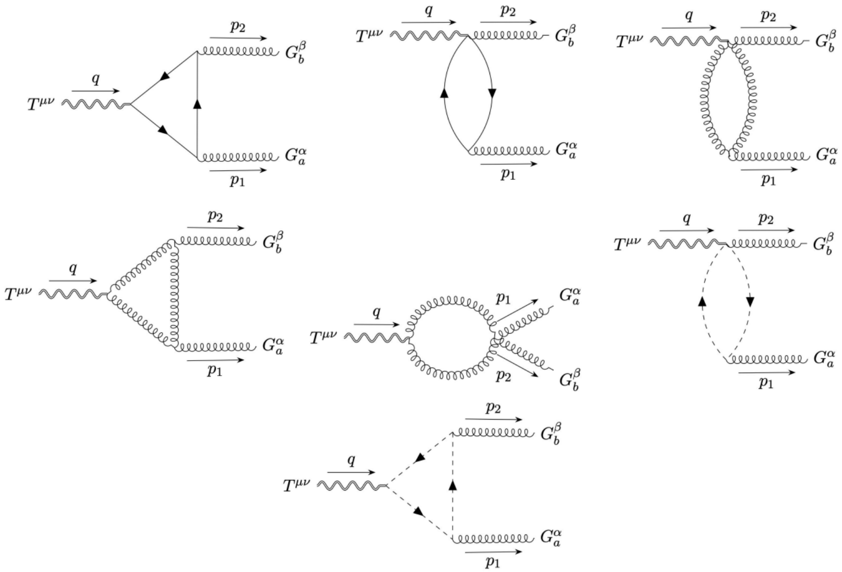
Collinear factorization in QCD

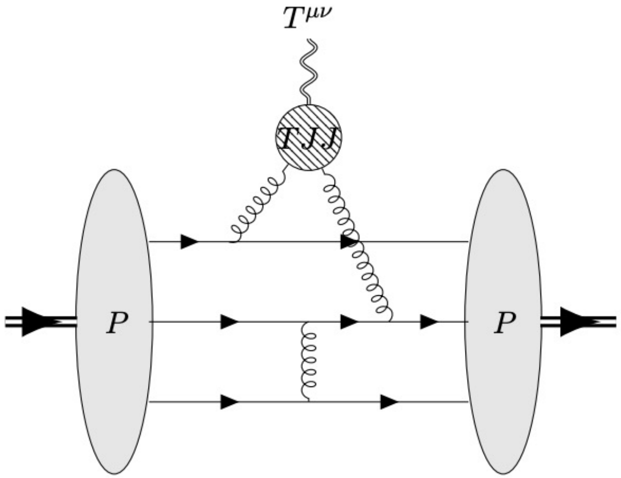


Lionetti, Melle, Tommasi, CC 2409.05609

Invariant amplitudes of DVCS
Related to form factors of
GFF of the proton.

A dilaton state is interpolating at higher orders





Lionetti, Melle, Tommasi, CC, 2024

Armillis, Delle Rose, CC. for QCD, for on-shell gluons

Giannotti, Mottola, QED

$$S_{pole} \;=\; \beta(g)\int d^4x\, d^4y\, R^{(1)}(x)\,\square^{-1}(x,y)\, F^a_{\alpha\beta}F^{a\alpha\beta}$$

$$\beta(g)=\frac{dg}{d\ln(\mu^2)}=-\beta_0\frac{g^3}{16\pi^2},\qquad \beta_0=\frac{11}{3}C_A-\frac{2}{3}n_f$$

Lionetti, Melle, Tommasi, CC 2409.05609

All anomalies are connected with anomalous Conformal Ward Identities

$$g_{\mu\nu}\langle T^{\mu\nu}\rangle = b_1\, E_4 + b_2\, C^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma} + b_3\, \nabla^2 R + b_4\, F^{\mu\nu}F_{\mu\nu},$$

where $C_{\mu\nu\rho\sigma}$ denotes the Weyl tensor and E_4 stands for the Gauss-Bonnet term:

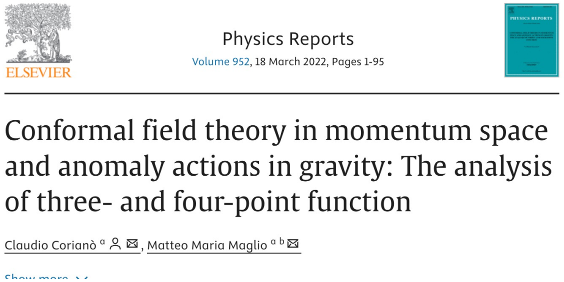
$$C^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma} = R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} - 2R^{\mu\nu}R_{\mu\nu} + \frac{1}{3}R^2,$$

$$E_4 \equiv E = R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} - 4R^{\mu\nu}R_{\mu\nu} + R^2.$$

$$\mathcal{A} = b_1\, E_4 + b_2\, C^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma} + b_3\, \nabla^2 R + b_4\, F^{\mu\nu}F_{\mu\nu} + f_1\, \varepsilon^{\mu\nu\rho\sigma}R_{\alpha\beta\mu\nu}R^{\alpha\beta}{}_{\rho\sigma} + f_2\, \varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma},$$

Correlators of higher order are affected by the anomaly and can be combined together in the conformal anomaly effective action

$$\mathcal{S}(g)_B \equiv \mathcal{S}(\bar{g})_B + \sum_{n=1}^{\infty}\frac{1}{2^n n!}\int d^dx_1\ldots d^dx_n\sqrt{-g_1}\ldots\sqrt{-g_n}\,\langle T^{\mu_1\nu_1}\ldots T^{\mu_n\nu_n}\rangle_{\bar{g}B}\,\delta g_{\mu_1\nu_1}(x_1)\ldots\delta g_{\mu_n\nu_n}(x_n)$$



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Physics Reports

Volume 952, 18 March 2022, Pages 1-95

Conformal field theory in momentum space and anomaly actions in gravity: The analysis of three- and four-point function

Claudio Corianò [a], Matteo Maria Maglio [a] [b]

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STEPS IN THE SOLUTION

$$p_{i\mu_i}\,\langle J^{\mu_1}(p_1)J^{\mu_2}(p_2)J_5^{\mu_3}(p_3)\rangle = 0, \qquad i = 1,2$$

$$p_{3\mu_3}\,\langle J^{\mu_1}(p_1)J^{\mu_2}(p_2)J_A^{\mu_3}(p_3)\rangle = -8\,a\,i\,\varepsilon^{p_1p_2\mu_1\mu_2}$$

Ward identities

$$\langle J^{\mu_1}(p_1)J^{\mu_2}(p_2)J_A^{\mu_3}(p_3)\rangle = \langle j^{\mu_1}(p_1)j^{\mu_2}(p_2)j_A^{\mu_3}(p_3)\rangle + \langle J^{\mu_1}(p_1)J^{\mu_2}(p_2)\,j_{5\,\text{long}}^{\mu_3}\,(p_3)\rangle$$

Decomp. Into L and T

Solve the long. sector by a pole

$$\langle J^{\mu_1}(p_1)J^{\mu_2}(p_2)\,j_{5\,\text{long}}^{\mu_3}\,(p_3)\rangle = \frac{p_3^{\mu_3}}{p_3^2}\,p_{3\,\alpha_3}\,\langle J^{\mu_1}(p_1)J^{\mu_2}(p_2)J_A^{\alpha_3}(p_3)\rangle = \Phi_0\,\varepsilon^{p_1p_2\mu_1\mu_2}\,p_3^{\mu_3}$$

$$\Phi_0 = -\frac{8\,a\,i}{p_3^2}$$

Anomaly form factor

$$\langle j^{\mu_1}(p_1)j^{\mu_2}(p_2)j_5^{\mu_3}(p_3)\rangle = \pi^{\mu_1}_{\alpha_1}(p_1)\pi^{\mu_2}_{\alpha_2}(p_2)\pi^{\mu_3}_{\alpha_3}(p_3)\,\Big[A_1(p_1,p_2,p_3)\,\varepsilon^{p_1p_2\alpha_1\alpha_2}p_1^{\alpha_3} + A_2(p_1,p_2,p_3)\,\varepsilon^{p_1\alpha_1\alpha_2\alpha_3} - A_2(p_2,p_1,p_3)\,\varepsilon^{p_2\alpha_1\alpha_2\alpha_3}\Big]$$

Introduce the transverse sector

$$\left(\sum_{i=1}\Delta_i - 2d - \sum_{i=1} p_i^{\mu}\frac{\partial}{\partial p_i^{\mu}}\right)\langle J^{\mu_1}(p_1)J^{\mu_2}(p_2)J_5^{\mu_3}(p_3)\rangle = 0.$$

$$\begin{aligned}0 = {}&\sum_{j=1}^{2}\left[-2\frac{\partial}{\partial p_{j\kappa}} - 2p_j^{\alpha}\frac{\partial^2}{\partial p_j^{\alpha}\,\partial p_{j\kappa}} + p_j^{\kappa}\frac{\partial^2}{\partial p_j^{\alpha}\,\partial p_{j\alpha}}\right]\langle J^{\mu_1}(p_1)J^{\mu_2}(p_2)J_A^{\mu_3}(p_3)\rangle\\ &+ 2\left(\delta^{\mu_1\kappa}\frac{\partial}{\partial p_1^{\alpha_1}} - \delta^{\kappa}_{\alpha_1}\frac{\partial}{\partial p_{1\mu_1}}\right)\langle J^{\alpha_1}(p_1)J^{\mu_2}(p_2)J_A^{\mu_3}(p_3)\rangle\\ &+ 2\left(\delta^{\mu_2\kappa}\frac{\partial}{\partial p_2^{\alpha_2}} - \delta^{\kappa}_{\alpha_2}\frac{\partial}{\partial p_{2\mu_2}}\right)\langle J^{\mu_1}(p_1)J^{\alpha_2}(p_2)J_A^{\mu_3}(p_3)\rangle \equiv \mathcal{K}^{\kappa}\,\langle J^{\mu_1}(p_1)J^{\mu_2}(p_2)J_A^{\mu_3}(p_3)\rangle\,.\end{aligned}$$

Impose the conformal constraints

Solve the conformal constraints

$$K_{31}\, A_1 = 0, \qquad K_{32}\, A_1 = 0,$$
$$K_{31}\, A_2 = 0, \qquad K_{32} K_{32}\, A_2 = 0$$

$$K_i = \frac{\partial^2}{\partial p_i^2} + \frac{(d+1-2\Delta_i)}{p_i}\frac{\partial}{\partial p_i}, \qquad K_{ij} = K_i - K_j$$

Appell Functions

$$A_1 = \alpha_1\, J_{3\{0,0,0\}},$$
$$A_2 = \alpha_2\, J_{1\{0,0,0\}} + \alpha_3\, J_{2\{0,1,0\}}.$$

Bzowski McFadden, Skenderis

They can be related to a class of parametric integrals of Bessel functs.

Delle Rose, Mottola, Serino, CC

$$I_{\alpha\{\beta_1\beta_2\beta_3\}}\left(p_1, p_2, p_3\right) = \int dx\, x^\alpha \prod_{j=1}^{3} p_j^{\beta_j} K_{\beta_j}\left(p_j x\right)$$

3K integrals

$$K_\nu(x) = \frac{\pi}{2}\frac{I_{-\nu}(x) - I_\nu(x)}{\sin(\nu\pi)}, \qquad \nu \notin \mathbb{Z} \qquad\qquad I_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+1)\Gamma(\nu+1+k)}\left(\frac{x}{2}\right)^{2k}$$

In 4-point functions 3K become 4K (generalized Lauricella) (Maglio, Theofilopoulos, CC)

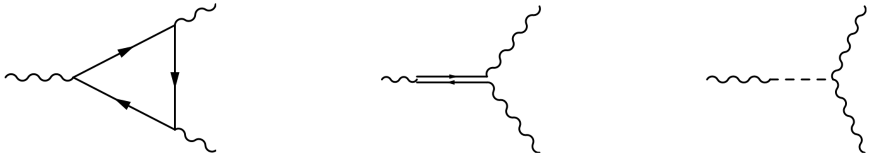
The solution (Lionetti, Maglio, CC, 2023. EPJC)

$$\begin{aligned}\langle J^{\mu_1}(p_1)J^{\mu_2}(p_2)J_5^{\mu_3}(p_3)\rangle =& -\frac{8\,a\,i}{p_3^2}\varepsilon^{p_1p_2\mu_1\mu_2}\,p_3^{\mu_3}\\ &+8ia\,\pi^{\mu_1}_{\alpha_1}(p_1)\pi^{\mu_2}_{\alpha_2}(p_2)\pi^{\mu_3}_{\alpha_3}(p_3)\left[p_2^2\,I_{3\{1,0,1\}}\,\varepsilon^{p_1\alpha_1\alpha_2\alpha_3}-p_1^2\,I_{3\{0,1,1\}}\,\varepsilon^{p_2\alpha_1\alpha_2\alpha_3}\right],\end{aligned}$$

Notice that the residue at the pole has propagated to the trnasverse sector

Related to ordinary perturbation theory integrals

$$\begin{aligned}I_{3\{1,0,1\}}(p_1^2,p_2^2,p_3^2) =& \frac{1}{\lambda^2}\Bigg\{-2p_1^2p_3^2\Big[p_1^2\left(p_2^2-2p_3^2\right)+p_1^4+p_2^2p_3^2-2p_2^4+p_3^4\Big]C_0\left(p_1^2,p_2^2,p_3^2\right)\\ &+p_1^2\left(\left(p_1^2-p_2^2\right)^2+4p_2^2p_3^2-p_3^4\right)\log\left(\frac{p_1^2}{p_2^2}\right)+4p_1^2p_3^2\left(p_1^2-p_3^2\right)\log\left(\frac{p_1^2}{p_3^2}\right)\\ &-p_3^2\Big((p_2^2-p_3^2)^2+4p_1^2p_2^2-p_1^4\Big)\log\left(\frac{p_2^2}{p_3^2}\right)-\lambda(p_1^2-p_2^2+p_3^2)\Bigg\}\end{aligned}$$



The pole defines an exchange of some sort

GRAVITATIONAL ANOMALY

<A TT>

Lionetti, Maglio CC 2024, PRD

At the conformal point

Two stress energy tensors and one chiral current J_A

Captured to lowest order by the ATT

$$\nabla_\mu \langle J_5^\mu \rangle = a_1 \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} + a_2 \varepsilon^{\mu\nu\rho\sigma} R^{\alpha\beta}{}_{\mu\nu} R_{\alpha\beta\rho\sigma},$$

Also in this case we reconstruct the entire correlator from
1 pole + CWI s

JUST IMPOSE

Anomalous WIs

$$\delta_{\mu_i\nu_i}\,\langle T^{\mu_1\nu_1}(p_1)T^{\mu_2\nu_2}(p_2)J_5^{\mu_3}(p_3)\rangle = 0,$$
$$p_{i\mu_i}\,\langle T^{\mu_1\nu_1}(p_1)T^{\mu_2\nu_2}(p_2)J_5^{\mu_3}(p_3)\rangle = 0,$$

$$p_{3\mu_3}\,\langle T^{\mu_1\nu_1}(p_1)T^{\mu_2\nu_2}(p_2)J_5^{\mu_3}(p_3)\rangle = 4\,i\,a_2\,(p_1\cdot p_2)\left\{\left[\varepsilon^{\nu_1\nu_2p_1p_2}\left(g^{\mu_1\mu_2}-\frac{p_1^{\mu_2}p_2^{\mu_1}}{p_1\cdot p_2}\right)+(\mu_1\leftrightarrow\nu_1)\right]+(\mu_2\leftrightarrow\nu_2)\right\}.$$

$$\int d^4x_1d^4x_2e^{i(-q\cdot x+p_1\cdot x_1+p_2\cdot x_2)}\frac{\delta^2R\tilde{R}(x)}{\delta g_{\mu_1\nu_1}(x_1)\delta g_{\mu_2\nu_2}(x_2)} =$$

The anomaly condition

$$(p_1\cdot p_2)\left\{\left[\varepsilon^{\nu_1\nu_2p_1p_2}\left(g^{\mu_1\mu_2}-\tfrac{p_1^{\mu_2}p_2^{\mu_1}}{p_1\cdot p_2}\right)+(\mu_1\leftrightarrow\nu_1)\right]+(\mu_2\leftrightarrow\nu_2)\right\}.$$

trace WI and conservation WI

$$\delta_{\mu_i\nu_i}\,\langle T^{\mu_1\nu_1}(p_1)T^{\mu_2\nu_2}(p_2)J_5^{\mu_3}(p_3)\rangle = 0,$$
$$p_{i\mu_i}\,\langle T^{\mu_1\nu_1}(p_1)T^{\mu_2\nu_2}(p_2)J_5^{\mu_3}(p_3)\rangle = 0,$$

From conservation of the stress-energy tensor

$$\nabla_\mu\langle T^{\mu\nu}\rangle = 0.$$

Solve the longitudinal sector by an anomaly pole

$$\left\langle t^{\mu_1\nu_1} t^{\mu_2\nu_2} j_{5\,loc}^{\mu_3} \right\rangle = 4ia_2 \frac{p_3^{\mu_3}}{p_3^2} \left(p_1 \cdot p_2\right) \left\{ \left[\varepsilon^{\nu_1\nu_2 p_1 p_2} \left(g^{\mu_1\mu_2} - \frac{p_1^{\mu_2} p_2^{\mu_1}}{p_1 \cdot p_2} \right) + (\mu_1 \leftrightarrow \nu_1) \right] + (\mu_2 \leftrightarrow \nu_2) \right\} .$$

Introduce the transverse sector

$$\begin{aligned} \langle t^{\mu_1\nu_1}(p_1) t^{\mu_2\nu_2}(p_2)\, j_5^{\mu_3}(p_3) \rangle = \Pi^{\mu_1\nu_1}_{\alpha_1\beta_1}(p_1)\, \Pi^{\mu_2\nu_2}_{\alpha_2\beta_2}(p_2)\, \pi^{\mu_3}_{\alpha_3}(p_3) \Big[& \\ A_1 \varepsilon^{p_1\alpha_1\alpha_2\alpha_3} p_2^{\beta_1} p_3^{\beta_2} - A_1(p_1 \leftrightarrow p_2) \varepsilon^{p_2\alpha_1\alpha_2\alpha_3} p_2^{\beta_1} p_3^{\beta_2} & \\ + A_2 \varepsilon^{p_1\alpha_1\alpha_2\alpha_3} \delta^{\beta_1\beta_2} - A_2(p_1 \leftrightarrow p_2) \varepsilon^{p_2\alpha_1\alpha_2\alpha_3} \delta^{\beta_1\beta_2} & \\ + A_3 \varepsilon^{p_1 p_2 \alpha_1\alpha_2} p_2^{\beta_1} p_3^{\beta_2} p_1^{\alpha_3} + A_4 \varepsilon^{p_1 p_2 \alpha_1\alpha_2} \delta^{\beta_1\beta_2} p_1^{\alpha_3} \Big], & \end{aligned}$$

Impose the CWIs

$$\begin{aligned} 0 = {} & \mathcal{K}^\kappa \langle T^{\mu_1\nu_1}(p_1)\, T^{\mu_2\nu_2}(p_2)\, J_5^{\mu_3}(p_3) \rangle \\ \equiv {} & \sum_{j=1}^{2} \left(2\left(\Delta_j - d\right) \frac{\partial}{\partial p_{j\kappa}} - 2p_j^\alpha \frac{\partial}{\partial p_j^\alpha} \frac{\partial}{\partial p_{j\kappa}} + (p_j)^\kappa \frac{\partial}{\partial p_j^\alpha} \frac{\partial}{\partial p_{j\alpha}} \right) \langle T^{\mu_1\nu_1}(p_1)\, T^{\mu_2\nu_2}(p_2)\, J_5^{\mu_3}(p_3) \rangle \\ & + 4 \left(\delta^{\kappa(\mu_1} \frac{\partial}{\partial p_1^{\alpha_1}} - \delta^\kappa_{\alpha_1} \delta^{(\mu_1}_\lambda \frac{\partial}{\partial p_{1\lambda}} \right) \left\langle T^{\nu_1)\alpha_1}(p_1)\, T^{\mu_2\nu_2}(p_2)\, J_5^{\mu_3}(p_3) \right\rangle \\ & + 4 \left(\delta^{\kappa(\mu_2} \frac{\partial}{\partial p_2^{\alpha_2}} - \delta^\kappa_{\alpha_2} \delta^{(\mu_2}_\lambda \frac{\partial}{\partial p_{2\lambda}} \right) \left\langle T^{\nu_2)\alpha_2}(p_2)\, T^{\mu_1\nu_1}(p_1)\, J_5^{\mu_3}(p_3) \right\rangle . \end{aligned}$$

$$
\begin{aligned}
0 &= \Pi^{\rho_1\sigma_1}_{\mu_1\nu_1}(p_1)\Pi^{\rho_2\sigma_2}_{\mu_2\nu_2}(p_2)\pi^{\rho_3}_{\mu_3}(p_3)\left(\mathcal{K}^\kappa\langle T^{\mu_1\nu_1}(p_1)T^{\mu_2\nu_2}(p_2)J_5^{\mu_3}(p_3)\rangle\right) \\
&= \Pi^{\rho_1\sigma_1}_{\mu_1\nu_1}(p_1)\Pi^{\rho_2\sigma_2}_{\mu_2\nu_2}(p_2)\pi^{\rho_3}_{\mu_3}(p_3)\big[p_1^\kappa\big(C_{11}\varepsilon^{p_1\mu_1\mu_2\mu_3}p_2^{\nu_1}p_3^{\nu_2} + C_{12}\varepsilon^{p_2\mu_1\mu_2\mu_3}p_2^{\nu_1}p_3^{\nu_2} \\
&\quad + C_{13}\varepsilon^{p_1\mu_1\mu_2\mu_3}\delta^{\nu_1\nu_2} + C_{14}\varepsilon^{p_2\mu_1\mu_2\mu_3}\delta^{\nu_1\nu_2} + C_{15}\varepsilon^{p_1p_2\mu_1\mu_2}p_2^{\nu_1}p_3^{\nu_2}p_1^{\mu_3} + C_{16}\varepsilon^{p_1p_2\mu_1\mu_2}\delta^{\nu_1\nu_2}p_1^{\mu_3}\big) \\
&\quad + p_2^\kappa\big(C_{21}\varepsilon^{p_1\mu_1\mu_2\mu_3}p_2^{\nu_1}p_3^{\nu_2} + C_{22}\varepsilon^{p_2\mu_1\mu_2\mu_3}p_2^{\nu_1}p_3^{\nu_2} + C_{23}\varepsilon^{p_1\mu_1\mu_2\mu_3}\delta^{\nu_1\nu_2} + C_{24}\varepsilon^{p_2\mu_1\mu_2\mu_3}\delta^{\nu_1\nu_2} \\
&\quad + C_{25}\varepsilon^{p_1p_2\mu_1\mu_2}p_2^{\nu_1}p_3^{\nu_2}p_1^{\mu_3} + C_{26}\varepsilon^{p_1p_2\mu_1\mu_2}\delta^{\nu_1\nu_2}p_1^{\mu_3}\big) + \delta^{\kappa\mu_1}\big(C_{31}\varepsilon^{p_1\mu_2\mu_3\nu_1}p_3^{\nu_2} + C_{32}\varepsilon^{p_2\mu_2\mu_3\nu_1}p_3^{\nu_2} \\
&\quad + C_{33}\varepsilon^{p_1p_2\mu_2\nu_1}p_1^{\mu_3}p_3^{\nu_2} + C_{34}\varepsilon^{p_1p_2\mu_2\mu_3}\delta^{\nu_1\nu_2}\big) + \delta^{\kappa\mu_2}\big(C_{41}\varepsilon^{p_1\mu_1\mu_3\nu_2}p_2^{\nu_1} + C_{42}\varepsilon^{p_2\mu_1\mu_3\nu_2}p_2^{\nu_1} + C_{43}\varepsilon^{p_1p_2\mu_1\nu_2}p_1^{\mu_3}p_2^{\nu_1} \\
&\quad + C_{44}\varepsilon^{p_1p_2\mu_1\mu_3}\delta^{\nu_1\nu_2}\big) + C_{51}\varepsilon^{\kappa\mu_1\mu_2\mu_3}\delta^{\nu_1\nu_2} + C_{52}\varepsilon^{\kappa\mu_1\mu_2\mu_3}p_2^{\nu_1}p_3^{\nu_2} + C_{53}\varepsilon^{p_1\kappa\mu_1\mu_2}p_1^{\mu_3}\delta^{\nu_1\nu_2} + C_{54}\varepsilon^{p_2\kappa\mu_1\mu_2}p_1^{\mu_3}\delta^{\nu_1\nu_2}\big],
\end{aligned}
$$

$$
\begin{aligned}
&0 = K_{31}A_1, \qquad 0 = K_{32}K_{32}A_1, \\
&0 = K_{31}K_{31}A_2, \qquad 0 = K_{32}K_{32}K_{32}A_2.
\end{aligned}
$$

Complicated set of differential operators, solved exactly.

$$
\begin{aligned}
A_1 &= -4\, i\, a_2\, p_2^2\, I_{5\{2,1,1\}} \\
A_2 &= -8\, i\, a_2\, p_2^2 \left(p_3^2\, I_{4\{2,1,0\}} - 1 \right) \\
A_3 &= 0 \\
A_4 &= 0.
\end{aligned}
$$

Unique determination of all the transverse form factors

$$a_2 = -\frac{ig}{384\pi^2}.$$

Expressed in terms of ordinary
Perturbative integrals

$$A_1 = \frac{g p_2^2}{24\pi^2 \lambda^4} \left\{ A_{11} + A_{12} \log\left(\frac{p_1^2}{p_2^2}\right) + A_{13} \log\left(\frac{p_1^2}{p_3^2}\right) + A_{14}\; C_0(p_1^2, p_2^2, p_3^2) \right\},$$

$$A_2 = \frac{g p_2^2}{48\pi^2 \lambda^3} \left\{ A_{21} + A_{22} \log\left(\frac{p_1^2}{p_2^2}\right) + A_{23} \log\left(\frac{p_1^2}{p_3^2}\right) + A_{24}\; C_0(p_1^2, p_2^2, p_3^2) \right\},$$

$$A_3 = 0,$$

$$A_4 = 0$$

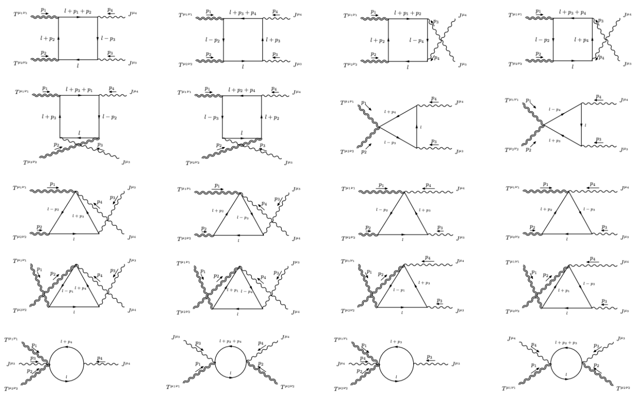
$$\lambda \equiv \lambda(p_1, p_2, p_3) = (p_1 - p_2 - p_3)\,(p_1 + p_2 - p_3)\,(p_1 - p_2 + p_3)\,.\,(p_1 + p_2 + p_3)$$

Four-Point Functions of Gravitons and Conserved Currents of CFT in Momentum Space: Testing the Nonlocal Action with the TTJJ

$$\begin{aligned}\langle T^{\mu_1\nu_1}T^{\mu_2\nu_2}J^{\mu_3}J^{\mu_4}\rangle^{(d=4)}_{anom} &= \langle t^{\mu_1\nu_1}_{loc}T^{\mu_2\nu_2}J^{\mu_3}J^{\mu_4}\rangle^{(d=4)}_{anom} + \langle T^{\mu_1\nu_1}t^{\mu_2\nu_2}_{loc}J^{\mu_3}J^{\mu_4}\rangle^{(d=4)}_{anom} - \langle t^{\mu_1\nu_1}_{loc}t^{\mu_2\nu_2}_{loc}J^{\mu_3}J^{\mu_4}\rangle^{(d=4)}_{anom}\\ &= \langle T^{\mu_1\nu_1}T^{\mu_2\nu_2}J^{\mu_3}J^{\mu_4}\rangle_{0-residue} + \langle T^{\mu_1\nu_1}T^{\mu_2\nu_2}J^{\mu_3}J^{\mu_4}\rangle_{pole}\,.\end{aligned}$$

$$\begin{aligned}\langle T^{\mu_1\nu_1}T^{\mu_2\nu_2}J^{\mu_3}J^{\mu_4}\rangle_{pole} = \beta_C \bigg\{ & 2\frac{\pi^{\mu_1\nu_1}(p_1)}{3}\left(\left[\sqrt{g}\,F^2\right]^{\mu_2\nu_2\mu_3\mu_4}(p_2,p_3,p_4) - \frac{\pi^{\mu_2\nu_2}(p_1+p_2)}{3}\left[F^2\right]^{\mu_3\mu_4}(p_3,p_4)\right)\\ &+ 2\frac{\pi^{\mu_2\nu_2}(p_2)}{3}\left(\left[\sqrt{g}\,F^2\right]^{\mu_1\nu_1\mu_3\mu_4}(p_1,p_3,p_4) - \frac{\pi^{\mu_1\nu_1}(p_1+p_2)}{3}\left[F^2\right]^{\mu_3\mu_4}(p_3,p_4)\right)\\ &+ 2\,\frac{\pi^{\mu_1\nu_1}(p_1)}{3}\,\frac{\pi^{\mu_2\nu_2}(p_2)}{3}\left[F^2\right]^{\mu_3\mu_4}(p_3,p_4)\bigg\}\end{aligned}$$

Bilinear mixings present



Th perturbative analysis differs from the derivation coming from

•Riegert action

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(G. Paci and O. Zanusso 2024)

Moving away from the conformal point

$$\Delta_{\alpha\mu\nu}(p_1,p_2) = \int \frac{d^4k}{(2\pi)^4}\mathrm{Tr}\left(\frac{1}{\not{k}-\not{p}_1-m}\gamma_\mu\frac{1}{\not{k}-m}\gamma_\nu\frac{1}{\not{k}+\not{p}_2-m}\gamma_\alpha\gamma_5\right)+[(p_1,\mu)\leftrightarrow(p_2,\nu)]$$

$$p_1^\mu\Delta_{\alpha\mu\nu}(p_1,p_2)=0, \quad p_2^\nu\Delta_{\alpha\mu\nu}(p_1,p_2)=0.$$

$$q^\alpha\Delta_{\alpha\mu\nu}(p_1,p_2)=2m\Delta_{\mu\nu}(p_1,p_2)+\frac{1}{2\pi^2}\epsilon_{\mu\nu\rho\sigma}p_1^\rho p_2^\sigma,$$

$$\Delta_{\alpha\mu\nu}(p_1,p_2)=F_1(s)q_\alpha\epsilon_{\mu\nu\rho\sigma}p_1^\rho p_2^\sigma+F_2(s)(p_{2\nu}\epsilon_{\alpha\mu\rho\sigma}-p_{1\mu}\epsilon_{\alpha\nu\rho\sigma})p_1^\rho p_2^\sigma$$

$$\Phi_0(p_1^2,p_2^2,p_3^2,m^2)\equiv F_1(s,s_1,s_2)=\frac{g^2m^2}{2\pi^2}\frac{1}{q^2}\mathrm{C}_0\left(q^2,p_1^2,p_2^2,m^2\right)+\frac{g^2}{4\pi^2}\frac{1}{q^2}.$$

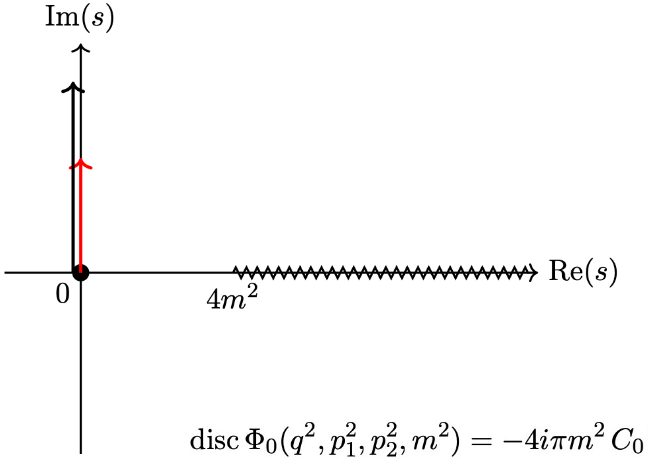
TOPOLOGICAL SUM RULES AND SPECTRAL FLOWS (AREA LAW)

Lionetti, Melle, CC.
to appear

$$\Phi_0(q^2, s_1, s_2, m^2) = \frac{1}{2\pi i}\oint_C \frac{\Phi_0(s, s_1, s_2, m^2)}{s - q^2}\, ds, \qquad \oint_C \Phi_0(s, s_1, s_2, m^2)\, ds = 0,$$

$$\begin{aligned}\Delta(s, p_1^2, p_2^2, m^2) &\equiv \mathrm{disc}\Phi_0 \\ &= \Phi_0(q^2 + i\epsilon, p_1^2, p_2^2, m^2) - \Phi_0(q^2 - i\epsilon, p_1^2, p_2^2, m^2)\end{aligned}$$

Complex function of 4 variables



INTERPLAY OF 3 CONTRIBUTIONS

A CUT
2 POLES
One of the pole is of fixed strength
the other is of variable strength

The anslysis of the spectral function is involved.

$$\mathrm{disc}\,\Phi_0(q^2, p_1^2, p_2^2, m^2) = -4i\pi m^2\, C_0(q^2, p_1^2, p_2^2, m^2)\,\delta(q^2) - 2\pi i\,\delta(q^2) + \frac{2m^2}{q^2}\mathrm{disc}\, C_0(q^2, p_1^2, p_2^2, m^2).$$

$$\frac{1}{\pi}\int_0^\infty \Delta\Phi_0\, ds = a_n$$

The integral of the absorbptive part of the anomaly form factor
Equals the numerical value of the anomaly.

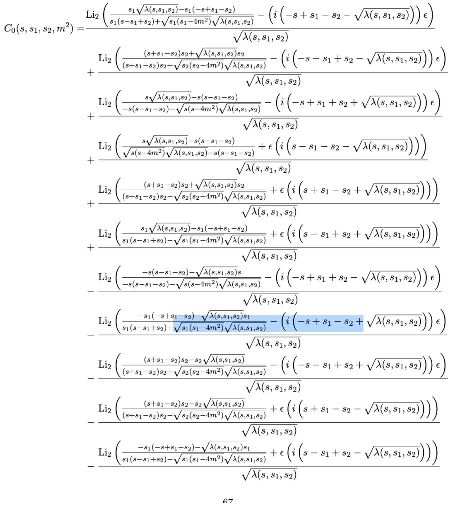
$$\begin{aligned}\rho(s,s_1,s_2,m^2) &\equiv \frac{2m^2}{q^2}\mathrm{disc}\, C_0(q^2,p_1^2,p_2^2,m^2)\theta(q^2-4m^2)\\ &= \frac{2m^2\pi\log\left(\frac{m^2-s(w-\bar z)(\bar w-z)}{m^2-s(w-z)(\bar w-\bar z)}\right)}{s^2(z-\bar z)},\end{aligned}$$

$$z=\frac{1}{2s}\left(\sqrt{\lambda}+s_1-s_2+s\right) \qquad \bar z = z = \frac{1}{2s}\left(-\sqrt{\lambda}+s_1-s_2+s\right)$$

$$w=\frac{1}{2}\left(1+\sqrt{1-\frac{4m^2}{s}}\right) \qquad \bar w=\frac{1}{2}\left(1-\sqrt{1-\frac{4m^2}{s}}\right)$$

The integral over the cut cancels
With the contribution of one of the
poles at s=0

$$4i\pi m^2\, C_0(0,p_1^2,p_2^2,m^2) = \int_{4m^2}^\infty ds\, \rho(s,s_1,s_2,m^2)$$



$$
\begin{aligned}
C_0(s,s_1,s_2,m^2) = {} & \frac{\mathrm{Li}_2\left(\frac{s_1\sqrt{\lambda(s,s_1,s_2)}-s_1(-s+s_1-s_2)}{s_1(s-s_1+s_2)+\sqrt{s_1(s_1-4m^2)}\sqrt{\lambda(s,s_1,s_2)}}-\left(i\left(-s+s_1-s_2-\sqrt{\lambda(s,s_1,s_2)}\right)\right)\epsilon\right)}{\sqrt{\lambda(s,s_1,s_2)}} \\
& + \frac{\mathrm{Li}_2\left(\frac{(s+s_1-s_2)s_2+\sqrt{\lambda(s,s_1,s_2)}s_2}{(s+s_1-s_2)s_2+\sqrt{s_2(s_2-4m^2)}\sqrt{\lambda(s,s_1,s_2)}}-\left(i\left(-s-s_1+s_2-\sqrt{\lambda(s,s_1,s_2)}\right)\right)\epsilon\right)}{\sqrt{\lambda(s,s_1,s_2)}} \\
& + \frac{\mathrm{Li}_2\left(\frac{s\sqrt{\lambda(s,s_1,s_2)}-s(s-s_1-s_2)}{-s(s-s_1-s_2)-\sqrt{s(s-4m^2)}\sqrt{\lambda(s,s_1,s_2)}}-\left(i\left(-s+s_1+s_2+\sqrt{\lambda(s,s_1,s_2)}\right)\right)\epsilon\right)}{\sqrt{\lambda(s,s_1,s_2)}} \\
& + \frac{\mathrm{Li}_2\left(\frac{s\sqrt{\lambda(s,s_1,s_2)}-s(s-s_1-s_2)}{\sqrt{s(s-4m^2)}\sqrt{\lambda(s,s_1,s_2)}-s(s-s_1-s_2)}+\epsilon\left(i\left(s-s_1-s_2-\sqrt{\lambda(s,s_1,s_2)}\right)\right)\right)}{\sqrt{\lambda(s,s_1,s_2)}} \\
& + \frac{\mathrm{Li}_2\left(\frac{(s+s_1-s_2)s_2+\sqrt{\lambda(s,s_1,s_2)}s_2}{(s+s_1-s_2)s_2-\sqrt{s_2(s_2-4m^2)}\sqrt{\lambda(s,s_1,s_2)}}+\epsilon\left(i\left(s+s_1-s_2+\sqrt{\lambda(s,s_1,s_2)}\right)\right)\right)}{\sqrt{\lambda(s,s_1,s_2)}} \\
& + \frac{\mathrm{Li}_2\left(\frac{s_1\sqrt{\lambda(s,s_1,s_2)}-s_1(-s+s_1-s_2)}{s_1(s-s_1+s_2)-\sqrt{s_1(s_1-4m^2)}\sqrt{\lambda(s,s_1,s_2)}}+\epsilon\left(i\left(s-s_1+s_2+\sqrt{\lambda(s,s_1,s_2)}\right)\right)\right)}{\sqrt{\lambda(s,s_1,s_2)}} \\
& - \frac{\mathrm{Li}_2\left(\frac{-s(s-s_1-s_2)-\sqrt{\lambda(s,s_1,s_2)}s}{-s(s-s_1-s_2)-\sqrt{s(s-4m^2)}\sqrt{\lambda(s,s_1,s_2)}}-\left(i\left(-s+s_1+s_2-\sqrt{\lambda(s,s_1,s_2)}\right)\right)\epsilon\right)}{\sqrt{\lambda(s,s_1,s_2)}} \\
& - \frac{\mathrm{Li}_2\left(\frac{-s_1(-s+s_1-s_2)-\sqrt{\lambda(s,s_1,s_2)}s_1}{s_1(s-s_1+s_2)+\sqrt{s_1(s_1-4m^2)}\sqrt{\lambda(s,s_1,s_2)}}-\left(i\left(-s+s_1-s_2+\sqrt{\lambda(s,s_1,s_2)}\right)\right)\epsilon\right)}{\sqrt{\lambda(s,s_1,s_2)}} \\
& - \frac{\mathrm{Li}_2\left(\frac{(s+s_1-s_2)s_2-s_2\sqrt{\lambda(s,s_1,s_2)}}{(s+s_1-s_2)s_2+\sqrt{s_2(s_2-4m^2)}\sqrt{\lambda(s,s_1,s_2)}}-\left(i\left(-s-s_1+s_2+\sqrt{\lambda(s,s_1,s_2)}\right)\right)\epsilon\right)}{\sqrt{\lambda(s,s_1,s_2)}} \\
& - \frac{\mathrm{Li}_2\left(\frac{(s+s_1-s_2)s_2-s_2\sqrt{\lambda(s,s_1,s_2)}}{(s+s_1-s_2)s_2-\sqrt{s_2(s_2-4m^2)}\sqrt{\lambda(s,s_1,s_2)}}+\epsilon\left(i\left(s+s_1-s_2-\sqrt{\lambda(s,s_1,s_2)}\right)\right)\right)}{\sqrt{\lambda(s,s_1,s_2)}} \\
& - \frac{\mathrm{Li}_2\left(\frac{-s_1(-s+s_1-s_2)-\sqrt{\lambda(s,s_1,s_2)}s_1}{s_1(s-s_1+s_2)-\sqrt{s_1(s_1-4m^2)}\sqrt{\lambda(s,s_1,s_2)}}+\epsilon\left(i\left(s-s_1+s_2-\sqrt{\lambda(s,s_1,s_2)}\right)\right)\right)}{\sqrt{\lambda(s,s_1,s_2)}}
\end{aligned}
$$

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These cancellations are true in full generality

(the "i eps" is. a signum function)

Things get more interesting if we consider the sum rule for on-shell photons (s1=s2=0)

In this case there is no pole

$$C_0(q^2,m^2) = -\frac{1}{2m^2}\,{}_3F_2\left(1,1,1;2,\frac{3}{2};x\right),$$

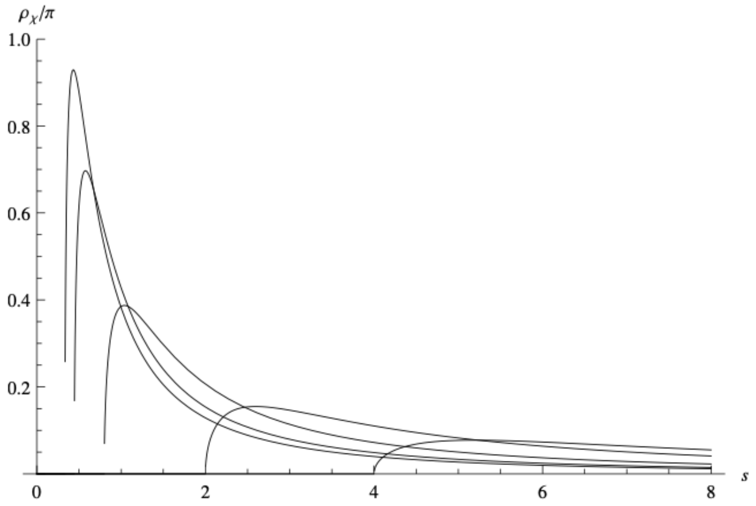
$${}_3F_2\left(1,1,1;2,\frac{3}{2};x\right) = \sum_{k=0}^{\infty}\frac{(1)_k(1)_k(1)_k}{(2)_k\left(\frac{3}{2}\right)_k}\frac{x^k}{k!},$$

$$C_0(q^2,m^2) = -\frac{1}{2m^2}\left(1+\frac{q^2}{12m^2}+\frac{q^4}{180m^4}+\mathcal{O}\left(\frac{q^6}{m^6}\right)\right),$$

$$\Phi_0 = -\frac{g^3}{4\pi^2}\frac{1}{12m^2} - \frac{g^3}{4\pi^2}\frac{q^2}{180m^4} + \ldots$$

No pole: just a cut from q^2 > 4m^2

Then the sum rule is an area law over the continuum



(a)

But the pole is reconstructed
In the conformal limit and no mass dependence

$$\text{disc}\,\Phi_0 = 4i\pi \frac{m^2}{(q^2)^2} \log \frac{1+\sqrt{\tau(q^2,m^2)}}{1-\sqrt{\tau(q^2,m^2)}} \theta(q^2-4m^2).$$

Send m to zero
The cut collapses
To a pole

$$\lim_{m_n^2\to 0} \Delta^{(n)}\Phi_0(s,m_n^2) = \lim_{m_n\to 0} \frac{2\pi m_n^2}{s^2} \log\left(\frac{1+\sqrt{\tau(s,m_n^2)}}{1-\sqrt{\tau(s,m_n^2)}}\right) \theta(s-4m_n^2) = \pi\delta(s),$$

$$\left\langle T^{\mu_1\nu_1}T^{\mu_2\nu_2}J_A^{\mu_3}\right\rangle = \left\langle t^{\mu_1\nu_1}t^{\mu_2\nu_2}j_A^{\mu_3}\right\rangle + \left\langle t^{\mu_1\nu_1}t^{\mu_2\nu_2}j_{A\,loc}^{\mu_3}\right\rangle + \left\langle t_{loc}^{\mu_1\nu_1}t^{\mu_2\nu_2}j_A^{\mu_3}\right\rangle + \left\langle t^{\mu_1\nu_1}t_{loc}^{\mu_2\nu_2}j_A^{\mu_3}\right\rangle$$

More sectors compared to the AVV

$$\begin{aligned} &g_{\mu_1\nu_1}\,g_{\mu_2\nu_2}\left\langle T^{\mu_1\nu_1}T^{\mu_2\nu_2}J_A^{\mu_3}\right\rangle = 0, \\ &p_{3\,\mu_3}\,g_{\mu_i\nu_i}\left\langle T^{\mu_1\nu_1}T^{\mu_2\nu_2}J_A^{\mu_3}\right\rangle = 0, \end{aligned}$$

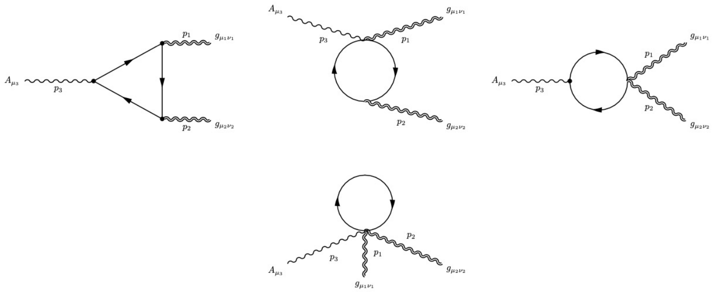
$$\left\langle t^{\mu_1\nu_1}t^{\mu_2\nu_2}j_{A\,loc}^{\mu_3}\right\rangle = p_3^{\mu_3}\Pi^{\mu_1\nu_1}_{\alpha_1\beta_1}(p_1)\Pi^{\mu_2\nu_2}_{\alpha_2\beta_2}(p_2)\epsilon^{\alpha_1\alpha_2p_1p_2}\left(F_1g^{\beta_1\beta_2} + F_2p_1^{\beta_2}p_2^{\beta_1}\right).$$

$$\begin{aligned} F_1 =& \frac{g\,(s-s_1-s_2)}{48\pi^2 s} + \frac{gm^2}{4\pi^2\lambda s}\Big\{2\big[\lambda m^2 + s s_1 s_2\big]C_0\left(s,s_1,s_2,m^2\right) + Q(s)\big[s(s_1+s_2)-(s_1-s_2)^2\big] \\ & - s_2Q(s_2)\big[s+s_1-s_2\big] - s_1Q(s_1)\big[s-s_1+s_2\big] + \lambda\Big\}, \\ F_2 =& -\frac{g}{24\pi^2 s} + \frac{gm^2}{2\pi^2\lambda^2 s}\Big\{2\big[\lambda\left(m^2(-s+s_1+s_2)-2s_1s_2\right) + 3s_1s_2\left((s_1-s_2)^2 - s(s_1+s_2)\right)\big]C_0\left(s,s_1,s_2,m^2\right) \\ & + s_1Q(s_1)\big[\lambda + 6s_2(s+s_1-s_2)\big] + s_2Q(s_2)\big[\lambda + 6s_1(s-s_1+s_2)\big] - Q(s)\big[12 s s_1 s_2 + \lambda(s_1+s_2)\big] \\ & + \lambda(-s+s_1+s_2)\Big\}. \end{aligned}$$

Lionetti, Melle, CC

Gravitational Anomaly sum rule away from the conformla point

Use perturbation theory



+ more..

$$\langle t^{\mu_1\nu_1} t^{\mu_2\nu_2} j_{5\,loc}^{\mu_3}\rangle = (\bar{F}_1 - \bar{F}_2)\frac{p_3^{\mu_3}}{p_3^2}\left\{\left[\varepsilon^{\nu_1\nu_2 p_1 p_2}\left(p_1\cdot p_2\, g^{\mu_1\mu_2} - p_1^{\mu_2} p_2^{\mu_1}\right) + (\mu_1 \leftrightarrow \nu_1)\right] + (\mu_2 \leftrightarrow \nu_2)\right\}$$

$$+(\bar{F}_1 + \bar{F}_2)\frac{p_3^{\mu_3}}{p_3^2}\left\{\left[\varepsilon^{\nu_1\nu_2 p_1 p_2}\left(p_1\cdot p_2\, g^{\mu_1\mu_2} + p_1^{\mu_2} p_2^{\mu_1}\right) + (\mu_1 \leftrightarrow \nu_1)\right] + (\mu_2 \leftrightarrow \nu_2)\right\} \qquad \text{(}$$

$$\begin{aligned}\phi_{ATT} = \frac{\bar{F}_1 - \bar{F}_2}{s} = &\frac{m^2 s_1\left(s_2\left(s_1^2 - s^2\right) + s_2^2(5s + s_1) + (s_1 - s)^3 - 3s_2^3\right)\mathrm{B}_0\left(s_1, m^2\right)}{\pi^2 s(s - s_1 - s_2)\lambda^2} \\ &+ \frac{m^2 s_2\left(s_2\left(3s^2 + s_1^2\right) + s_2^2(s_1 - 3s) - (s - s_1)^2(s + 3s_1) + s_2^3\right)\mathrm{B}_0\left(s_2, m^2\right)}{\pi^2 s(s - s_1 - s_2)\lambda^2} \\ &+ \frac{m^2 K_1(s, s_1, s_2)\mathrm{B}_0\left(s, m^2\right)}{\pi^2 s(s - s_1 - s_2)\lambda^2} + \frac{m^2 K_2(s, s_1, s_2)\mathrm{C}_0\left(s, s_1, s_2, m^2\right)}{\pi^2 s(s - s_1 - s_2)\lambda^2} \\ &\frac{m^2\left(s^2 - 2s(s_1 + s_2) + s_1^2 + s_2^2\right)}{\pi^2 s(s - s_1 - s_2)\lambda} + \frac{1}{12\pi^2 s}. \qquad \text{(11}\end{aligned}$$

$$\begin{aligned}\Phi_{\mathrm{ATT}} =&\frac{c_0}{s}+\frac{c_1(s,s_2)}{s}+(\text{continuum terms})+\frac{c_2(s,s_1,s_2)}{s-\bar{s}}+\frac{c_3(s,s_1,s_2)}{(s-s_-)^2}\\&+\frac{c_4(s,s_1,s_2)}{(s-s_+)^2}+\frac{c_5(s,s_1,s_2)}{s-s_-}+\frac{c_6(s,s_1,s_2)}{s-s_+},\end{aligned}$$

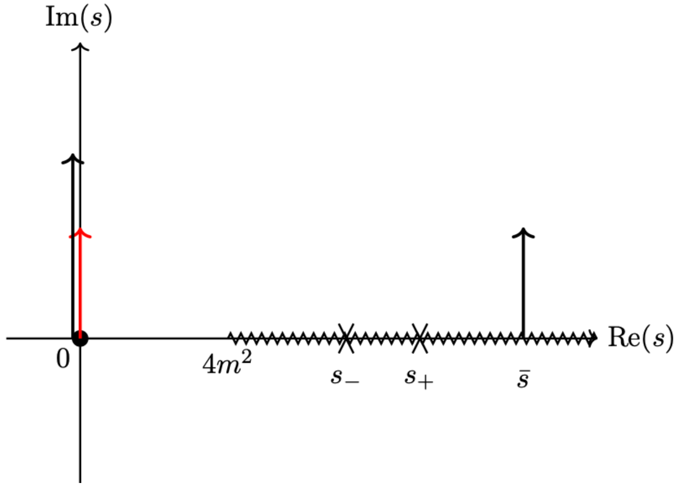
Spectral density
Gravit. anomaly form factor

$$\lambda=(s-s_-)(s-s_+),$$

$$\begin{aligned}\mathrm{disc}\Phi_{\mathrm{ATT}} =&c_0\delta(s)+c_1(s,s_1,s_2)\theta(s-4m^2)H(s)+c_2(s,s_1,s_2)\delta(s-s_1-s_2)\\&+c_3(s,s_1,s_2)\delta'(s-s_-)+c_4\delta'(s-s_+)+c_5\delta(s-s_-)+c_6\delta(s-s_+),\end{aligned}$$

Cancelation of extra thresholds

$$\Phi^{(\lambda)}_{\mathrm{ATT}}(q^2=s_-)=-\frac{d}{ds}\left(\frac{c_3(s,s_1,s_2,m^2)}{(s-q^2)}\right)\Bigg|_{s=s_-}-\frac{c_5(s_-,s_1,s_2,m^2)}{s_--q^2}.$$



Ordinary thresholds
plus extra thresholds.
The two black spikes cancel the contribution
of the cut after integration, leaving the sum rule to
Be attributed only to the red constant
Resonance.

$$s_\pm = (\sqrt{s_1} \pm \sqrt{s_2})^2$$

$$\mathrm{disc}(\phi_{ATT})_{pole} = \mathrm{disc}(\phi_{ATT})_0 + \mathrm{disc}(\phi_{ATT})_{s_1+s_2} + \mathrm{disc}(\phi_{ATT})_{s_-} + \mathrm{disc}(\phi_{ATT})_{s_+}.$$

TOPOLOGICAL SUM RULE

$$\mathrm{disc}(\phi_{ATT})_0 = 2\pi i \frac{\delta(s)}{12\pi^2} - \frac{2\pi i \delta(s)}{12\pi^2 (s_1 - s_2)^3 (s_1 + s_2)} \Psi(s, s_1, s_2, m^2)$$

$$\begin{aligned}\Psi(s, s_1, s_2, m^2) \equiv \Bigg(& 12m^2 \left(- \left((s_1 - s_2)\left(s_1^2 + s_2^2\right) \mathrm{B}_0\left(s, m^2\right)\right)\right. \\ & + s_1 \left(s_1^2 + 2 s_1 s_2 + 3 s_2^2\right) \mathrm{B}_0\left(s_1, m^2\right) - s_2 \left(3 s_1^2 + 2 s_1 s_2 + s_2^2\right) \mathrm{B}_0\left(s_2, m^2\right) \\ & \left. + (s_1 - s_2)\left(2m^2\left(s_1^2 + s_2^2\right) + s_1 s_2 (s_1 + s_2)\right) \mathrm{C}_0\left(s, s_1, s_2, m^2\right)\right) \\ & + 12 m^2 \left(s_1^2 + s_2^2\right)(s_1 - s_2)\Bigg).\end{aligned}$$

Also in this case we have a cancelation
between the cut and the the two black spikes

The effective action

One needs to compute the residue of a vertex in order to identify the character of the coupling

$$\lim_{q^2 \to 0} q^2 G(q^2, p_1^2, p_2^2) = g_p(p_1^2, p_2^2).$$

This limit has to be performed on the entire vertex
Not just on the anomaly form factor

$$G(q^2, p_1^2, p_2^2) = \frac{1}{q^2} \frac{1}{p_1^2} \frac{1}{p_3^2} F(q^2, p_1^2, p_2^2),$$

If the residue is nonzero, then there is an interpolating
State.
But this state is asymptotic only if the coupling
Is nonzero quite generally

For chiral anomalies

Absence of a particle pole for $q^2 \to 0$ $(s_1, s_2, m^2) \neq 0$

GENERAL DECOUPLING

$$\lim_{q^2 \to 0} q^2 \langle J^{\mu_1}(p_1) J^{\mu_2}(p_2) J^{\mu_3}(q) \rangle = 0.$$

$$\Delta_{TTJ_f}(q^2,m^2) \equiv \operatorname{Im} f_1(q^2) = \frac{d_{TTJ_f}}{q^2}(1-v^2)^2 \log\frac{1+v}{1-v}\theta(q^2-4m^2),$$

Also in this case you get a
Spectral flow with an area law.

$$\int_{4m^2}^{\infty} ds\, \Delta_{AVV}(s,m) = 2\, d_{AVV},$$
$$\int_{4m^2}^{\infty} ds\, \Delta_{TTJ_f}(s,m) = \frac{2}{3}\, d_{TTJ_f},$$

Sum rules

When we consider states on shell and we are at the conformal point

Grav anomaly

$$\mathcal{S}_{anom} \sim \int d^4x\, d^4y\, \partial_\lambda A^\lambda\, \frac{1}{\Box}(x,y) R\tilde{R}(y) + \ldots$$

Conf. anomaly

$$\mathcal{S}_{TJJ} \int d^4x d^4y R^{(1)}(x)\, \Box^{-1}(x,y) F_V F_V(y) + \ldots$$

Chiral anomaly

$$\mathcal{S}_{JJJ_5} = \int d^4x d^4y \partial \cdot A\, \Box^{-1}(x,y) F_V \tilde{F}_V(y) + \ldots$$

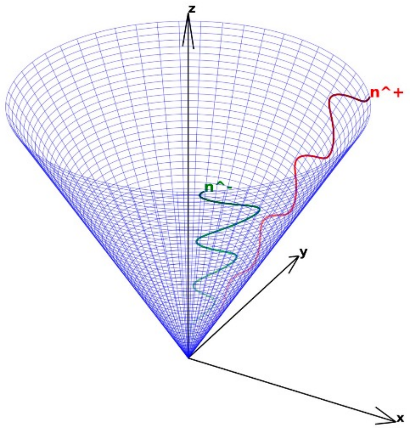
KINETIC MIXING

$$\mathcal{S}_{eff}[\eta,\chi;A,B] = \int d^4x \left\{(\partial^\mu\eta)\,(\partial_\mu\chi) - \chi\,\partial^\mu B_\mu + \frac{e^2}{8\pi^2}\,\eta\,F_{\mu\nu}\tilde{F}^{\mu\nu}\right\}$$

Giannotti and Mottola

An analogous action for the gravitational anomaly pole

$$\begin{aligned}&\Box\,\eta = -\partial^\lambda B_\lambda\,,\\&\Box\,\chi = \frac{e^2}{8\pi^2}\,F_{\mu\nu}\tilde{F}^{\mu\nu} = \frac{e^2}{16\pi^2}\,\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\,.\end{aligned}$$



ENTANGLEMENT

$$\begin{aligned}\mathcal{S}_{eff}[\bar\eta,\bar\chi;A,B] \;&=\; \int d^4x\,\{(\partial^\mu\bar\eta)\,(\partial_\mu\bar\chi) + M\partial^\mu\bar\chi\,B_\mu\\&\quad+\alpha\frac{\bar\eta}{M}\,F_{\mu\nu}\tilde{F}^{\mu\nu}\Big\}\end{aligned}$$

$$\bar\eta = a(x) - b(x) \qquad \bar\chi = b(x) + a(x)$$

$$\mathcal{S}_{eff}[a,b;A,B] \;=\; \int d^4x\left(\frac{1}{2}(\partial_\mu a - \bar{M}B_\mu)^2 - \frac{1}{2}(\partial_\mu b - \bar{M}B_\mu)^2 + \frac{a-b}{\bar{M}}F\tilde{F}\right)$$

$$a \to a - \bar{M}\theta(x), \qquad b \to b + \bar{M}\theta, \qquad B_\mu \to B_\mu + \partial_\mu\theta$$

Conclusions

Axion like interactions may not allow asymptotic axion-like particles, if a confining phase is absent.
Equating an axion-like particle to an ordinary Nambu Goldstone mode is not automatic.
An anomaly pole is not an asymptotic state, it is an interpolating scalar (or pseudoscalar) state.
It decouples in the IR as on ordinary NG mode, but it is richer than that: it is tightly connected to the transverse sector of the anomaly interaction by a sum rule.
We have shown that anomaly interactions satisfy sum rule for general kinematics.

The same is also true for dilaton form factors, extracted from the gravitational form factor of the proton