Ginsparg–Wilson, Overlap Fermion, and extensions

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Introduction

- **Quantum Field Theory**: universal framework in theoretical physics
- **Lattice Field Theory**: non-perturbative formulation of QFT
  - confinement in lattice gauge theory: Wilson's formulation
  - computational precision science: QCD, Standard Model
  - microscopic model for cond-mat physics
- **Fermion on a lattice**: doubling vs (chiral) anomaly
  - theoretical challenges: chiral gauge theory, topological phases
  - many avatars: Wilson, staggered, SLAC, domain-wall, overlap...

Lattice fermion and doubling

- **Discretization of Dirac operator**: $D = \gamma^\mu \nabla_\mu - m$
  - difference operator: $\nabla_\mu \psi(x) = \dfrac{\psi(x + a\hat{\mu}) - \psi(x - a\hat{\mu})}{2a} \xrightarrow{a \to 0} \partial_\mu \psi(x)$
  - (Euclidian) gamma matrices: γ^μ, lattice spacing: a, unit vector: $\hat{\mu}$
- **Momentum space representation**: $(p_\mu \in [0, 2\pi);\ a = 1)$

$$D(p) = \mathrm{i}\gamma^\mu \sin p_\mu - m \longrightarrow \begin{cases} +\mathrm{i}\gamma^\mu p_\mu - m & (p_\mu \ll 1) \\ -\mathrm{i}\gamma^\mu q_\mu - m & (p_\mu = \pi + q_\mu, q_\mu \ll 1) \end{cases}$$

Nielsen–Ninomiya theorem

Imposing **translational invariance**, **hermiticity**, and **locality**, there exists an equal number of left-handed and right-handed chiral fermions.

- Chiral theories?: chiral anomaly, chiral effects, chiral gauge theory...

Realizations

- Naive fermion: (many) doublers
- Wilson fermion: no doublers, but chiral symmetry violated
- Staggered fermion: (reduced) doublers, "flavored" chiral symmetry
- SLAC fermion: non-local formulation
- Minimal-doubling fermion: $\#\{\text{doublers}\} = 2$, lower spatial symmetry
- Domain-wall fermion: boundary of the extra dimention
- Overlap fermion: modified chiral symmetry

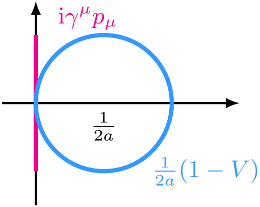
Ginsparg–Wilson relation

$$\gamma_5 D + D\gamma_5 = 2aD\gamma_5 D$$

[Ginsparg–Wilson '82]

Overlap fermion

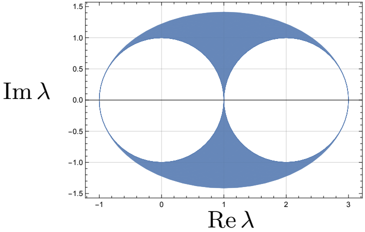
- **Overlap Dirac operator**: $D_{\text{ov}} = \frac{1}{2a}(1 - V)$ [Neuberger '98]
  - unitary op.: $\gamma_5 V \gamma_5 = V^\dagger = V^{-1}$
  - lattice spacing: a
  - γ_5-hermiticity: $\gamma_5 D \gamma_5 = D^\dagger$
  - GW relation: $\gamma_5 D_{\text{ov}} + D_{\text{ov}} \gamma_5 = 2a D_{\text{ov}} \gamma_5 D_{\text{ov}}$



- **Construction of D_{ov} in d-dim**
  1. Start with d-dim system (Wilson-Dirac operator D_{W} with mass)
  2. Add an extra dimension of size N_{ext} with open b.c.
  3. Integrate the extra dimension and take $aN_{\text{ext}} \to \infty$, $a \to 0$

$$V = \frac{D_{\text{W}}}{\sqrt{D_{\text{W}}^\dagger D_{\text{W}}}} = \gamma_5 \frac{H_{\text{W}}}{\sqrt{H_{\text{W}}^2}}, \quad H_{\text{W}} = \gamma_5 D_{\text{W}}$$

- Complex spectrum of Wilson–Dirac operator in $d = 2$:



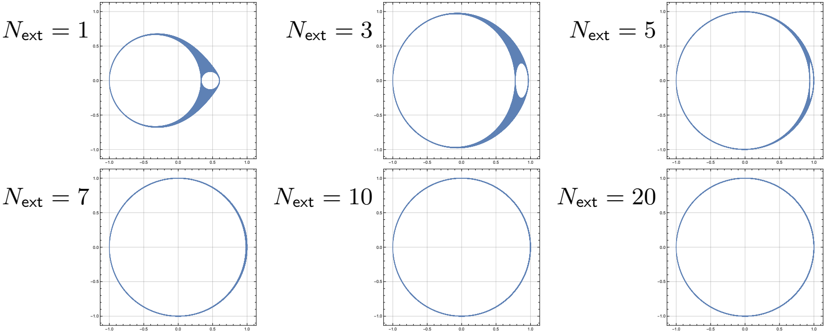
- **Point-gap non-hermitian sys.**

  → deformable to **unitary op.**

  [Kawabata et al.]

- Overlap formalism = band flatten

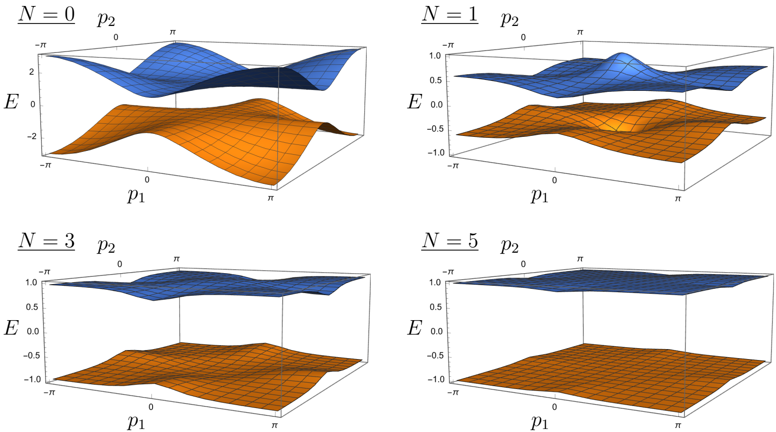
- N_{ext} dependence of the spectrum: $V^{(N_{\text{ext}})} \xrightarrow{N_{\text{ext}} \to \infty} V$



- Wilson–Dirac Hamiltonian in $d = 2$ (momentum space rep.):

$$H_{\mathsf{W}} = \sigma_1 \sin p_1 + \sigma_2 \sin p_2 + (m + 2 - \cos p_1 - \cos p_2)\sigma_3$$

- Band flattening: $H_{\mathsf{W}}^{(N_{\mathrm{ext}})} = \gamma_5 V^{(N_{\mathrm{ext}})} \xrightarrow{N_{\mathrm{ext}} \to \infty} \dfrac{H_{\mathsf{W}}}{\sqrt{H_{\mathsf{W}}^2}} = \mathrm{sgn}(H_{\mathsf{W}})$



Chiral anomaly and Dirac index

- **Chiral transformation**: $(\psi, \bar{\psi}) \mapsto (\gamma_5 \psi, \bar{\psi}\gamma_5)$, $\bar{\psi}\gamma^\mu \partial_\mu \psi \mapsto \bar{\psi}\gamma^\mu \partial_\mu \psi$
  - Fujikawa method: $\mathscr{D}\bar{\psi}\mathscr{D}\psi \mapsto \mathscr{D}\bar{\psi}\mathscr{D}\psi \times$ Jacobian

- **Ginsparg–Wilson relation**: $\gamma_5 D + D\gamma_5 = 2aD\gamma_5 D \iff \gamma_5 D + D\hat{\gamma}_5 = 0$

  where $\hat{\gamma}_5 = \gamma_5(1 - 2aD) = \gamma_5 V$ implying modification: [Lüscher]

$$(\psi, \bar{\psi}) \mapsto (\hat{\gamma}_5 \psi, \bar{\psi}\gamma_5) = \left(\gamma_5 V \psi, \bar{\psi}\gamma_5\right) , \quad \mathscr{D}\bar{\psi}\mathscr{D}\psi \mapsto \mathscr{D}\bar{\psi}\mathscr{D}\psi \times \det V^{-1}$$

- **Atiyah–Singer index theorem and chiral anomaly**: $\operatorname{ind} D = \partial^\mu j^5_\mu$

$$\operatorname{ind} D = \dim \ker D - \dim \operatorname{coker} D = \frac{1}{8\pi^2} \int_M \operatorname{tr} F \wedge F + (R\text{-term}) \in \mathbb{Z}$$

Overlap operator index

$$\operatorname{ind} D_{\mathsf{ov}} = \operatorname{tr} \gamma_5 D_{\mathsf{ov}} = -\frac{1}{2} \operatorname{tr} \gamma_5 V = -\frac{1}{2} \operatorname{tr} \left(\frac{H_{\mathsf{W}}}{\sqrt{H_{\mathsf{W}}^2}} \right) = -\frac{1}{2} \eta(H_{\mathsf{W}})$$

[Hasenfratz–Laliena–Niedermayer] [Lüscher] [Adams]

- **Q.** Is it all about the overlap fermion?
- **Hint**: 10-fold way classification of free fermion systems [Altland–Zirnbauer]

  ⟶ Wilson–Dirac system belongs to class A ($m \neq 0$)

- **Q.** Overlap fermion for other symmetry classes with $\mathcal{C}$, $\mathcal{T}$?
  - $\mathcal{C} = CK$, $\mathcal{T} = TK$

    where $CHC^{-1} = -H^*$, $THT^{-1} = +H^*$, $K\mathcal{O}K = \mathcal{O}^*$

    with $\mathcal{C}^2 = \pm 1$, $\mathcal{T}^2 = \pm 1$

- Overlap fermion in odd-dimension: [Bietenholz–Nishimura]

$$D = \frac{1}{2a}(1 - V)\,, \qquad V^\dagger = V^{-1} \qquad \text{(no } \gamma_5\text{-hermiticity condition)}$$

  - Remark: $H_{3d} = \begin{pmatrix} 0 & m + \mathrm{i}\vec{\sigma}\cdot\vec{p} \\ m - \mathrm{i}\vec{\sigma}\cdot\vec{p} & 0 \end{pmatrix} \Longrightarrow \{\gamma_5, H_{3d}\} = 0$ (class AIII)

- Unitary operator:

$$V^\dagger = V^{-1} = \gamma_5 V \gamma_5 \implies V \in \frac{\mathrm{U}}{\mathrm{U} \times \mathrm{U}}\,, \quad V^\dagger = V^{-1} \implies V \in \mathrm{U}$$

Symmetry class	$\mathscr{C}$	Classifying space $S_{\mathscr{C}}$	T-evolution operator $U_{\mathscr{C}}$	$\mathcal{T}^2$	$\mathcal{C}^2$	χ
A	C_0	$\mathrm{U}/\mathrm{U}\times\mathrm{U}$	U	0	0	0
AIII	C_1	U	$\mathrm{U}/\mathrm{U}\times\mathrm{U}$	0	0	1
AI	R_0	$\mathrm{O}/\mathrm{O}\times\mathrm{O}$	U/O	$+1$	0	0
BDI	R_1	O	$\mathrm{O}/\mathrm{O}\times\mathrm{O}$	$+1$	$+1$	1
D	R_2	O/U	O	0	$+1$	0
DIII	R_3	U/Sp	O/U	-1	$+1$	1
AII	R_4	$\mathrm{Sp}/\mathrm{Sp}\times\mathrm{Sp}$	U/Sp	-1	0	0
CII	R_5	Sp	$\mathrm{Sp}/\mathrm{Sp}\times\mathrm{Sp}$	-1	-1	1
C	R_6	Sp/U	Sp	0	-1	0
CI	R_7	U/O	Sp/U	$+1$	-1	1

10-fold way classification of overlap operator

For class $\mathscr{C}$ system, **the unitary operator** in the overlap operator takes a value in **the classifying space** $S_{\mathscr{C}}$:

$$D_{\mathrm{ov}} = \frac{1}{2a}(1-V) \qquad \text{where} \qquad V \in S_{\mathscr{C}}$$

[K–Watanabe]

Ginsparg–Wilson relation revisited

- Ginsparg–Wilson relation: $\gamma_5 D + D\gamma_5 = 2aD\gamma_5 D$

  → non-linear deformation of **chiral symmetry** $\{\gamma_5, D\} = 0$

- Reformulation: [Bietenholz–Nishimura]

$$\gamma_5 D + D\gamma_5 = 2aD\gamma_5 D \xrightarrow{\gamma_5 D\gamma_5 = D^\dagger} D + D^\dagger = 2aDD^\dagger$$

  → deformation of **anti-hermiticity** $D + D^\dagger = 0$ (masslessness)

Ginsparg–Wilson relation for $\mathcal{C}, \mathcal{T}$ symmetry

Ginsparg–Wilson relation for $\mathcal{C}, \mathcal{T}$ symmetric system is given by

$$CD + D^\mathsf{T} C = 2aD^\mathsf{T} CD\,, \qquad TD + D^* T = 2aD^* TD$$

Defining $\hat{C} = CV$ and $\hat{T} = TV$, it is equivalent to

$$CD + D^\mathsf{T} \hat{C} = 0\,, \qquad TD + D^* \hat{T} = 0$$

which implies anomalous $\mathcal{C}, \mathcal{T}$ transformations,

$$\mathcal{C} : (\psi, \bar{\psi}) \mapsto (\hat{C}\bar{\psi}^\mathsf{T}, \psi^\mathsf{T} C^{-1})\,, \qquad \mathcal{T} : (\psi, \bar{\psi}) \mapsto (\hat{T}\psi, \bar{\psi}T^{-1})$$

[Clancy–Kaplan–Singh] [K–Watanabe]

$\mathcal{C}, \mathcal{T}$ anomaly and mod-two index

- Ginsparg–Wilson relation for $\mathcal{C}, \mathcal{T}$ symmetry implies anomalies of:
  - Majorana(-Weyl) fermion [Huet–Narayanan–Neuberger] [Inagaki–Suzuki]
  - $\mathcal{T}$-invariant topological insulator [Fukui–Fujiwara] [Ringel–Stern]

Mod-two overlap index

Let $\nu = \operatorname{ind} D_{\text{ov}} = \dim\ker D_{\text{ov}} \in \mathbb{Z}_2$ be **the mod-two index** of D_{ov} for **the class with $V \in \mathrm{O}$, O/U**. Then, we have

$$(-1)^{\nu} = \det V$$

[K–Watanabe]

- Remark: mod-two index in the domain-wall fermion formalism

[Fukaya et al.]

Summary

- Ginsparg–Wilson relation and overlap fermion:

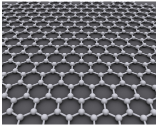
$$D = \frac{1}{2a}(1 - V)\,, \qquad \gamma_5 D + D\gamma_5 = 2aD\gamma_5 D$$

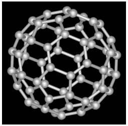
- 10-foldway of overlap fermion: $V \in S_{\mathscr{C}}$ (classifying space)

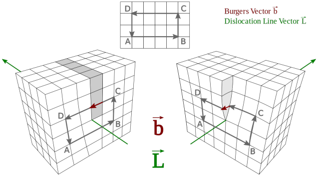
- Ginsparg–Wilson relation for $\mathcal{C}, \mathcal{T}$ symmetric system:

$$CD + D^{\mathsf{T}}C = 2aD^{\mathsf{T}}CD\,, \qquad TD + D^{*}T = 2aD^{*}TD$$

- Curved space-time (with torsion)?







[wikipedia]