Gravitational S-matrix, infrared divergences and BMS representations

Tours (24/01/25),
Avenues of Quantum Field Theory In Curved Spacetime 2025

Yannick Herfray, Institut Denis Poisson Tours

Based on

X. Bekaert -- L.Donnay -- YH, 2024

J.Borthwick -- M.Chantreau -- YH, 2024

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Mathématiques & Physique Théorique

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0- Overview

- In a recent work *(Bekaert– Donnay —YH, 2024)* we investigated BMS particles *i.e. unitary irreducible representations of the BMS group. (See Xavier's talk)*

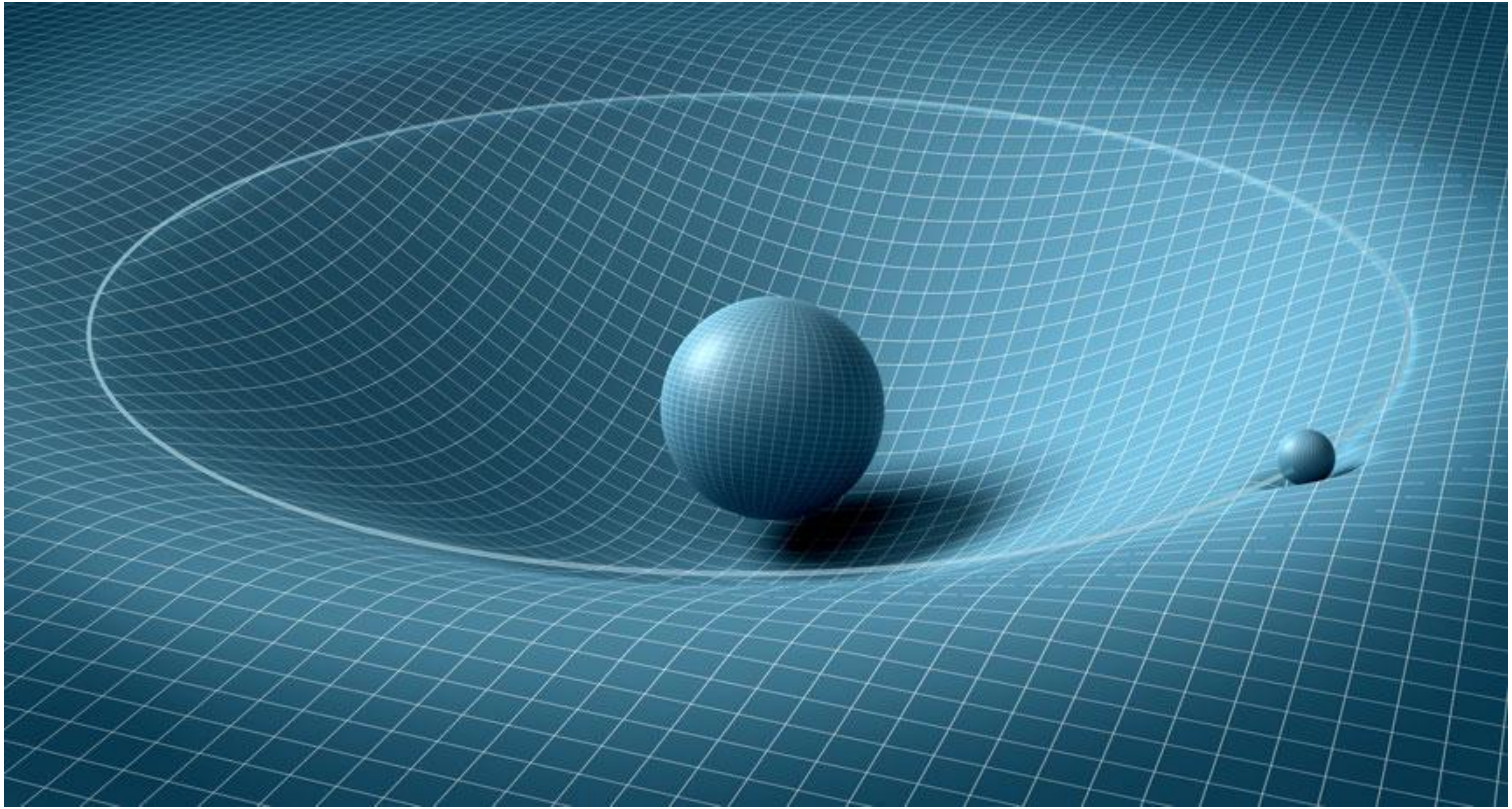
- This presentation aims at giving physical motivations as well as preliminary background for the construction

→ BMS group as asymptotic symmetry group of gravity

→ Relations to infrared divergences and soft theorems in QFT

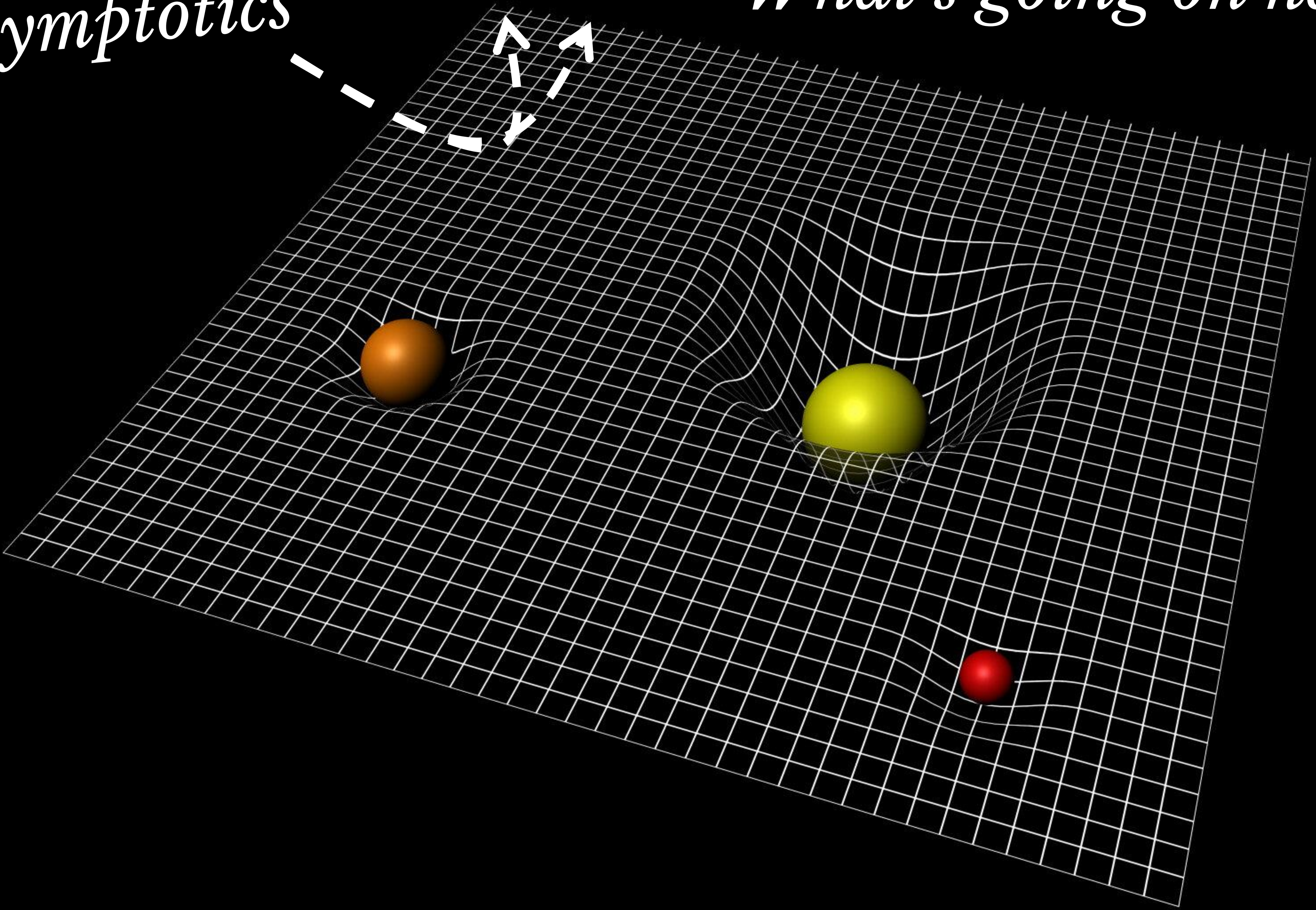
with a twist of rep. theory !

1- Asymptotics in General Relativity



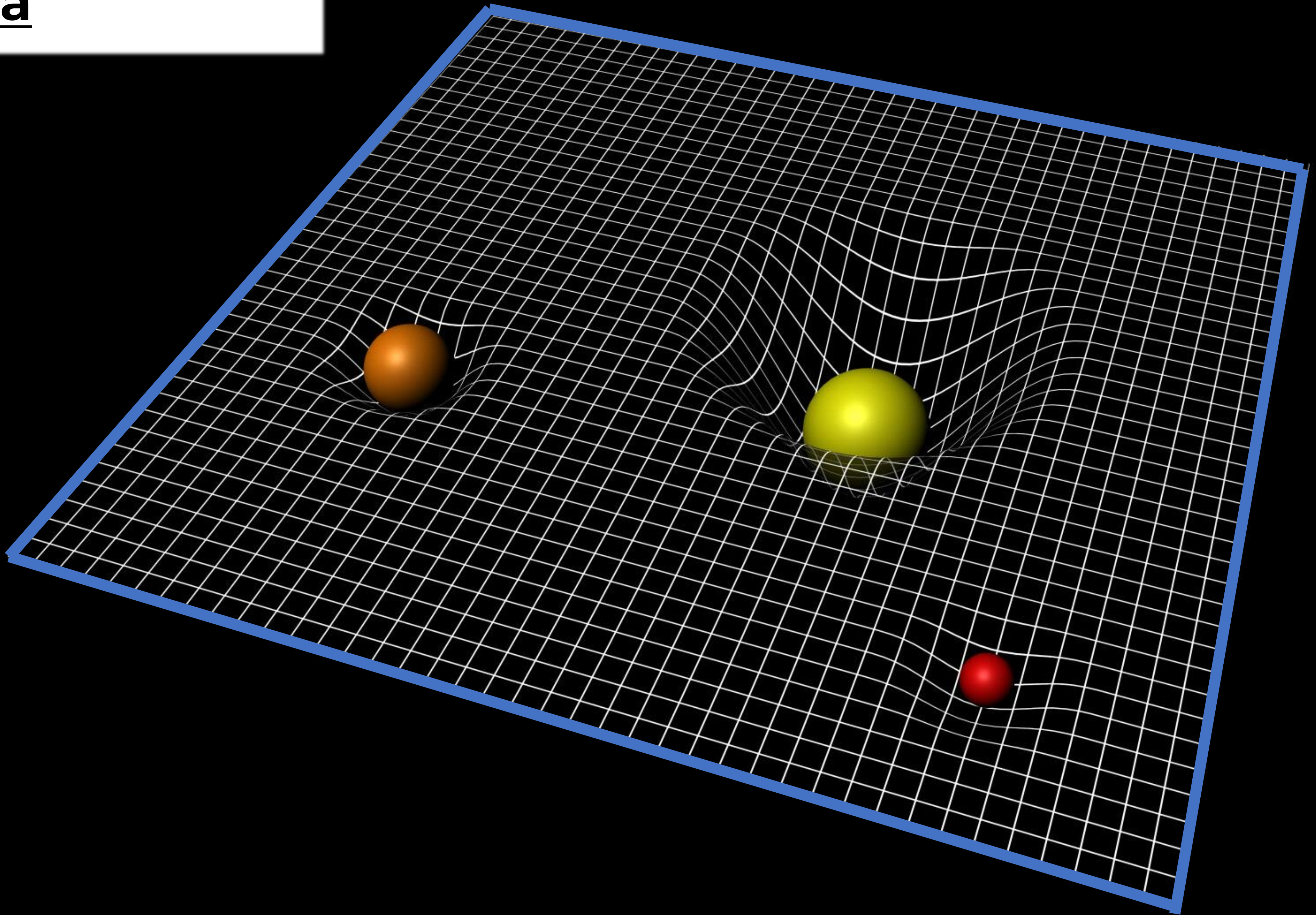
What's going on here ?

Asymptotics



Essential Idea

1) Introduce *M*, manifold with boundary

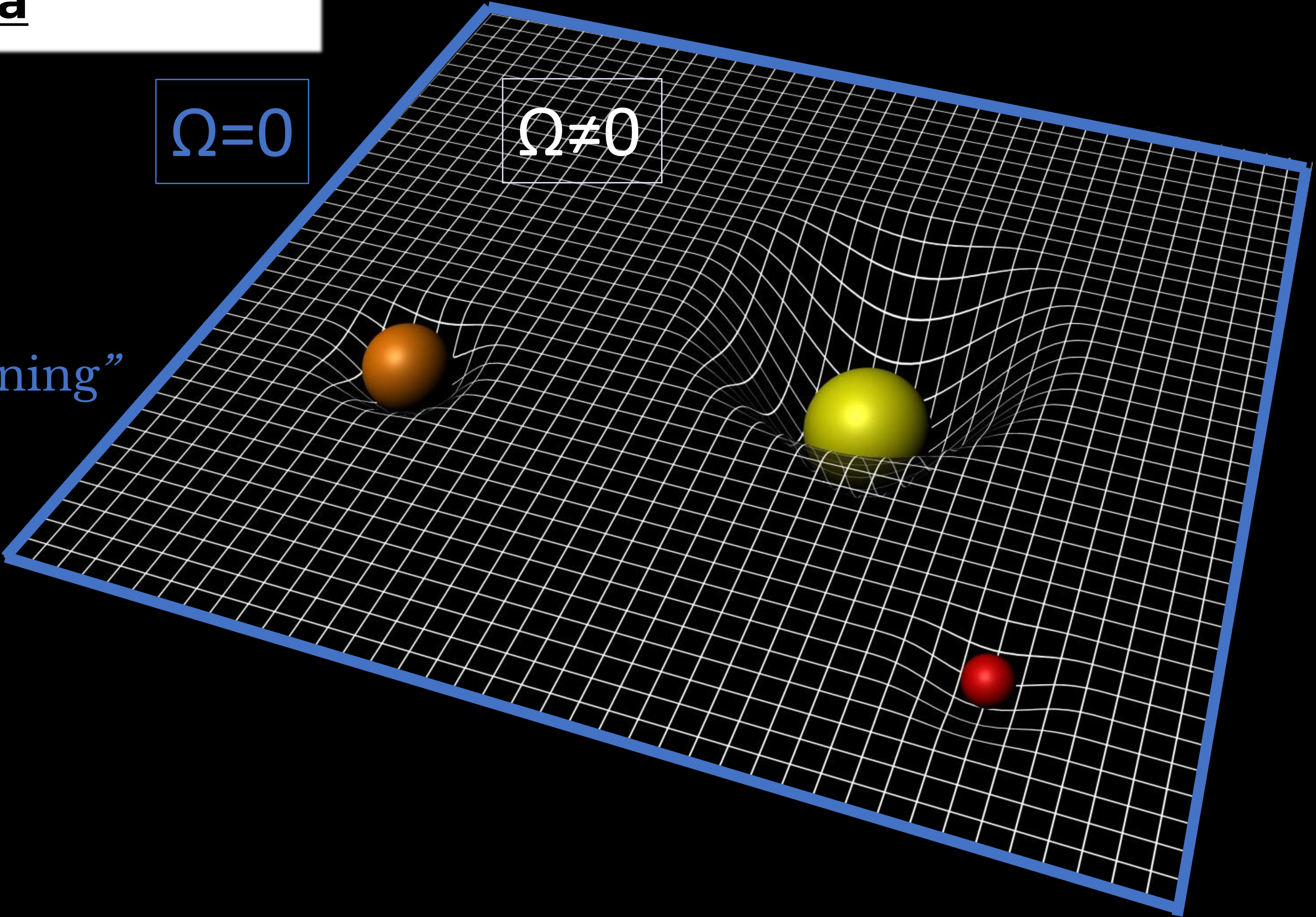


Essential Idea

1) Introduce M, manifold with boundary

2) Introduce a "boundary defining" function Ω

$\Omega=0$



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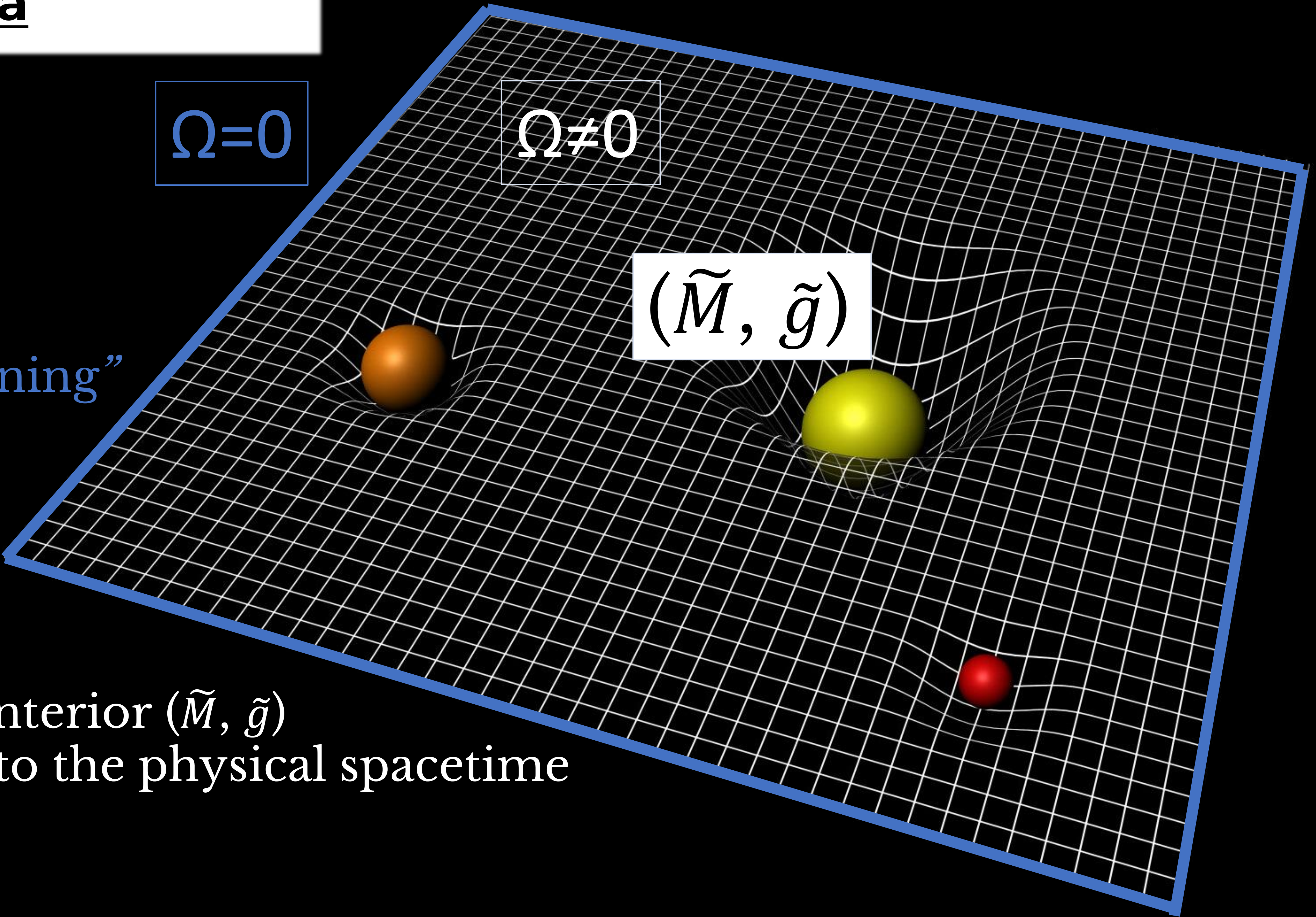

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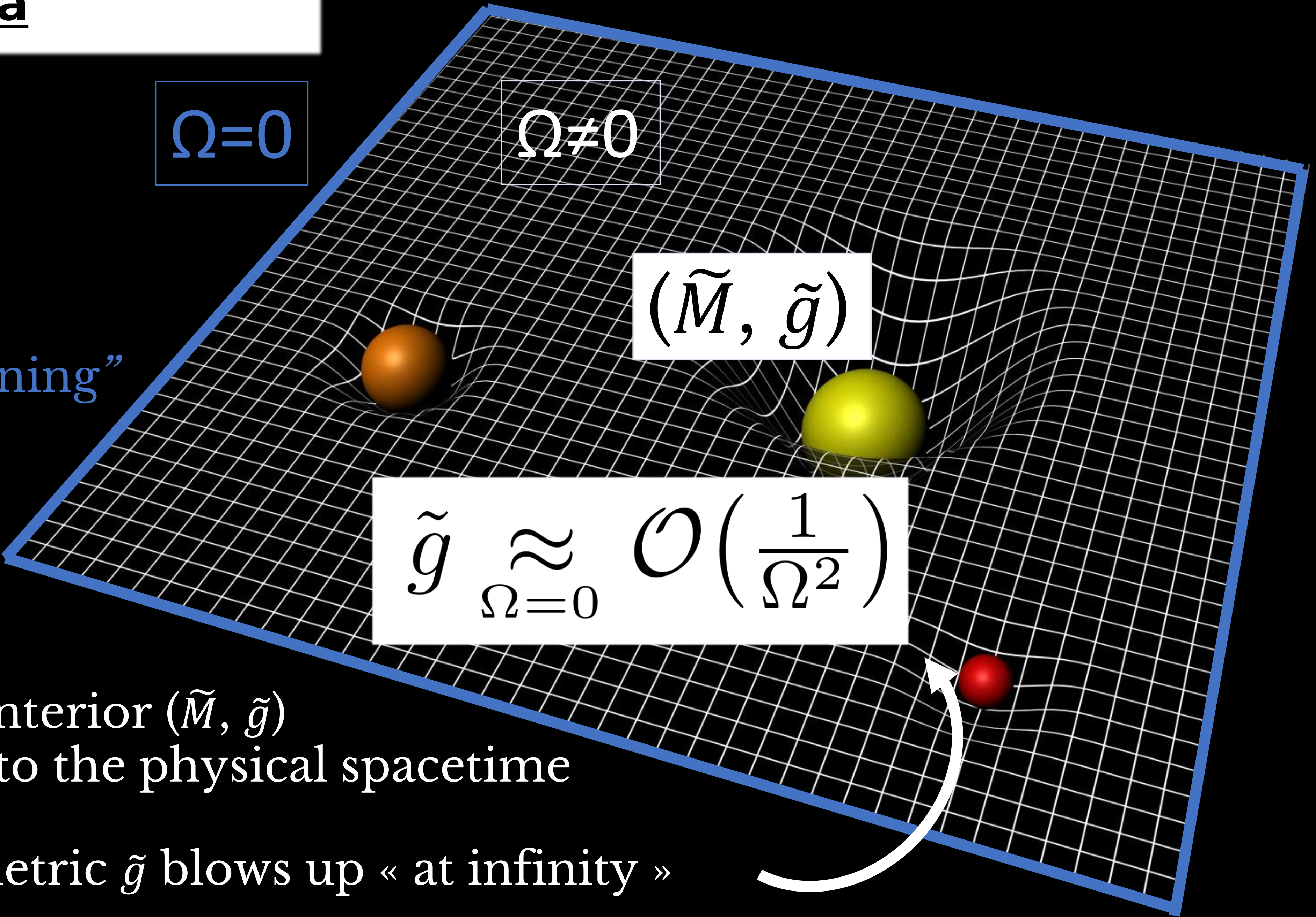
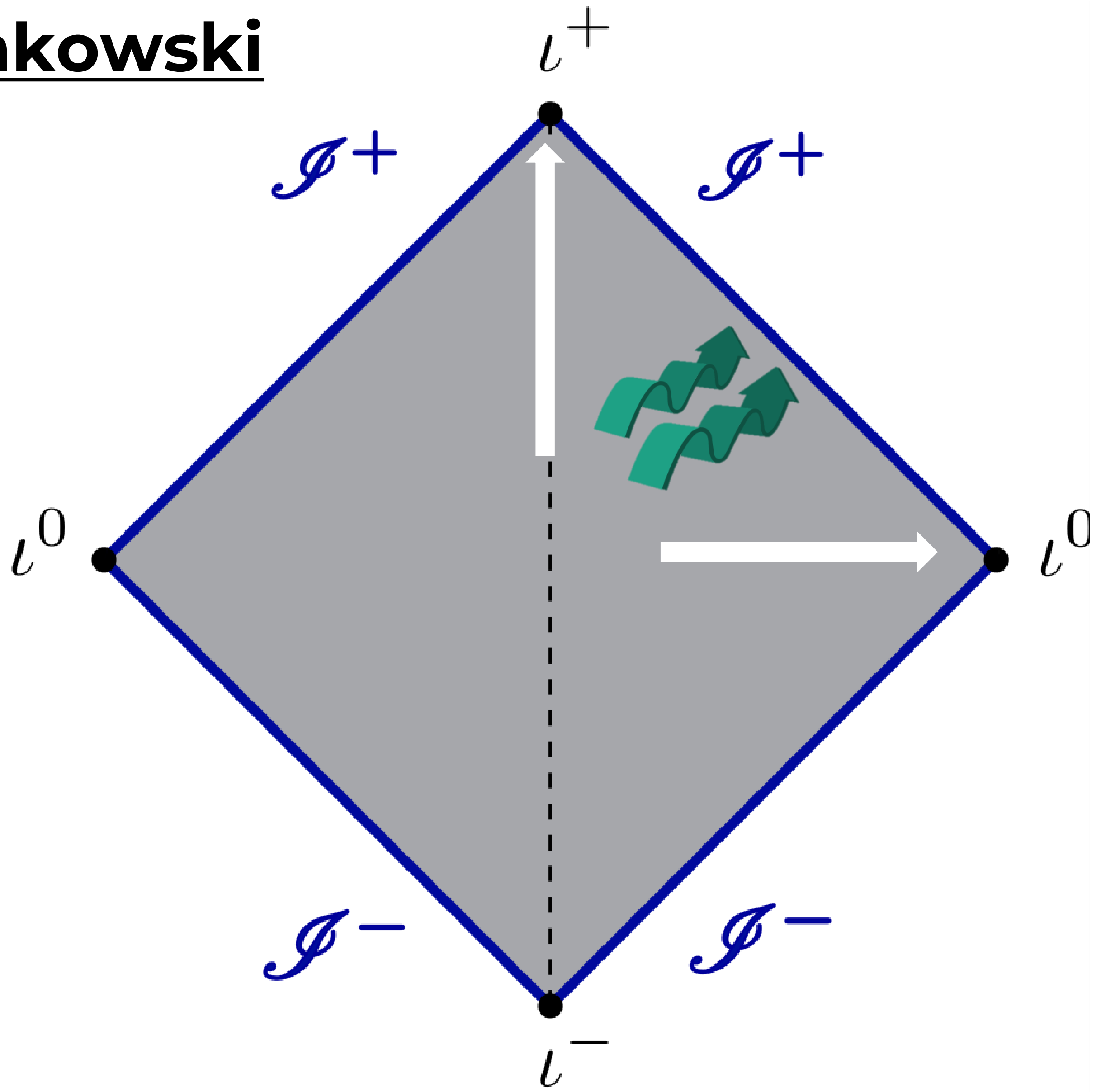


3) Such that the interior $(\widetilde{M}, \tilde{g})$ of M is isometric to the physical spacetime

4) The physical metric $\tilde{g}$ blows up « at infinity »

Conformal diagram

Minkowski



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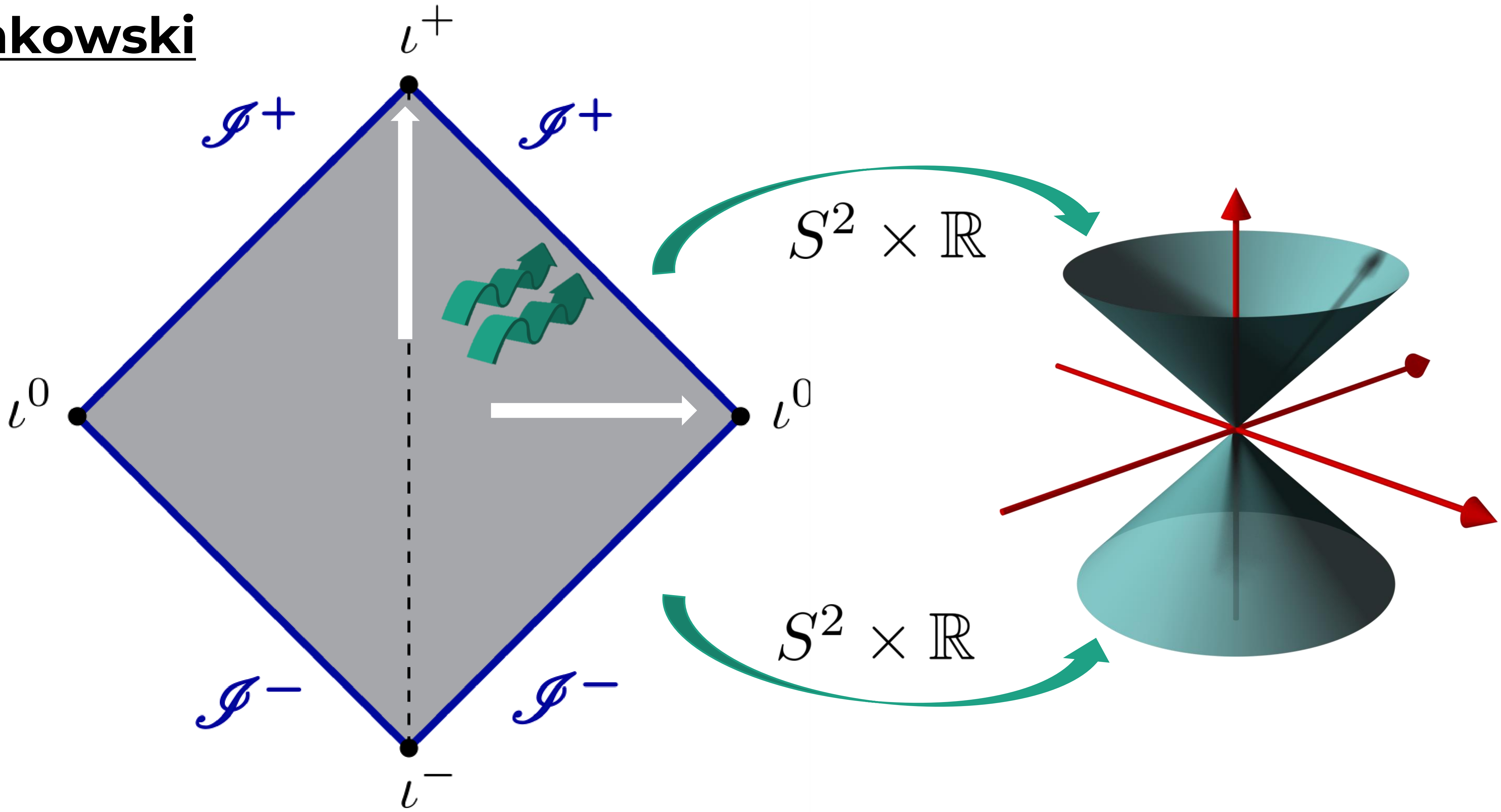

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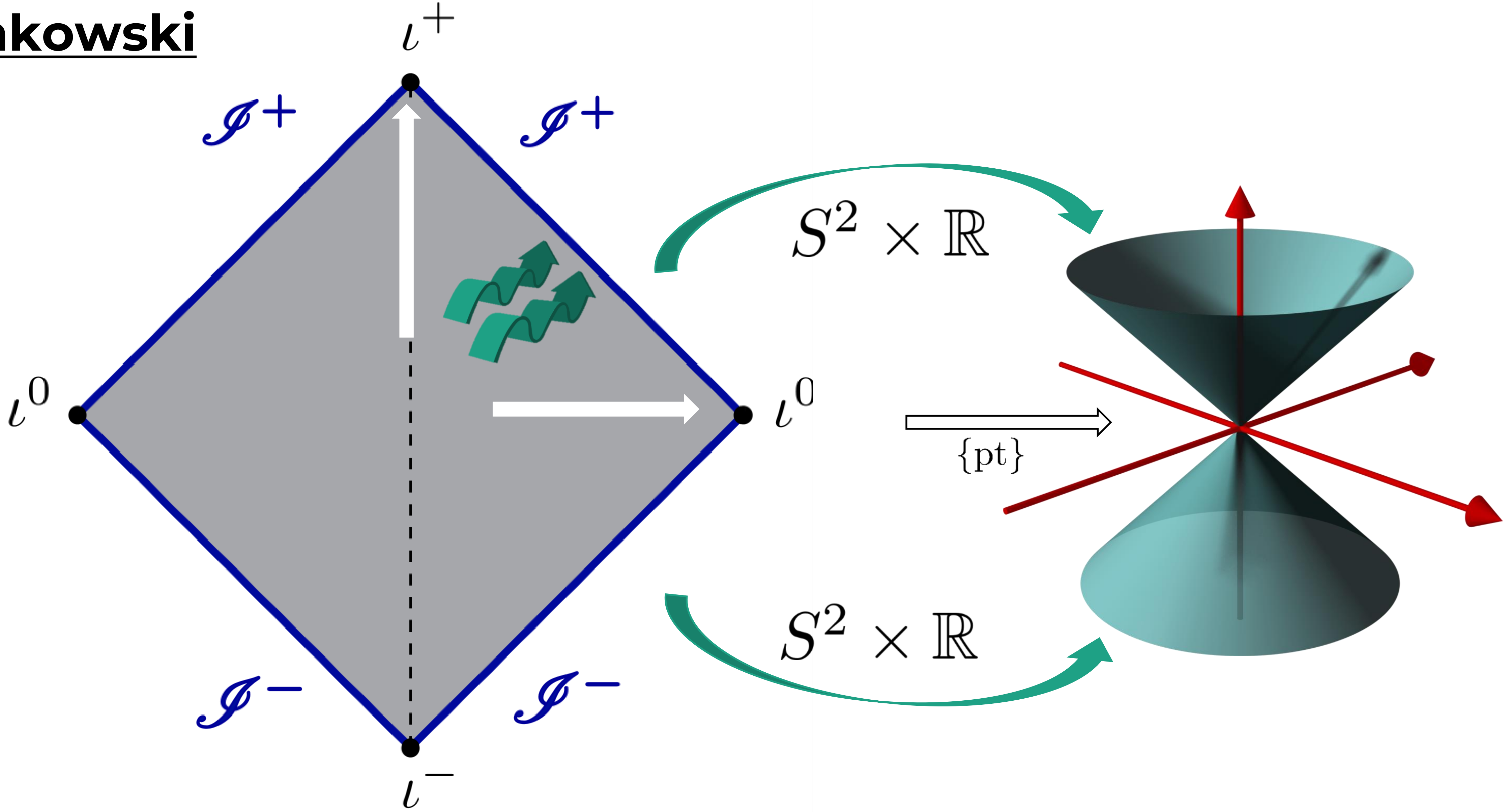
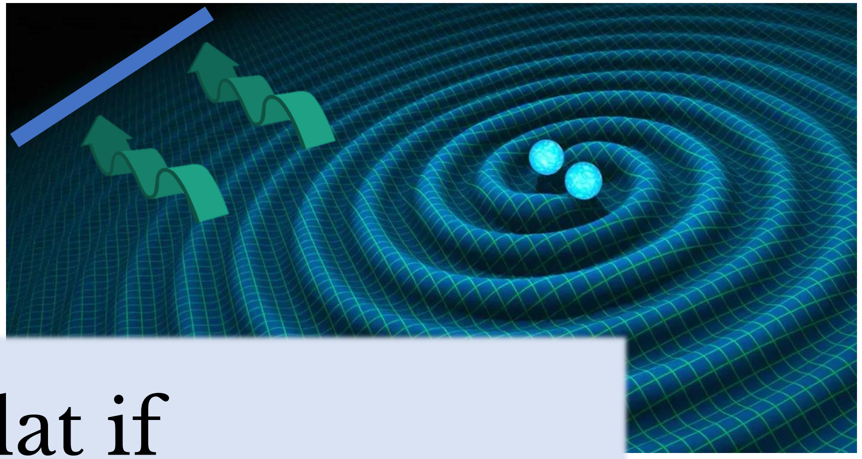


Definition :

Asymptotic flatness at null infinity

[Penrose (1963)]



A spacetime $(\tilde{M}, \tilde{g})$, is asymptotically flat if

- There exists a spacetime with boundary (M, g)
- A “boundary defining” function Ω : $\Omega|_\partial = 0, \quad d\Omega|_\partial \neq 0$
- The interior of *M* is isometric to $\tilde{M}$ with

$$\tilde{g} = \frac{1}{\Omega^2} g$$

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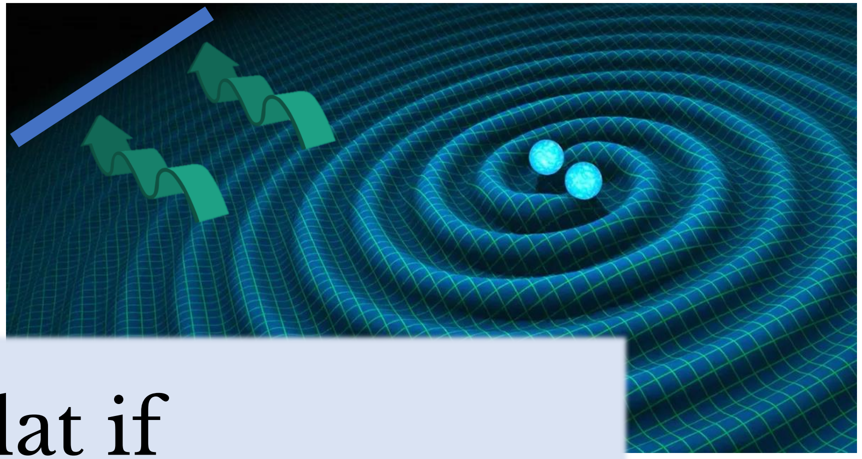

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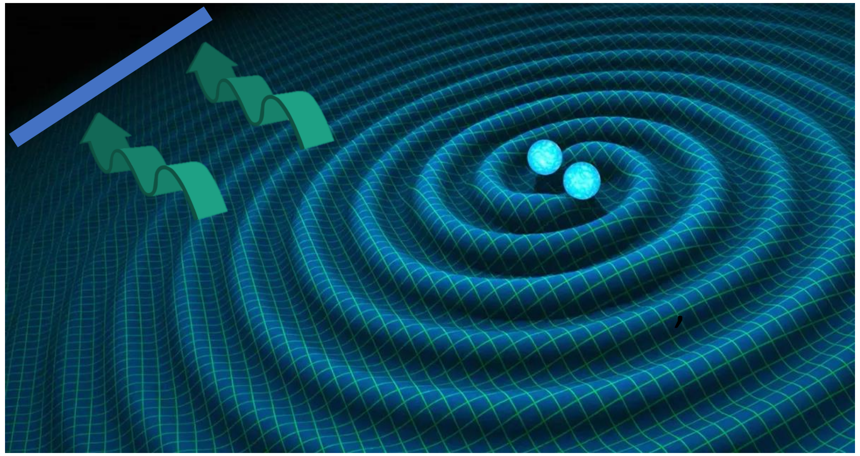
$$\boxed{\tilde{g} = \frac{1}{\Omega^2} g}$$

- $\tilde{g}$ is Einstein
- The normal $n^\mu = \Omega^{-2} \tilde{g}^{\mu\nu} \nabla_\nu \Omega\big|_\partial$ is null $\boxed{n^2 = 0}$

Adapted coordinates :



BMS coordinates

(Bondi -- Van der Burg -- Metzner -- Sachs 62)

One can always choose a coordinate system (u, Ω, x^A) $(A \in \{1,2\})$ such that

$$\tilde{g} = \frac{1}{\Omega^2} \Big[\quad 2dud\Omega + \tilde{h}_{AB}(x)dx^A dx^B + \Omega\Big(\, C_{AB}(u,x)dx^A dx^B \, \Big) + \mathcal{O}(\Omega^2)\Big]$$

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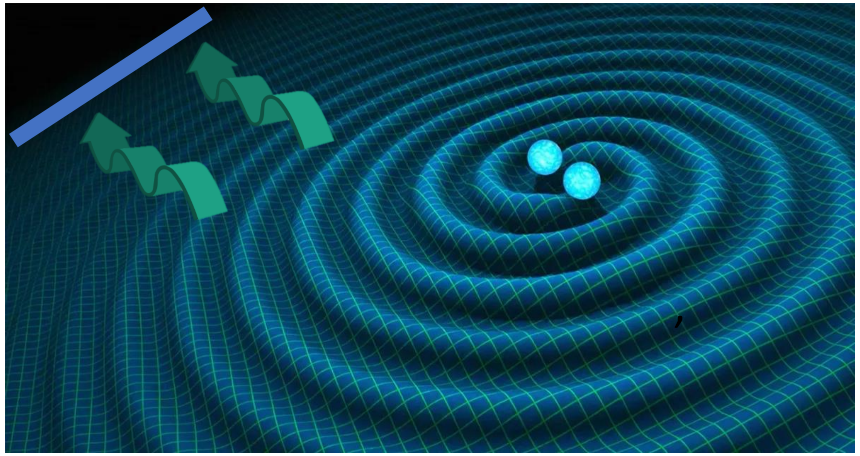

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"Universal" boundary geometry

$$(\tilde{h}_{AB}(x), \tilde{n} = \partial_u)$$

"asymptotic shear", encodes the dynamical part of the geometry

Why we care, again :

Asymptotically flat spacetime (in Penrose sense) are a vast class of objects which ...

- model isolated systems
 (“ to which one can associate energy, momentum, etc ”)

- are associated to an invariant, nonlinear, notion of gravitational waves

- contain a large class of spacetimes (Minkowksi, Schwarschild, Kerr, *Friedrich (1986), Christodoulou—Klainerman (1993), Chrusciel—Delay (2002)*, ...)

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→ Possible starting point for a holographic duality

Arcioni—Dappiaggi (2004),… Bagchi et al (2016),…

More recently "Celestial (or Carroll) holography"

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➡ But is this a symmetry of the S-matrix in the first place? Yes.

Strominger et al (2014)

The BMS group $BMS_4 \simeq SO(3,1) \ltimes C^{\infty}(S^2)$

is the group of asymptotic symmetry of asymptotically flat space-time

$$\tilde{g} = \frac{1}{\Omega^2}\left[\; 2dud\Omega + \tilde{h}_{AB}(x)dx^A dx^B + \quad \Omega\Big(\, C_{AB}(u,x)dx^A dx^B \,\Big) \quad + \mathcal{O}(\Omega^2)\right]$$

$$\xi^{\mu}\partial_{\mu} = \left(\mathcal{T}(z,\bar{z}) + \frac{u}{2}\left(\partial_z \mathcal{Y}^z + \partial_{\bar{z}}\bar{\mathcal{Y}}^{\bar{z}}\right)\right)\partial_u + \mathcal{Y}^z(z,\bar{z})\partial_z + \bar{\mathcal{Y}}^{\bar{z}}(z,\bar{z})\partial_{\bar{z}} \quad + O(\Omega)$$

(Infinitesimal) asymptotic symmetry

generated by $\Big(\mathcal{Y}^z(z,\bar{z}) = \alpha + \beta z + \gamma z^2 \,,\, \mathcal{T}(z,\bar{z})\Big) \in SL(2,\mathbb{C}) \ltimes C^{\infty}(S^2)$

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(Infinitesimal) diffeomorphism along null infinity

(Infinitesimal) asymptotic symmetry

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Important remarks

- The Poincaré group $ISO(3,1) \simeq SO(3,1) \ltimes \mathbb{R}^{3,1}$

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Super-translations: $$\mathcal{T}(z,\bar{z}) = \sum_{l,m}^{l=\infty} \mathcal{T}_{l,m} Y_{l,m}(z,\bar{z})$$

Translations: $$T^\mu \simeq T^0 Y_{0,0}(z,\bar{z}) + \sum_{m=-1}^{m=1} T^m Y_{1,m}(z,\bar{z})$$

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Important remarks

- The Poincaré group $ISO(3,1) \simeq SO(3,1) \ltimes \mathbb{R}^{3,1}$

sits inside BMS: $ISO(3,1) \subset BMS_4$

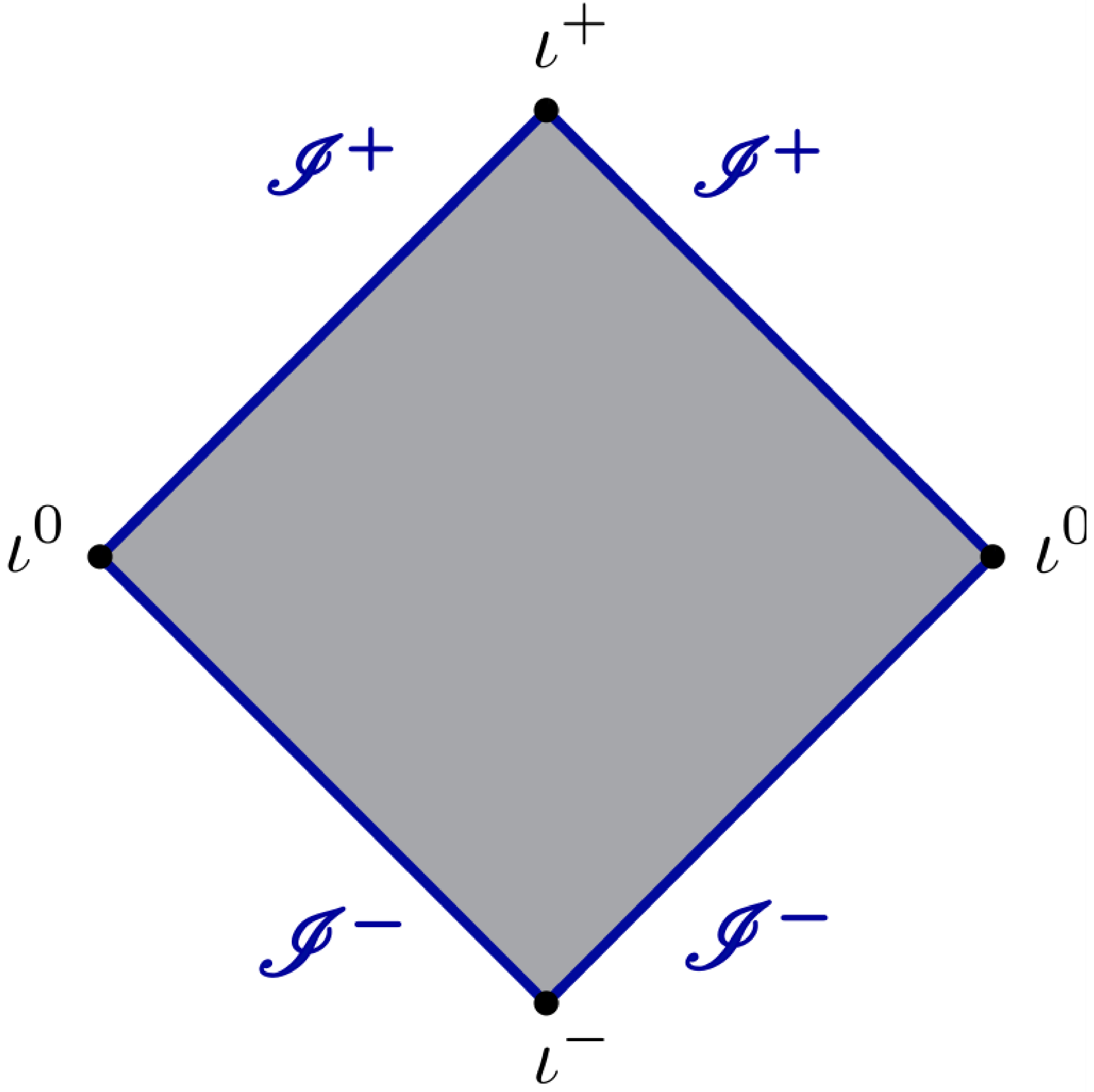
- However the inclusion is **not unique**.

⟹ **Many non equivalent Poincaré groups inside BMS**

3- BMS and the S-matrix: Asymptotic states

$$\langle \mathrm{out}|S|\mathrm{in}\rangle$$

Quantum field theory ?



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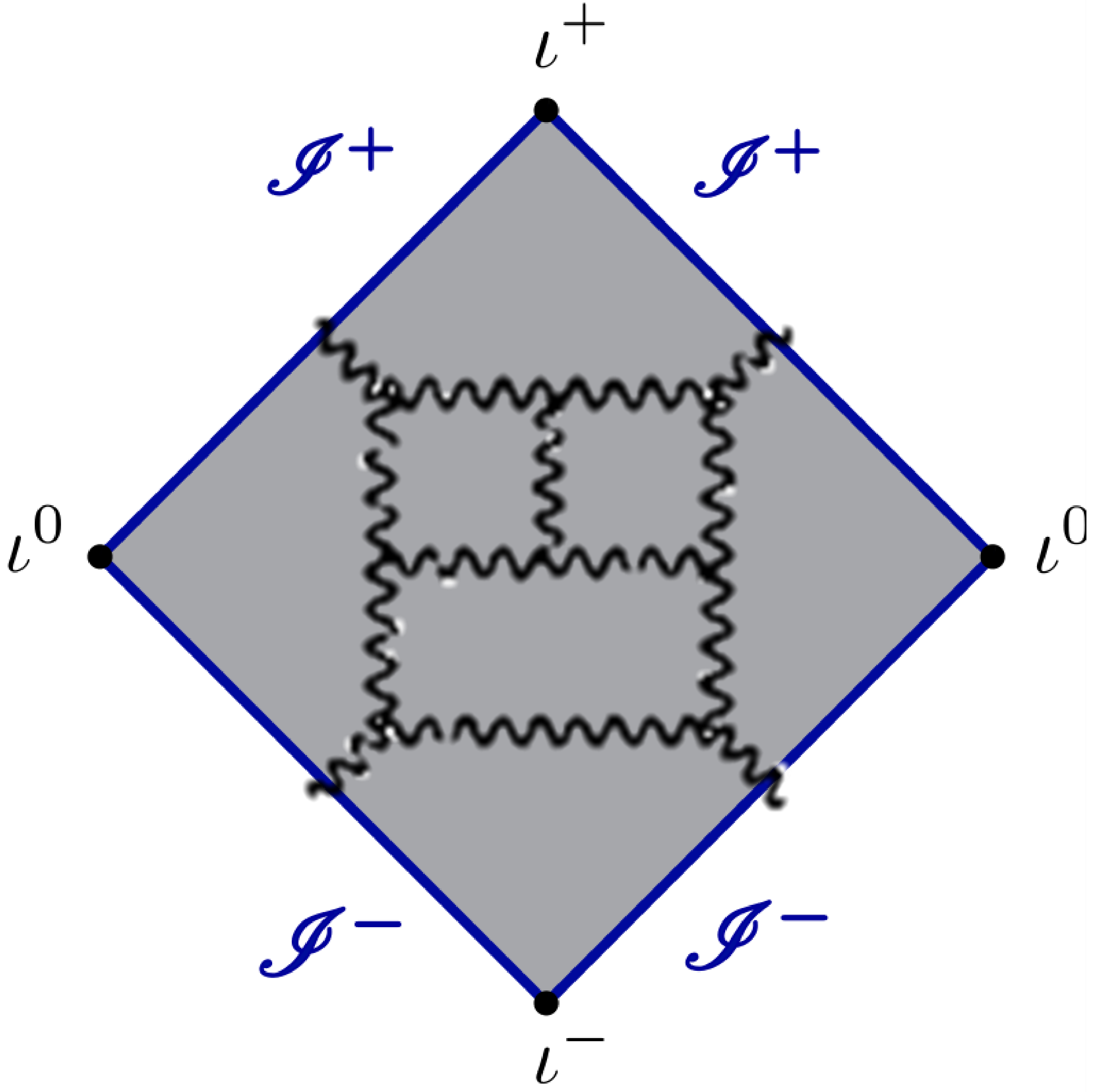

Asymptotically flat spacetimes give a natural geometrical setup to the “interaction picture” of QFT :

Asymptotically free states are “ at infinity ”.

Spin 2 field (of positive helicity):

$$h_{\mu\nu}(x)dx^\mu dx^\nu \quad = \left(\frac{\kappa}{(2\pi)^3} \int \frac{d^3\boldsymbol{p}}{2p^0}\, \epsilon^{(+)}_{\mu\nu}(\boldsymbol{p}) \left(e^{-ix^\mu P_\mu(\boldsymbol{p})}\, a_-(\boldsymbol{p}) + e^{ix^\mu P_\mu(\boldsymbol{p})} a^\dagger_+((\boldsymbol{p})) \right) \right) dx^\mu dx^\nu$$

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Introduce BMS coordinates $\left(r = \Omega^{-1}, u, z, \bar{z}\right)$ on Minkowski space:

$$X^\mu = u\partial_z\partial_{\bar{z}} q^\mu(z, \bar{z}) + r q^\mu(z, \bar{z})$$

$$\begin{aligned} ds^2 = dX^\mu dX^\nu \eta_{\mu\nu} &= -2dudr + 2r^2 dzd\bar{z} \\ &= \frac{1}{\Omega^2}\left(2dud\Omega + 2dzd\bar{z}\right) \end{aligned}$$

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.... and take the limit $r \to \infty \Leftrightarrow \Omega \to 0$

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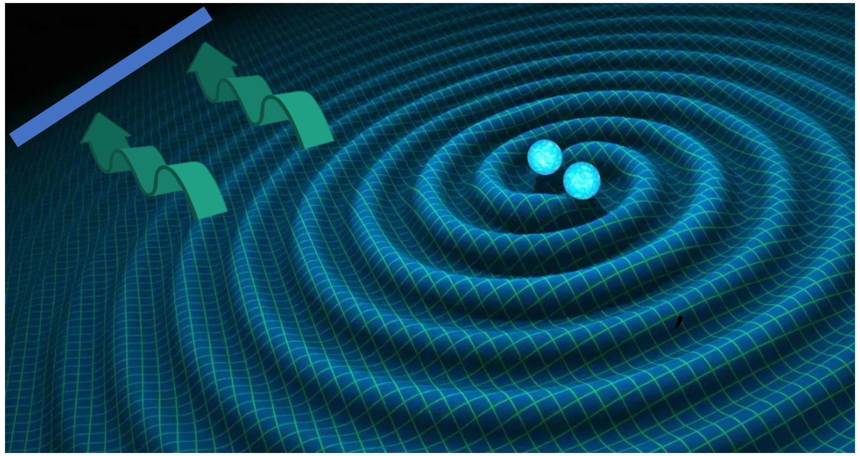
Minkowski space:

$$ds^2 = -2dudr + 2r^2 dz d\bar{z}$$
$$= \frac{1}{\Omega^2}\left(2dud\Omega + 2dzd\bar{z}\right)$$

Linearized perturbation:

$$h = r\left(C_{zz}dz^2 + c.c.\right) + O(r^0)$$
$$= \frac{1}{\Omega^2}\left(\Omega C_{zz}dz^2 + c.c. + O(\Omega^2)\right)$$

Adapted coordinates :



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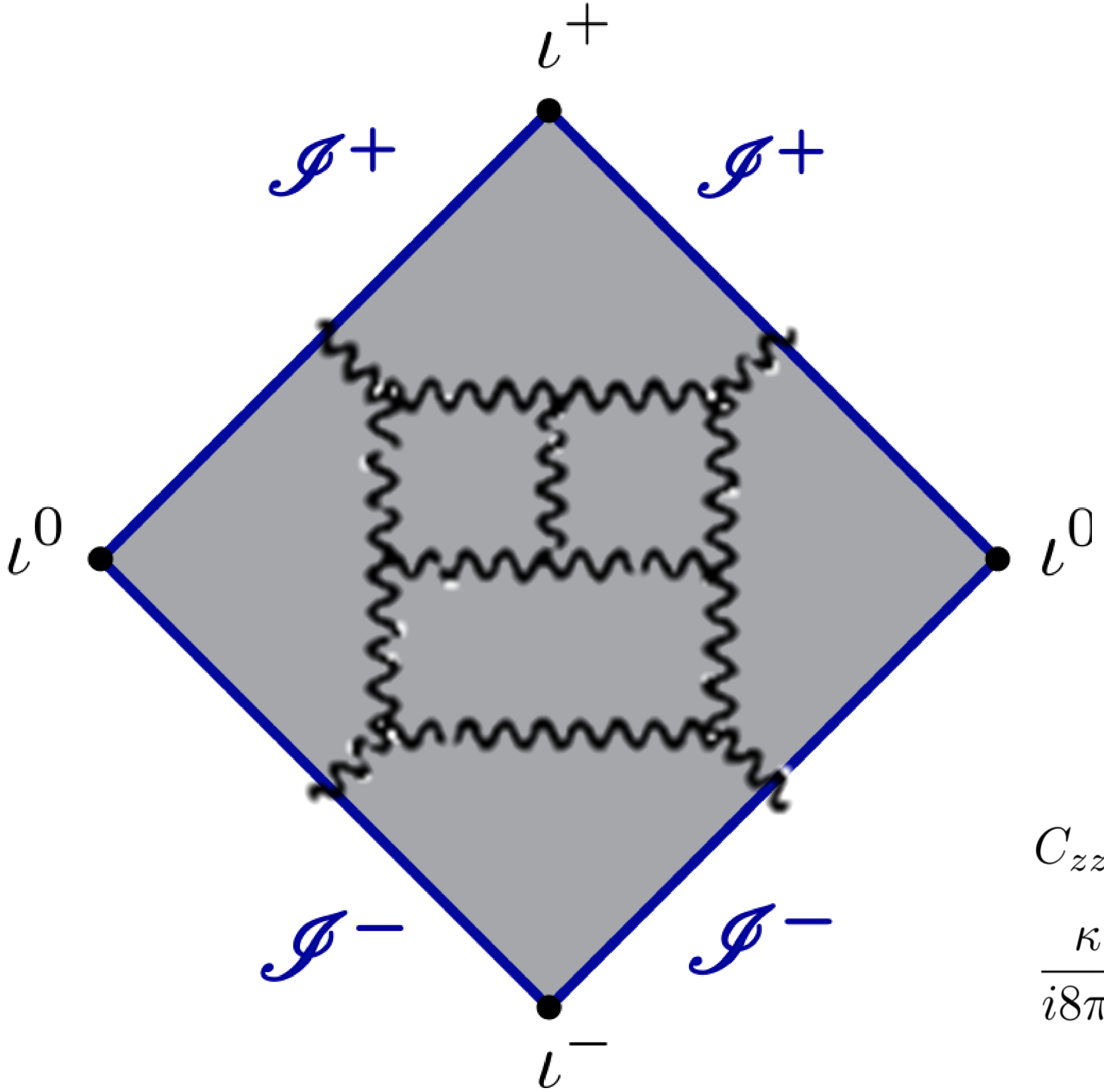

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$$C_{zz}(u,z,\bar{z}) =$$
$$\frac{\kappa}{i8\pi^2}\int_0^\infty \omega d\omega \left(e^{-i\omega u}\, a_+(\omega,\zeta,\bar{\zeta}) - e^{i\omega u} a_-^\dagger(\omega,\zeta,\bar{\zeta})\right)$$

What did we gain ?

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Scattering data of a massless field

... is a **BMS representation** (as a field on $\mathscr{I} = S^2 \times \mathbb{R}$).

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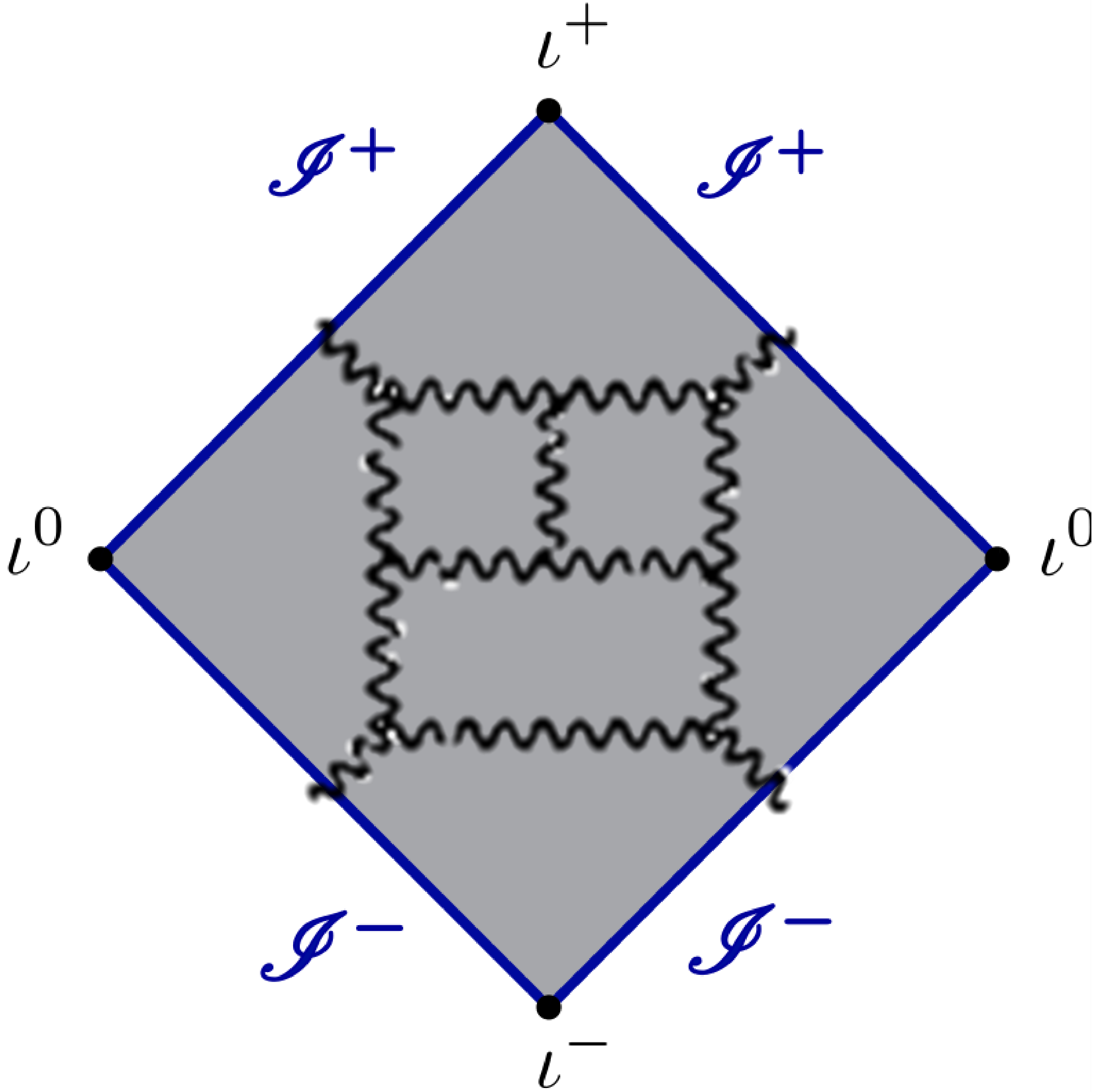
$$\left(\mathcal{T}(z,\bar{z}) \, , \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) \in C^\infty(S^2) \rtimes SL(2,\mathbb{C})$$

$$C_{zz}(u,z,\bar{z}) \underset{BMS_4}{\longmapsto} C_{z'z'}(u',z',\bar{z}')$$

$$u' = u + \mathcal{T}(z,\bar{z})$$

$$z' = \frac{a+bz}{c+dz}$$

Quantum field theory ?



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They form a representation of the BMS group

$$\phi(u, z, \bar{z}) = \frac{\kappa}{i8\pi^2} \int_0^\infty \omega d\omega \left(e^{-i\omega u}\, a(\omega, z, \bar{z}) - e^{i\omega u} a^\dagger(\omega, z, \bar{z})\right)$$

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(field on $\mathscr{I} = S^2 \times \mathbb{R}$)

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Hard Massless BMS (unitary irreducible) representation

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Scattering data of a massless field

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$$a(\omega,z,\bar{z}) \underset{BMS_4}{\longmapsto} e^{-i\omega\mathcal{T}(z,z)} a(\omega',z',\bar{z}')$$

$$u' = u + \mathcal{T}(z,\bar{z})$$

$$z' = \frac{a+bz}{c+dz}$$

Sachs (62)

$$\phi(u, y^\alpha) = \frac{\sqrt{m}}{2(2\pi)^{3/2}} a(y^\alpha) e^{-imu}$$

See M. Chantreau's poster
Borthwick—Chantreau—YH (2024)

Scattering data
of a massive field

(field on $Ti = H^3 \times \mathbb{R}$)

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Hard Massive BMS (unitary irreducible) representation

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$$\left(\mathcal{T}(z,\bar{z}) \,,\, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) \in C^\infty(S^2) \rtimes SL(2,\mathbb{C})$$

$$a(y^\alpha) \quad \underset{BMS_4}{\longmapsto} \quad e^{-im\omega(y)} a((y^\alpha)')$$

$$u' = u + \omega(y)$$

$$\omega(y) := \int dz d\bar{z} \frac{\mathcal{T}(z,\bar{z})}{(q(z,\bar{z}) \cdot \tilde{p}(y))^3}$$

Longhi--Materassi (99)

Hard BMS (unitary irreducible) representation

$$\Big(\mathcal{T}(z,\bar{z})\,,\,M\Big) \in C^\infty(S^2) \rtimes SL(2,\mathbb{C})$$

$$a(p) \quad \underset{BMS_4}{\longmapsto} \quad e^{-i\langle\mathcal{P},\mathcal{T}\rangle} a(p') \qquad (p')^\mu = M^\mu{}_\nu p^\nu$$

$$\langle\mathcal{P},\mathcal{T}\rangle := \int dz d\bar{z}\, \mathcal{P}(z,\bar{z})\mathcal{T}(z,\bar{z})$$

supermomentum

Hard massive

$$\mathcal{P}(z,\bar{z}) := m^4\, (q(z,\bar{z})\cdot p)^{-3}$$

Hard massless $\quad p^\mu = \omega q^\mu(\zeta,\bar{\zeta})$

$$\mathcal{P}(z,\bar{z}) := \omega\delta^{(2)}(z-\zeta)$$

Beyond hard representations?

Bekaert—Donnay—YH (2024)

Decomposition of supermomenta *Bekaert—Donnay—YH (2024)*

$$\mathcal{P}(z,\bar{z}) = P(z,\bar{z}) + \partial_z^2 \partial_{\bar{z}}^2 \mathcal{N}$$

Hard contribution

$\simeq p^\mu$

Hard massive

$P(z,\bar{z}) := (q(z,\bar{z}) \cdot p)^{-3}$

Hard massless

$P(z,\bar{z}) := \omega \delta^{(2)}(z - \zeta)$

Soft charges

$\partial_z^2 \partial_{\bar{z}}^2 \mathcal{N}$

$\simeq$ memory effect

$\simeq$ extra degrees of freedom (in the IR)

4- BMS and the S-matrix: Infrared divergences

$$\langle \mathrm{out}|S|\mathrm{in}\rangle$$

Strominger (2014) **showed the following result :**

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Weinberg's soft theorems *Weinberg (65)* …

… can be understood as *Ward identities* …

… for BMS asymptotic symmetries
Bondi—Van der Burg—Metzner-Sachs (62).

➡ The gravitational S-matrix is BMS invariant !

Sketch of the proof (rewritten to fit with rep. theory language)

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$$0 = \langle \text{out} | \, [\hat{\mathcal{P}}(z,\bar{z}), \hat{S}] \, | \text{in} \rangle$$

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Expression of BMS charges

Barnich--Troessaert (2011)

+ insight from

Strominger (2014)

$$= \int \omega d\omega a^\dagger(\omega, z, \bar{z}) a(\omega, z, \bar{z}) + \partial_{\bar{z}}^2 \left(\lim_{\omega \to 0} \omega \hat{a}(\omega, z, \bar{z}) \right)$$

Sketch of the proof (rewritten to fit with rep. theory language)

$$0 = \langle \mathrm{out}| \, [\hat{P}(z,\bar{z}), \hat{S}] \, |\mathrm{in}\rangle + \lim_{\omega \to 0} \omega \, \partial_{\bar{z}}^2 \, \langle \mathrm{out}| \, [a(\omega, z, \bar{z}), \hat{S}] \, |\mathrm{in}\rangle$$

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$$= \partial_{\bar{z}}^2 \left(\Big(\sum_f \frac{\epsilon \cdot p_i}{q \cdot p_i} - \frac{\epsilon \cdot p_f}{q \cdot f} \Big) \langle \mathrm{out}| \, \hat{S} \, |\mathrm{in}\rangle + \lim_{\omega\to 0} \omega \, \langle \mathrm{out}| \, [a(\omega, z, \bar{z}), \hat{S}] \, |\mathrm{in}\rangle \right)$$

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$$= 0$$ *Weinberg (65)*

Infrared divergences

- It is well known that, in presence of gravitational interactions, all S-matrix elements are infrared divergent at one loop (due to infrared gravitons running in the loop)

➡ "S-matrix coupled with gravity is ill-defined" S-matrix $\neq$ Observables

Infrared divergences

- It is well known that, in presence of gravitational interactions, all S-matrix elements are infrared divergent at one loop (due to infrared gravitons running in the loop)

➡ "S-matrix coupled with gravity is ill-defined" S-matrix $\neq$ Observables

- However, *Weinberg (65)* showed that inclusive cross sections, where infinitely many external soft gravitons are included, are finite.

➡ "Observables are given by inclusive cross sections"

Infrared divergences

- *Weinberg (65)* 's result boils down to soft theorem

$$0 = \partial_{\bar{z}}^2 \left(\left(\sum_f \frac{\epsilon \cdot p_i}{q \cdot p_i} - \frac{\epsilon \cdot p_f}{q \cdot f} \right) \langle \text{out} | \hat{S} | \text{in} \rangle + \lim_{\omega \to 0} \omega \langle \text{out} | [a(\omega, z, \bar{z}), \hat{S}] | \text{in} \rangle \right)$$

- *Strominger et all (2017)* : infrared divergences arise due to non conservation of BMS charges of the usual S-matrix elements

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$$\neq 0$$

sum and difference of hard supermomenta

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sum and difference of hard supermomenta

$$\neq 0$$

Bekaert—Donnay—YH (2025)

Hard supermomenta cannot ensure conservations of supermometa

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sum and difference of hard supermomenta

Bekaert—Donnay—YH (2025)

Hard supermomenta cannot ensure conservations of supermometa

➡ infrared divergences arise due to the fact that
hard UIR cannot be preserved by interactions

➡ any notion of BMS-invariant S-matrix will need to include all other (non hard) BMS UIR representations !

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- any notion of BMS-inv ... to include all other
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How to understand these generic BMS particles ? See Xavier's talk.

Conclusion

- Usual asymptotic states of QFT are BMS representations (Hard UIR)

→ however they cannot by themselves fulfill supermomentum conservations

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Conclusion

- Usual asymptotic states of QFT are BMS representations (Hard UIR)

→ however they cannot by themselves fulfill supermomentum conservations

- Weinberg's soft theorems can be read as saying that

Infrared divergences arise due to non conservation of supermomenta

- A fully BMS invariant extension of QFT has a chance to define infrared finite S-matrix elements !

→ will require a generic notion of BMS particles !

Bekaert—Donnay—YH (2025)

Thank you for your attention !