Poor people's quantum gravity: Gibbsian Hydrodynamics with statistical fluctuations

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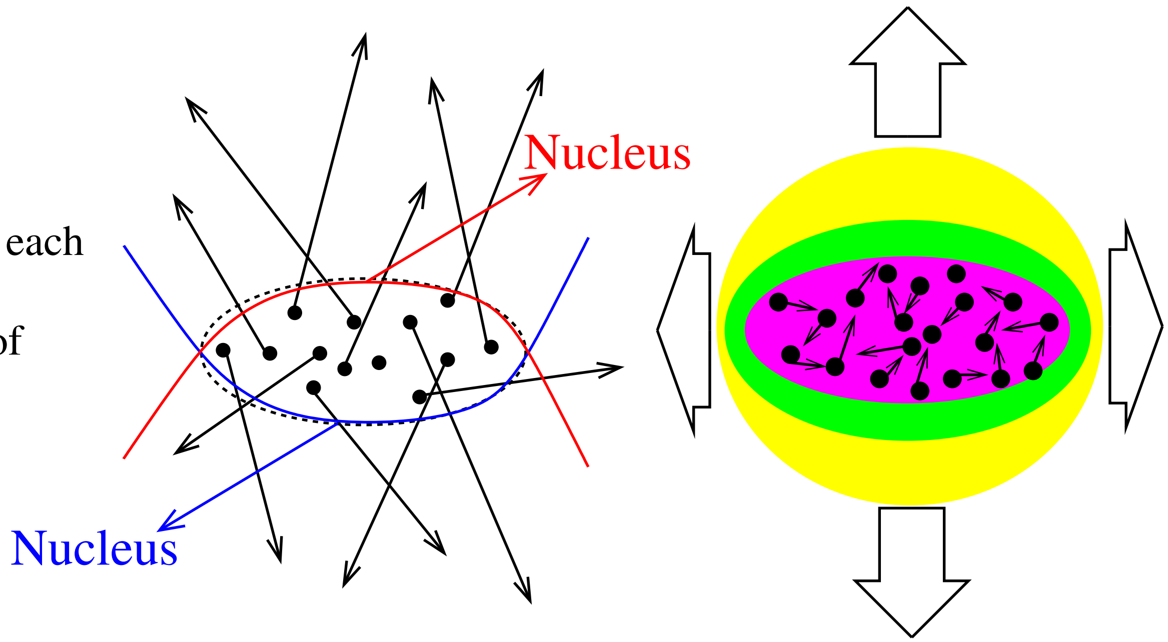
2109.06389 (Annals of Physics, With T.Dore,M.Shokri,L.Gavassino,D.Montenegro)
Answers somewhat speculative... but I think I am asking good questions!

Heavy ion physicists found the perfect liquid! our field largely redefined to this

Why do we believe this?

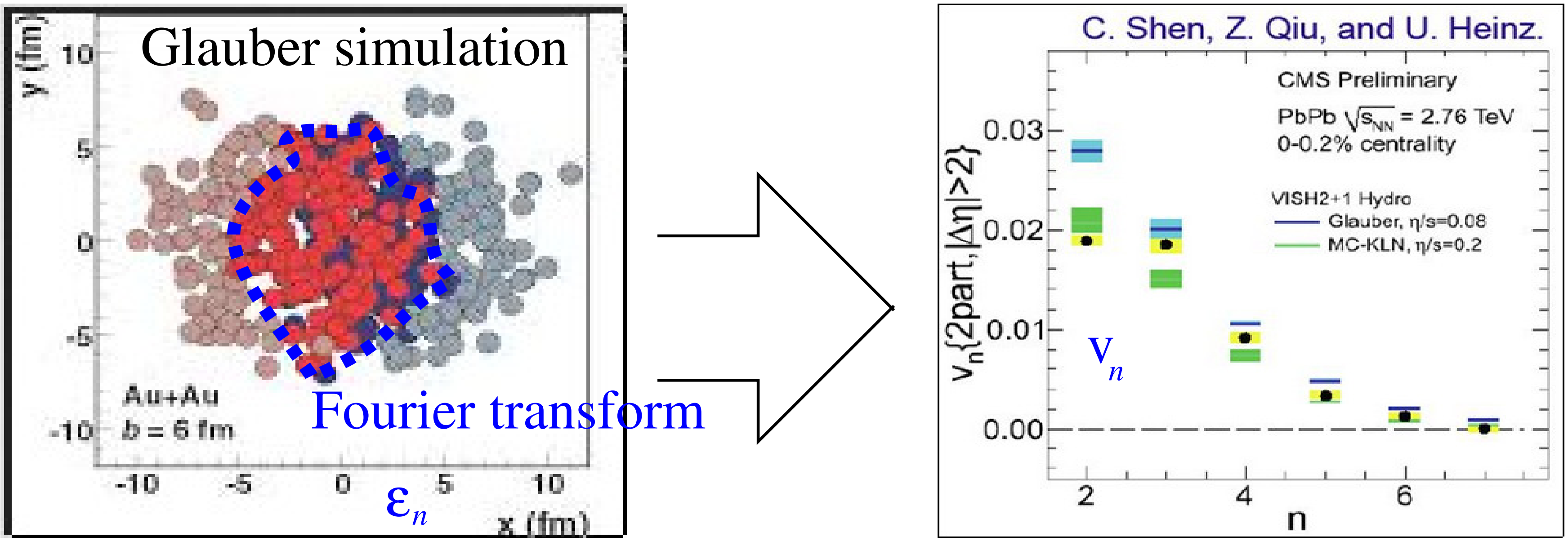
A "dust"
Particles ignore each other, their path is independent of initial shape



A "fluid"
Particles continuously interact. Expansion determined by density gradient (shape)

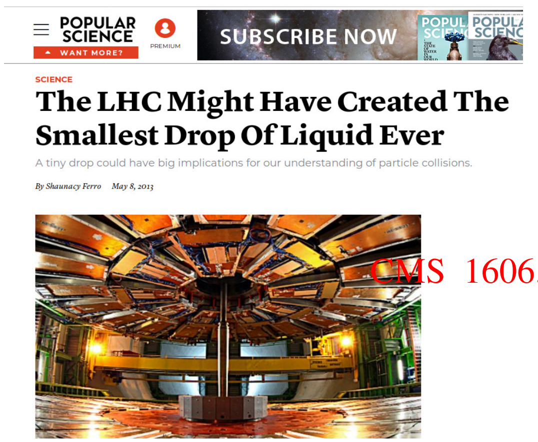
Observable: $\frac{dN}{p_T dp_T dy d\phi} = \frac{dN}{p_T dp_T dy}\left[1 + 2v_n(p_T, y)\cos\left(n\left(\phi - \phi_0\left(n, p_T, y\right)\right)\right)\right]$

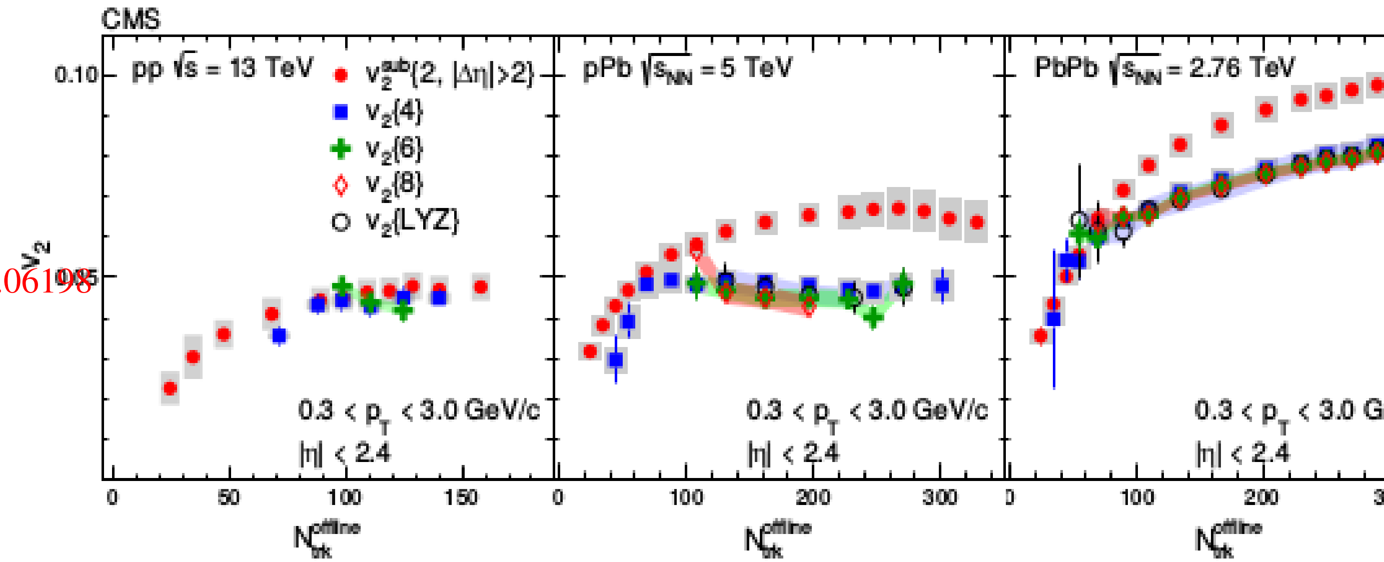
"Collectivity" Same v_n <u>appears</u> in $\forall$ n-particle correlations , $\left\langle \frac{dN}{d\phi_1}\frac{dN}{d\phi_2}...\right\rangle$



Fits ideal hydro , fitted upper limit on viscosity low Spurned a lot of theoretical and numerical/phenomenological development of relativistic hydrodynamics. Restarted the controversy over viscous relativistic hydrodynamics of the 70s

But then LHC switched on and we got a surprise and a conceptual challenge!





1606.06198 (CMS) : When you consider geometry differences and multi-particle cumulants (remove momentum conservation), hydro with $\mathcal{O}(20)$ particles "just as collective" as for 1000.

Hydrodynamics: an "effective theory" of averages $\langle ... \rangle$ using coarse-graining and "fast thermalization" w.r.t. Gradients of coarse-grained variables
If thermalization instantaneus, then isotropy,EoS enough to close evolution

$$\langle T_{\mu\nu} \rangle = (e + P(e)) u_\mu u_\nu + P(e) g_{\mu\nu} \quad , \quad \langle J^\mu \rangle = \rho u^\mu$$

In rest-frame at rest w.r.t. u^μ

$$\langle T_{\mu\nu} \rangle = \mathrm{Diag}\left(e(p,\mu), p, p, p\right) \quad , \quad \langle J_\mu \rangle = \left(\rho(p,\mu), \vec{0}\right)$$

Makes sysem solvable just from conservation laws:

$$\partial_\mu \langle T^{\mu\nu} \rangle = \partial_\mu \langle J^\mu \rangle = 0, p = p(e,\mu), \rho = \rho(e,\mu)$$

A beautiful, rigorous theory with a direct connection to statistical mechanics, i.e. fundamental physics,maths. Exciting that HIC can be described by it!

If thermalization not instantaneus,

$$\langle T_{\mu\nu}\rangle = \langle T_{\mu\nu}(e, u_\mu, \partial u, ..)\rangle \quad , \quad \langle J_\mu\rangle = (\rho, u, \partial\rho, \partial u, ...)$$

Basically one decomposes non-equilibrium part into gradients and relaxes

$$\langle T_{\mu\nu}\rangle = T_{\mu\nu}^{eq} + \Pi_{\mu\nu} \quad , \quad u_\mu \Pi^{\mu\nu} = 0 \quad , \quad \lim_{t\to\infty} \left(\langle T^{`}\mu\nu\rangle - \langle T_{eq}^{\mu\nu}\rangle\right) = 0$$

$$\sum_n \tau_{n\Pi}\partial_\tau^n \Pi_{\mu\nu} = -\Pi_{\mu\nu} + \mathcal{O}\left(\partial u\right) + \mathcal{O}\left((\partial u)^2\right) + ...$$

A series whose "small parameter" $K \sim \frac{l_{micro}}{l_{macro}} \sim \frac{\eta}{sT}\nabla u$ and the transport coefficients calculable from asymptotic correlators of microscopic theory Navier-Stokes $\sim K$, Israel-Stewart $\sim K^2$ etc.

Non-relativistic version still considered beautiful and profound, but with relativity...

What's wrong with this?

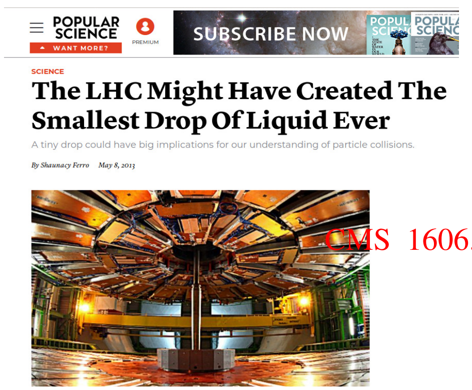
u_μ **ambiguus** many definitions (Landau, Eckart,BDNK...)
We think flow is "clear", so this is a bit strange . choices supposed to be field redefinitions but give slightly different dynamics

$\Pi_{\mu\nu}$ **ambiguus** can even be eliminated as a DOF ($\sim \partial u$ by carefully choosing u_μ (BDNK)

Entropy is ambiguus it's definition depends on the definitions above. Yet from statistical mechanics , as long as microstates are local, it should not be ambiguus!

Fluctuations... $\left\langle (\Delta T_{\mu\nu})^2 \right\rangle$ Is not the same as $\langle T_{\mu\nu}\rangle - \langle T_{\mu\nu}\rangle_{eq}$

- One can define linearly, whith a Langevin-like fluctuation-dissipation relation but contradicts experiment!



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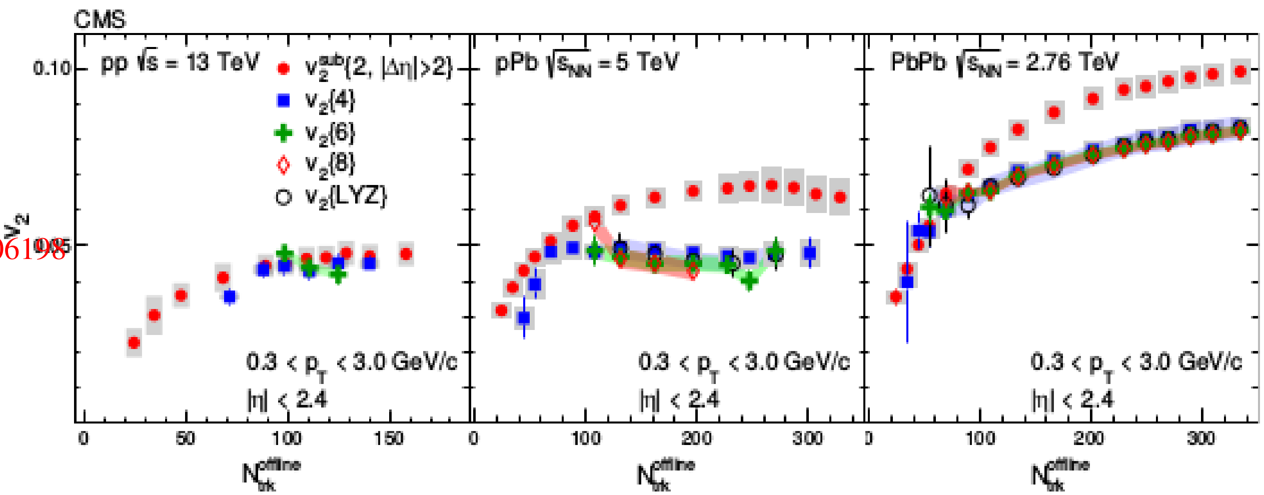


- Exact theory strongly depends on u_μ convention! Also on pseudogauge! but if field redefinition, does "everything" fluctuate? What if fluctuation of $u_\mu, T, \Pi_{\mu\nu}$ leave $T_{\mu\nu}$ invariant?

More concretely

A theorist (Romatschke,Kovtun,...) will say that fluctuations of e.g. $\delta\Pi_{\mu\nu}, \delta f(x,p)$ produce "non-hydrodynamic modes", sensitive to underlying theries, and hydrodynamics is easy to break down to a non-universal dynamics.

An experimentalist measures neither $\Pi_{\mu\nu}$ nor f but rather, e.g.

$$\frac{dN}{dyp_T dp_T d\phi} \equiv \frac{dN}{dyp_T dp_T} \left[1 + 2 v_n(p_T, y) \cos\left(n\left(\phi - \phi_{0n}\right)\right)\right]$$

i.e. gradients of $T_{\mu\nu}$,entropy : $v_n \equiv \langle cos\left(n\left(\phi - \phi_0\right)\right)\rangle$
Most theorists treat it as an average, but This is a cumulant of $\mathcal{O}\left(\infty\right)$ so sensitive to non-hydrodynamic modes. Yet experiment finds hydro everywhere they look! Can your non-hydro mode be my fluctuating sound-wave? Can we tell,in principle?

Hydrodynamics from microscopic theories

QFT transport coefficients plagued by divergences, need truncation (Schwinger-Keldysh separates "fast", "slow", Kadanoff-Baym needs truncation)

Boltzmann equation Sequential scattering and molecular chaos. Weak coupling, Lose microscopic correlations

AdS/CFT strong coupling and large N_c, lose microscopic correlations

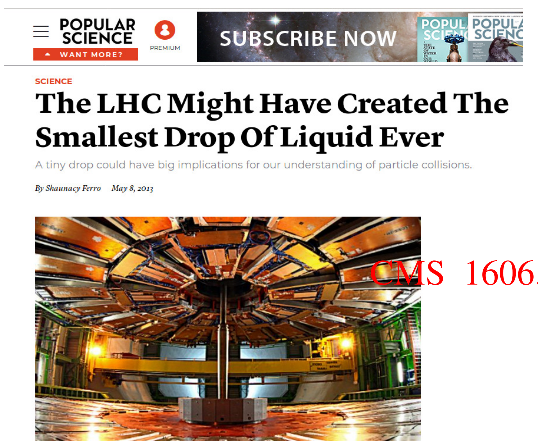
Molecular dynamics keeps microscopic correlations, lose Lorentz invariance (in practice not a problem)

Basic problem with either Lorentz invariance or correlations on scale of gradients! Ambiguity in flow, $\Pi_{\mu\nu}$ comes from here!

In brief most microscopic approaches to EFT hydrodynamics assume that

$$\underbrace{l_{micro}}_{\sim s^{-1/3},\, n^{-1/3}} \ll \underbrace{l_{mfp}}_{\sim \eta/(sT)} \ll L_{macro}$$

But this seems falsified by hydrodynamics in small systems!



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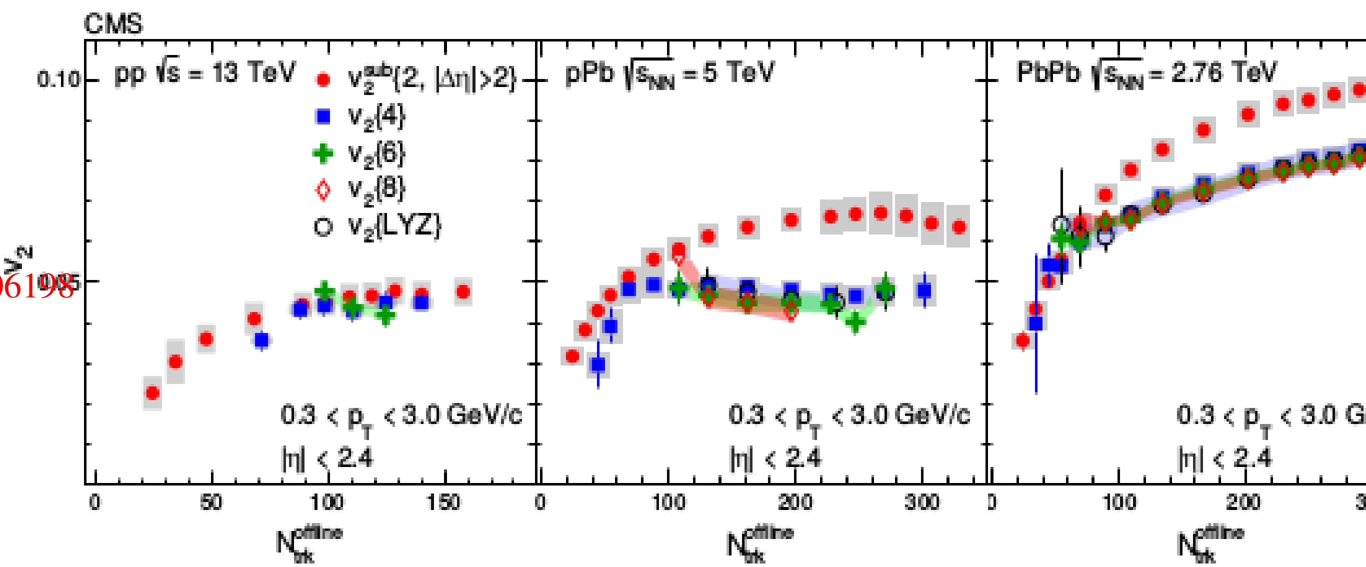
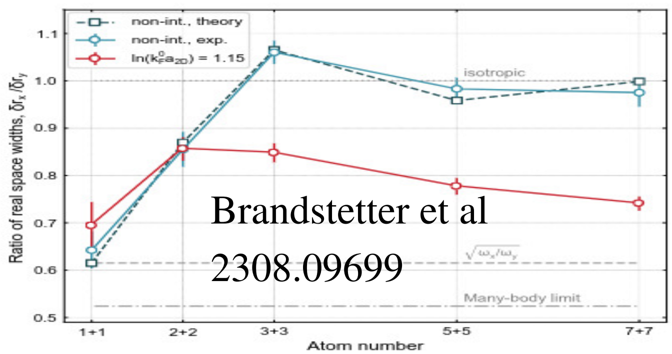


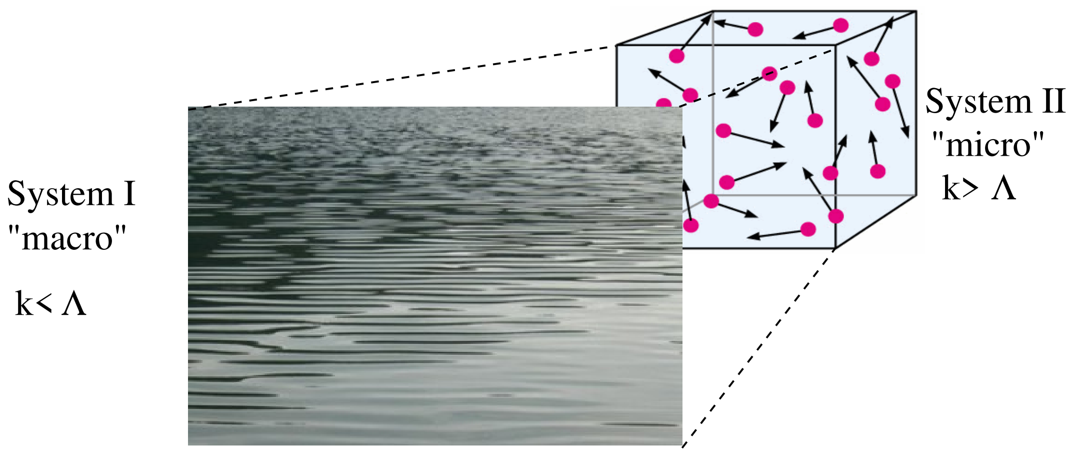
Not just in heavy ions

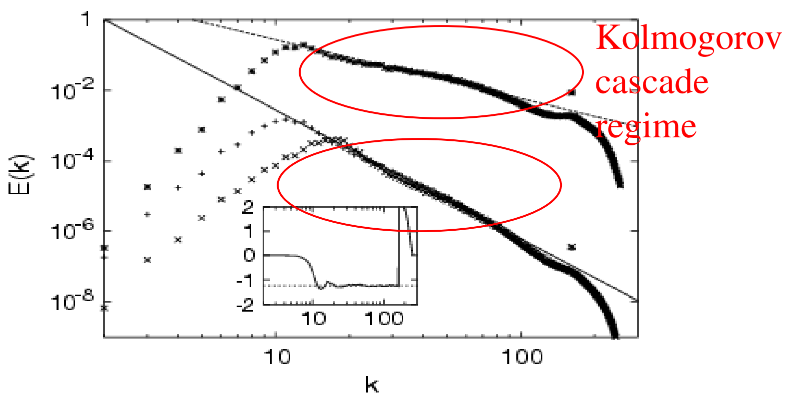


The Brazil nut effect

Empirically, strongly coupled systems with enough thermal energy seem to be "fluid" even with a small number of DoFs. EFT does not explain this! The role of fluctuations in hydrodynamics, and of the exast relation of statistical physics and hydrodynamics, are still ambiguous and this is related to experimental puzzles

A final issue: Entropy current not clearly connected to energy-momentum current, need microscopic theory to "select good EFT" (2nd law)





At best related to stability (sound waves don't explode) and causality (sound waves $dw/dk \leq c$)

Hydrodynamics and statistical mechanics

Equation of state $p(E)$ comes from basic statistical mechanics

$$p = T \ln \mathcal{Z} \quad , \quad \frac{dP}{dT} = \frac{dS}{dV} = \frac{p + e - \mu n}{T}$$

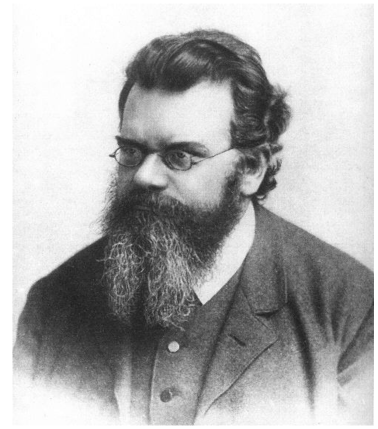
But the same partition function also predicts fluctuations

$$\langle (\Delta E)^2 \rangle = \frac{\partial \ln \mathcal{Z}}{\partial \beta^2} \sim \frac{1}{(\Delta V) \times s}$$

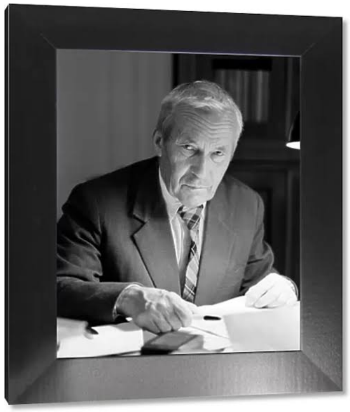
which in a deterministic theory are completely neglected. could this have something to do with the above ambiguity?

the battle

of the entropies



Boltzmann entropy (associated with frequentist probability) a property of the "DoF", and is "kinetic" subject to the H-theorem which is really a consequence of the not-so-justified molecular chaos assumption. Gibbsian entropy (more Bayesian) is the log of the area of phase space, and is justified from coarse-graining and ergodicity . The two are different even in equilibrium, with interactions! (Khinchin,stat.mech.) Note, Von Neumann $\langle ln\hat{\rho}\rangle$ Gibbsian . Gibbs is more general, but...

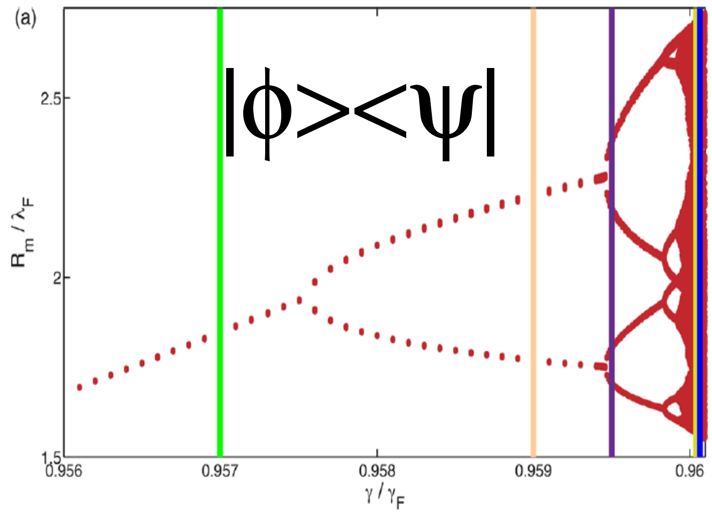


the unreasonable effectiveness of stat mech

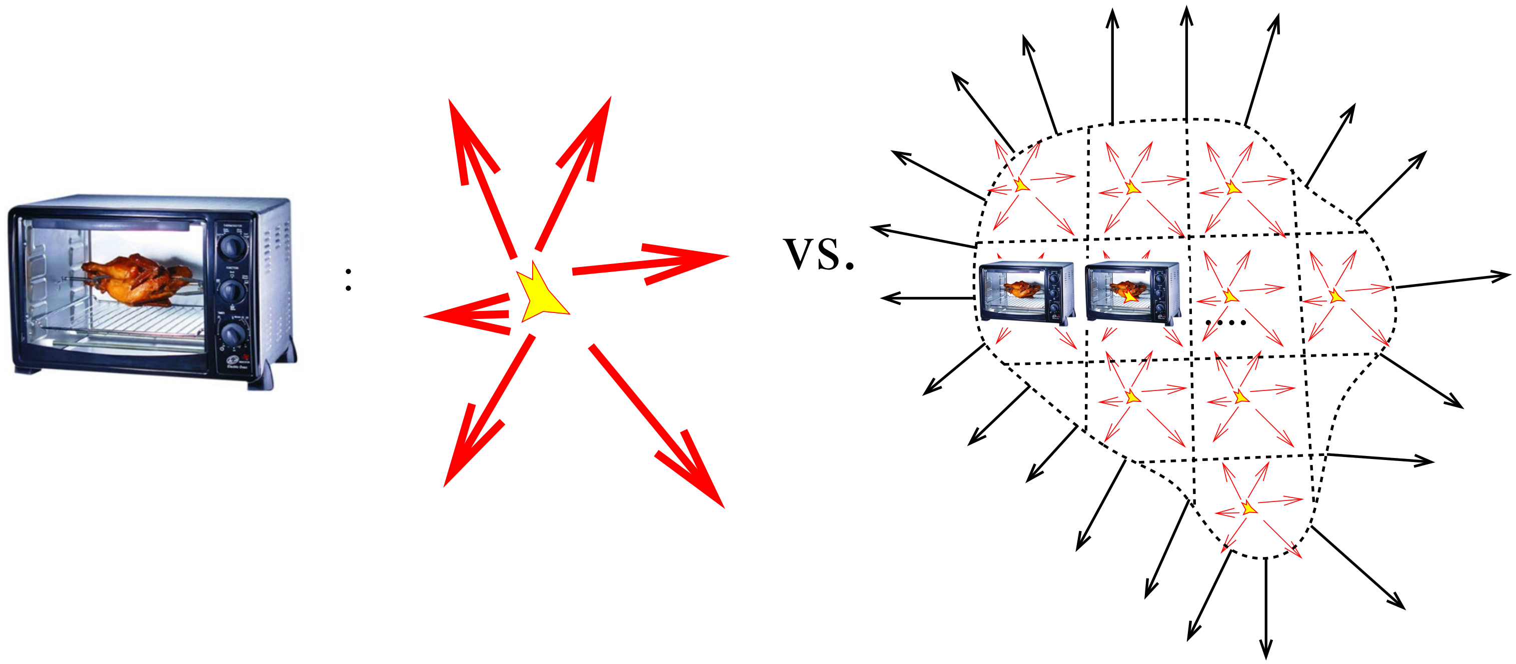
Non-ideal hydrodynamics is based around approximate local equilibrium . Boltzmannian global and local equilibrium are defined, but they depend on Boltzmannian physics Only Global equilibrium well defined in Gibbs (what is "approximate maxiumum" Gibbsian entropy?)

Khinchin's "failed" PhD: Stat Mech just seems wrong but seems to apply everywhere! Just like hydro?

QM to rescue? Berry/Bohigas/Eigenstate thermalization



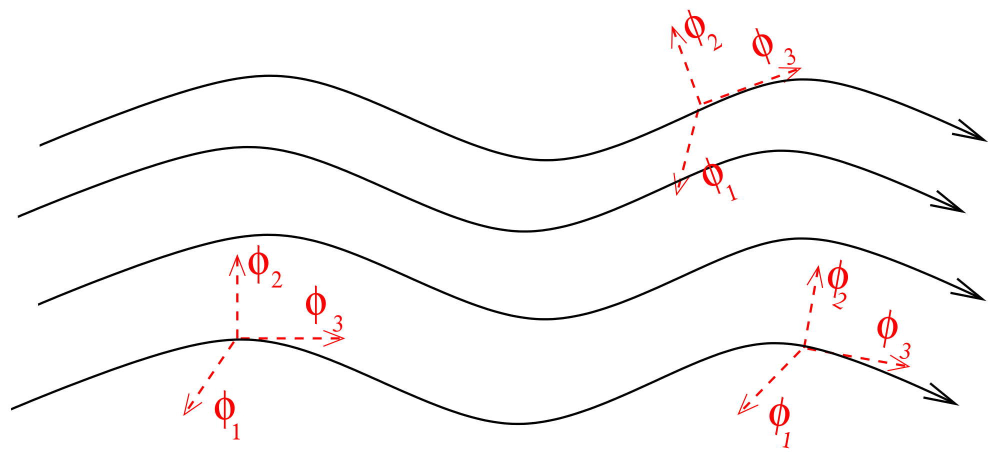
$\underline{E_{n>>1}}$ of quantum systems whose classical correspondent is chaotic have density matrices that look like pseudo-random. If off-diagonal elements oscillate fast or observables simple, indistinguishable from MCE!



But need to coarse-grain, impose causality, and build hydro-like EFT out of this. could be very different from usual EFT expansion!

Let's look at this ambiguity a bit deeper: Lagrangian and Eulerian hydrodynamics Hydro as fields: (Nicolis et al,1011.6396 (JHEP))

Continuus mechanics (fluids, solids, jellies,...) is written in terms of 3-coordinates $\phi_I(x^\mu), I = 1...3$ of the position of a fluid cell originally at $\phi_I(t=0, x^i), I = 1...3$. (Lagrangian hydro . NB: no conserved charges)



The system is a Fluid if it's Lagrangian obeys some symmetries (Ideal hydrodynamics $\leftrightarrow$ Isotropy in comoving frame) Excitations (Sound waves, vortices etc) can be thought of as "Goldstone bosons"

Translation invariance at Lagrangian level $\leftrightarrow$ Lagrangian can only be a function of $B^{IJ} = \partial_\mu \phi^I \partial^\mu \phi^J$ Now we have a "continuus material"!

Homogeneity/Isotropy means the Lagrangian can only be a function of $B = \det B^{IJ}, \mathrm{diag} B^{IJ}$
The comoving fluid cell must not see a "preferred" direction $\Leftarrow SO(3)$ invariance

Invariance under Volume-preserving diffeomorphisms means the Lagrangian can only be a function of B
In all fluids a cell can be infinitesimally deformed
(with this, we have a fluid. If this last requirement is not met, Nicolis et all call this a "Jelly")

A few exercises for the bored public Check that L = -F(B) leads to

$$T_{\mu\nu} = (P + \rho)u_\mu u_\nu - P g_{\mu\nu}$$

provided that

$$\rho = F(B) \; , \qquad p = F(B) - 2F'(B)B \; , \qquad u^\mu = \frac{1}{6\sqrt{B}} \epsilon^{\mu\alpha\beta\gamma} \epsilon_{IJK} \, \partial_\alpha \phi^I \partial_\beta \phi^J \partial_\gamma \phi^K$$

Equation of state chosen by specifying $F(B)$. "Ideal": $\Leftrightarrow F(B) = B^{4/3}$
$\sqrt{B}$ is identified with the entropy and $\sqrt{B}\frac{dF(B)}{dB}$ with the microscopic temperature. u^μ fixed by $u^\mu \partial_\mu \phi^{\forall I} = 0$

Conserved charges (Dubovsky et al, 1107.0731(PRD))

Within Lagrangian field theory a scalar chemical potential is added by adding a $U(1)$ symmetry to system.

$$\phi_I \to \phi_I e^{i\alpha} \quad , \quad L(\phi_I, \alpha) = L(\phi_I, \alpha + y) \quad , \quad J^\mu = \frac{dL}{d\partial_\mu \alpha}$$

generally flow of b and of J not in same direction. Can impose a well-defined u^μ by adding chemical shift symmetry

$$L(\phi_I, \alpha) = L(\phi_I, \alpha + y(\phi_I)) \to L = L\left(b, y = u_\mu \partial^\mu \alpha\right)$$

A comparison with the usual thermodynamics gives us

$$\mu = y \quad , \quad n = dF/dy$$

obviously can generalize to more complicated groups

This looks a bit like GR and this is not a coincidence!

4D local Lorentz invariance becomes local SO(3) invariance

Vierbein $g_{\mu\nu} = \eta^{\alpha\beta} e_\mu^\alpha e_\nu^\beta$ is $\frac{\partial x_I^{comoving}}{\partial x_\mu} = \partial_\mu \phi_I$ (with Gauge phase for μ)

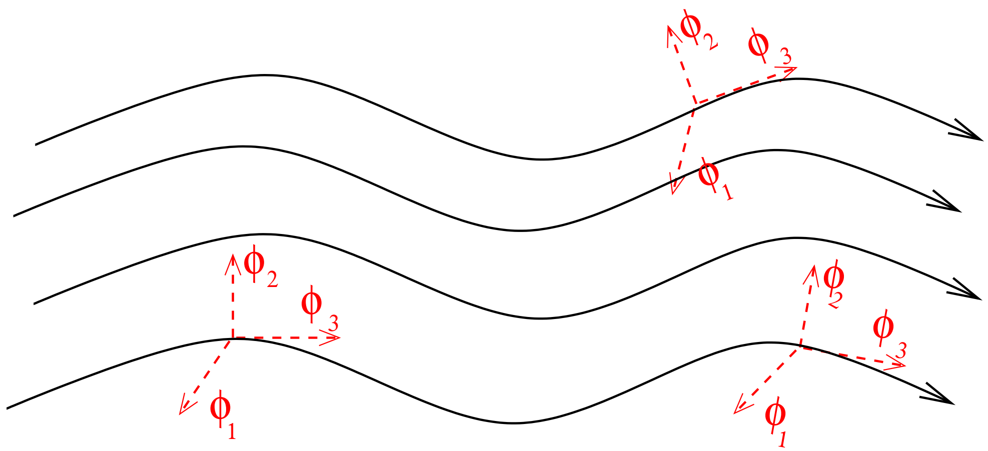
Entropy $\sim \sqrt{b}$, diffeomorphism invariant

Killing vector becomes u_μ

$\mathcal{L} \sim \sqrt{-g}\,(\Lambda + R + ...)$ becomes $\mathcal{L} \sim F(B) \equiv f(\sqrt{-g})$ Just cosmological constant, expanding fluid $\equiv$ dS space

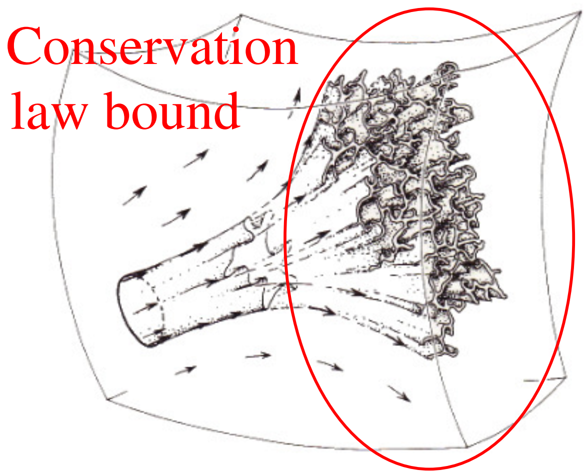
Very nice... but the ambiguities beyond ideal hydro generally break this .
Who cares? Should beyond idel hydrodynamics have this general covariance?

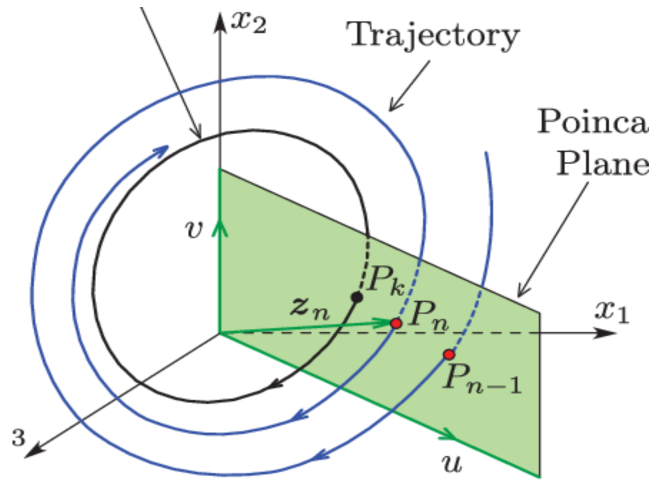
The poor people's quantum gravity: How can fluctuations and dissipation keep hydrodynamic's diffeomorphism invariance? Perhaps has a role to answer how come fluctuation/dissipation experimentally breaks down and fluids exist for 20 particles!



First step: Lagrangian hydrodynamics very elegant, but where is the connection to local thermalization? Statistical mechanics? Transport? Hint from D.T.Son: it is the largest group of diffeomorphisms where time plays no role!

Where does statistical mechanics come from? Ergodicity





Classical evolution via Hamilton's equations

$$\dot{x} = \frac{\partial H}{\partial p} \quad , \quad \dot{p} = -\frac{\partial H}{\partial x} \quad , \quad \dot{O} = \{O, H\}$$

"Chaos",conservation laws$\rightarrow$ phase space more "fractal", recurring

"After some time", for any observable ergodic limit applies

$$\int_0^{(large)\ T} \dot{O}(p,q)dt = \int P(O(p,q))dqdp$$

where $P(...)$ probability independent of time. This probability can only be given by conservation laws

$$P(O) = \frac{\left(\sum_i O_i\right)\delta^4\left(\sum_i P_i^\mu - P^\mu\right)\delta\left(\sum_i Q_i - Q\right)}{N} \quad , \quad N = \int P(O)dO = 1$$

this is the microcanonicanal ensemble. In thermodynamic limit

$$P(O) \to \delta(O - \langle O \rangle)$$

Hydrodynamics is “thermodynamics in every cell

$$\int_0^{(large)\ \ T} \dot{O}(p,q)dt \to \frac{\Delta\phi}{\Delta t}$$

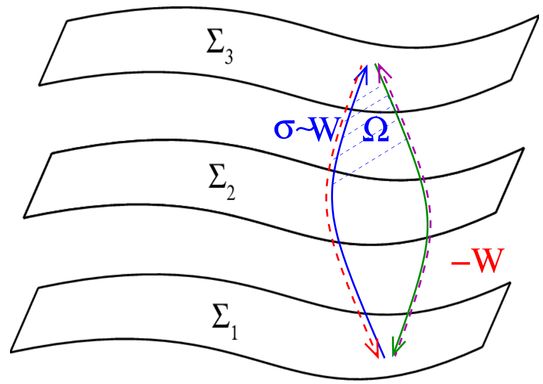
where ϕ is some local observable.

$$\left.\frac{\Delta\phi}{\Delta t}\right|_{t-t'=\Delta} \simeq \frac{1}{d\Omega(Q,E)}\times$$

$$\times \sum \delta^4_{P^\mu, P^\mu_{macro}(t)} \delta_{Q, Q_{macro}(t)} \delta\left(\sum_j^\infty p_j^\mu - P^\mu\right)\delta\left(\sum_j^\infty Q_j - Q\right)$$

Problem: This is not relativistically covariant!

Solution: Foliation!



$$t \to \Sigma_0 \quad , \quad x_\mu \to \Sigma_\mu \quad , \quad \Delta \to \text{"}smooth\text{"} \; \frac{\partial \Sigma_\mu}{\partial \Sigma_\nu}$$

Smooth: $R_{curvature}$ of metric change smaller than "cell size" (New l_{mfp})

$$\frac{\Delta \phi}{\Delta \Sigma_0} = \int P(\phi, \Sigma_\mu) d\Sigma_i \quad , \quad \Sigma_\mu \to \Sigma'_\mu \quad , \quad \frac{\Delta \phi}{\Delta \Sigma'_0} = \frac{\Delta \phi}{\Delta \Sigma_0}$$

What kind of effective lagrangian would enforce

$$\frac{\Delta\phi}{\Delta\Sigma_0} = \int P(\phi,\Sigma_\mu) d\Sigma_i \quad , \quad \frac{\Delta\phi}{\Delta\Sigma_0'} = \frac{\Delta\phi}{\Delta\Sigma_0}$$

with

$$P(...) \sim \delta(\sum_i P_i^\mu - P)\delta(\sum_i Q_i - Q)$$

Now Remember Noether's theorem!

$$p_\mu = \int d^3\Sigma^\nu T_{\mu\nu} \quad , \quad T_{\mu\nu} = \frac{\partial L}{\partial\partial^\mu\phi}\Delta_\nu\phi - g_{\mu\nu}L \quad , \quad \Delta_\nu\phi(x_\mu) = \phi(x_\mu + dx_\nu)$$

$$Q = \int d^3\Sigma^\nu j_\nu \quad , \quad j_\nu = \frac{\partial L}{\partial\partial^\mu\phi}\Delta_\psi\phi \quad , \quad \Delta_\psi\phi = |\phi(x)| e^{i(\psi(x)+\delta\psi(x))}$$

momentum generates spatial translations, conserved charges generate complex rotations!

Space-like foliations decompose

$$d\Sigma_\mu = \epsilon_{\mu\nu\alpha\beta}\frac{\partial\Sigma^\nu}{\partial\Phi_1}\frac{\partial\Sigma^\alpha}{\partial\Phi_2}\frac{\partial\Sigma^\beta}{\partial\Phi_3}d\Phi_1 d\Phi_2 d\Phi_3$$

where the determinant (needed for integrating out $\delta - functions$ is only in the volume part

$$\frac{\partial\Sigma'_\mu}{\partial\Sigma_\nu} = \Lambda_\mu^\nu \det\frac{d\Phi'_I}{d\Phi_J} \quad , \quad \det\Lambda_\mu^\nu = 1$$

Physically, Λ_μ^ν moves between the frame $d\Sigma_{rest}^\mu = d\Phi_1 d\Phi_2 d\Phi_3(1,\vec{0})$

so lets try

$$\underbrace{L(\phi)}_{microscopic\ \ DoFs} \simeq L_{eff}(\Phi_{1,2,3})$$

with

$$\frac{\Delta\phi}{\Delta\Sigma_0} = \int P(\phi, \Sigma_\mu) d\Sigma_i \quad , \quad P(...) = \delta(...)\delta(...)$$

the general covariance requirement of $\frac{\Delta\phi}{\Delta\Sigma_0} = \frac{\Delta\phi}{\Delta\Sigma_0'}$ means the invariance of the RHS

$$\frac{d\Omega(dP'_\mu, dQ', \Sigma_0')}{d\Omega(dP_\mu, dQ, \Sigma_0)} =$$

$$= \frac{d\Sigma_0'}{d\Sigma_0} \frac{\int da_\mu d\psi \delta^4 \left(d\Sigma^\nu a_\alpha \partial^\alpha \left(\delta^\mu_\nu L\right) - dP^\mu(\Sigma_0)\right) \delta \left(d\Sigma^\mu \psi \partial_\mu L - dQ(\Sigma_0)\right)}{\int da'_\mu d\psi' \delta^4 \left(d\Sigma'_\nu a'_\alpha \partial^\alpha \left(\delta^\mu_\nu L\right) - dP'_\mu(\Sigma_0')\right) \delta \left(d\Sigma'_\mu \psi' \partial^\mu L - dQ'(\Sigma_0')\right)}$$

It is then easy to see,via

$$\delta((f(x_i))) = \sum_i \frac{\delta(x_i - a_i)}{\underbrace{f'(x_i = a_i)}_{f(a_i)=0}} \quad , \quad \phi'_I = \frac{\partial_\alpha \Sigma'_I}{\partial^\alpha \Sigma^J} \Phi_J \quad , \quad \delta^4(\Sigma_\mu) = \det \left| \frac{\partial \Sigma^\mu}{\partial \Sigma^\nu} \right| \delta^4$$

that for general covariance to hold

$$L(\Phi_I, \psi) = L(\Phi'_I, \psi') \quad , \quad \det \frac{\partial \phi_I}{\partial \phi_J} = 1 \quad , \quad \psi' = \psi + f(\phi_I)$$

the symmetries of perfect fluid dynamics are equivalent to requiring the ergodic hypothesys to hold for generally covariant causal spacetime foliations!!!! Quantum: $\Delta t_{micro-sampling} \to \rho_{ij} e^{i \Delta t E_{ij}}$ and proof similar!

The crucial question: Does this extend to non-ideal hydrodynamics?

Generating functionals , not constitutive relations Every cell corresponds to a partition function , not a conserved current Near-maximum entropy related to this,and diffeo-invariant! Covariant, metric $g_{\mu\nu} \leftrightarrow \partial\Sigma_\mu / \partial\Sigma^\nu$

Close to local equilibrium is not on gradient expansion but the approximate applicability of fluctuation-dissipation (not the same!) Refoliations in $\Sigma_\mu \to$ Changes in $g_{\mu\nu} \leftrightarrow$ reshuffling in interpretation

NB: Global equilibrium , defined as $\mathrm{Max}\left[\langle \ln \hat{\rho} \rangle\right]_{\beta_\mu,\mu,...}$ ill defined if $\nabla \delta_\mu \simeq 1/R, 1/T$ since hydrodynamic turbulence, statistical fluctuations talk ("unstable" equilibrium is not in equilibrium!). local equilibrium well-defined!, solid basis of an EFT. This ambiguity is due to entropy in Global equilibrium being Boltzmannian ("micro" Dofs) and not Gibbsian (covariantly "coarse-grained" Dofs,fluctuation-generated soundwaves,...)

In summary,what we need is a hydrodynamics...

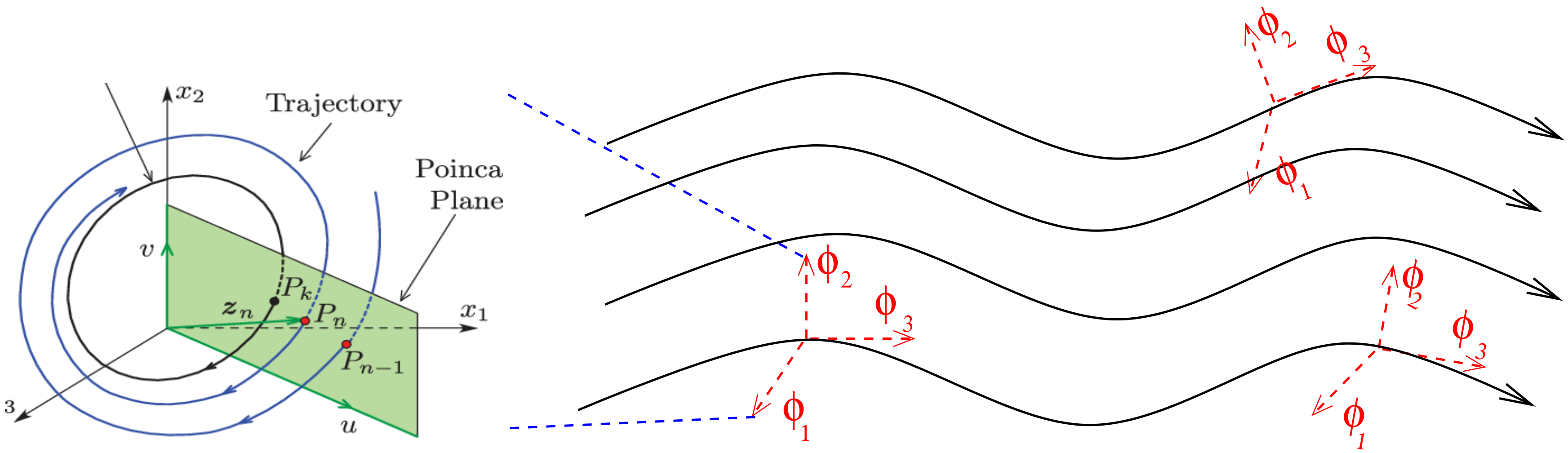
Manifestly in terms of probability distributions of observable quantities $T_{\mu\nu}, J_\mu, \Omega_{\alpha\mu\nu}$,Cells defined by full generating functionals,

A diffeffeomorphism-invariant GC ensemble at the level of fluctuations equivalent $e, u_\mu, \beta_\mu, \Pi^{\mu\nu}, ...$ choices leaving $\langle T_{\mu\nu} \rangle$ invariant! Equivalent to choosing foliations Σ_μ

Entropy content a scalar w.r.t. Σ_μ changes. Possibly order-by order, Different Boltzmannian entropy $\forall$ counted as Gibbsian entropy

Ambiguity from fluctuations makes system look like a fluid, If many equivalent choices of $e, u_\mu, J^\mu, \Pi^{\mu\nu}, ...$ likely in one its "small"! Ideal hydro behavior.

The physical intuition Ergodicity/Poncaire cycles meet relativity slightly away from equilibrium!



Gibbs entropy level+relativity : Lack of equilibrium is equivalent to "loss of phase" of Poncaire cycles. one can see a slightly out of equilibrium cell <u>either</u> as a "mismatched u_μ" (fluctuation) or as lack of genuine equilibrium (dissipation)

What is a gauge theory,exactly?

$$\mathcal{Z} = \int \mathcal{D}A^\mu \exp\left[S[F_{\mu\nu}\right] \equiv \int \mathcal{D}A_1^\mu \mathcal{D}A_2^\mu \exp\left[S[A_1^\mu]\right]$$

$A^\mu_{1,2}$ can be separated since physics sensitive to derivatives of $\ln \mathcal{Z}$

$$\ln \mathcal{Z} = \Lambda + \ln \mathcal{Z}_G \quad , \quad Z_G = \int \mathcal{D}\mathcal{A}^\mu \delta\left(G(A^\mu)\right) \exp\left[S(A_\mu)\right]$$

Ghosts come from expanding $\delta(...)$ term. In KMS condition/Zubarev

$$Z = \int \mathcal{D}\phi \quad , \quad "S" \rightarrow d\Sigma_\nu \beta_\mu T^{\mu\nu}$$

Multiple $T_{\mu\nu}(\phi)$ $\rightarrow$ Gauge-like configuration . Related to Phase space fluctuations of ϕ

Zubarev partition function for local equilibrium: think of Eigenstate thermalization...

Let us generalize the GC ensemble to a co-moving frame $E/T \rightarrow \beta_\mu T^\mu_\nu$

$$\hat{\rho}(T_0^{\mu\nu}(x), \Sigma_\mu, \beta_\mu) = \frac{1}{Z(\Sigma_\mu, \beta_\mu)} \exp\left[- \int_{\Sigma(\tau)} d\Sigma_\mu \beta_\nu \hat{T}_0^{\mu\nu} \right]$$

Z is a partition function with a field of Lagrange multiplies β_μ , with microscopic and quantum fluctuations included.

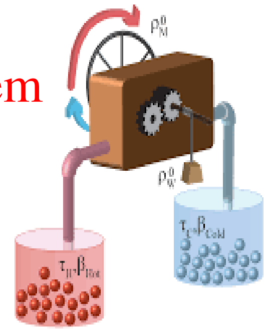
Effective action from $\ln[Z]$. Correction to Lagrangian picture?

All normalizations diverge but hey, it's QFT! (Later we resolve this!)

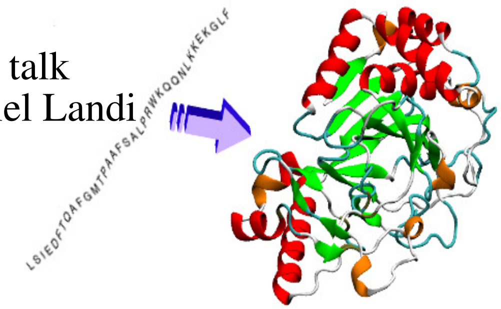
How to go forward... Crooks fluctuation theorem

Crooks fluctuation theorem

$P(W)/P(-W)=e^{\Delta s}$



From talk
Gabriel Landi



Relates fluctuations, entropy in <u>small</u> <u>fluctuating</u> systems (Nano,proteins)

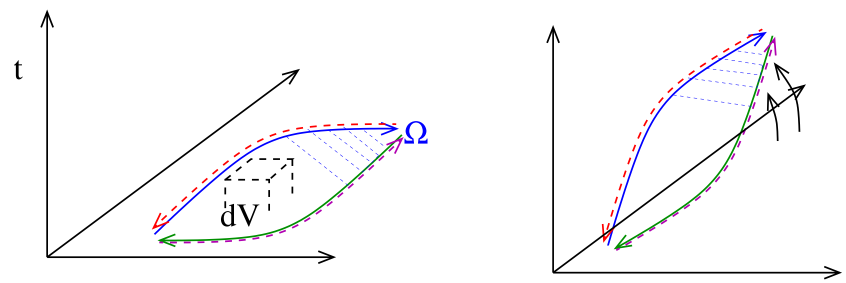
P(W) Probability system doing work in its usual thermal evolution

P(-W) Probability of the same system “running in reverse” and decreasing entropy due to a <u>thermal fluctuation</u>

ΔS Entropy produced by $P(W)$

A non-perturbative operator equation,divergences cancel out...

$$\left.\frac{\hat{\Pi}^{\mu\nu}}{T}\right|_{\sigma} = \left(\frac{1}{\partial_\mu \beta_\nu}\right) \frac{\delta}{\delta\sigma} \left[\int_{\sigma(\tau)} d\Sigma_\mu \beta_\nu \hat{T}^{\mu\nu} - \int_{-\sigma(\tau)} d\Sigma_\mu \beta_\nu \hat{T}^{\mu\nu} \right]$$



A sanity check: For a an equilibrium spacelike $d\Sigma_\mu = (dV, \vec{0})$ (left-panel) we recover Boltzmann's $\Pi^{\mu\nu} \Rightarrow \Delta S = \frac{dQ}{T} = \ln\left(\frac{N_1}{N_2}\right)$, for an analytically continued "tilted" panel, Kubo's formula

But highly non-local and non-linear, "lattice" , but there might be an analytically sovable Gaussian approximation

General covariance via the Gravitational Ward identity

Gaussian approximation from Zubarev hydrodynamics

Covariantized Gibbs-Duhem to define entropy in terms of $d\Sigma_\mu$

Kramers-Konig to enforce fluctuation-dissipation

The gravitational ward identity $\nabla \mathcal{W} = 0$

$$\mathcal{W} = G^{\mu\nu,\alpha\beta}\left(\Sigma_\mu, \Sigma'_\nu\right) - \frac{1}{\sqrt{g}}\delta\left(\Sigma' - \Sigma\right) \times$$

$$\times \left(g^{\beta\mu} \left\langle \hat{T}^{\alpha\nu}\left(x'\right) \right\rangle_\Sigma + g^{\beta\nu} \left\langle \hat{T}^{\alpha\mu}\left(x'\right) \right\rangle_\Sigma - g^{\beta\alpha} \left\langle \hat{T}^{\mu\nu}\left(x'\right) \right\rangle_\Sigma \right)$$

Fancy name and complicated but consequence of elementary properties of the metric and energy conservation

$$\partial_\mu T^{\mu\nu} + \Gamma_{\nu\alpha\beta} T^{\alpha\beta} = 0 \quad , \quad \left\langle T^n_{\mu\nu} \right\rangle = \frac{\delta^n}{\sqrt{-g}\,\delta g^{\mu\nu(n)}} \ln \mathcal{Z}$$

Note: gradient expansion and linearized fluctuations inherently break this!

The mere fact that thermodynamic quantities can be described via a $\ln \mathcal{Z}$ gives rise to the Gibbs-Duhem relation

$$s = T \ln \mathcal{Z} = P + e - \mu n$$

Enforce invariance under Σ_μ refoliations, a <u>scalar</u> $\ln \mathcal{Z}$

$$-\Delta \ln \mathcal{Z} = -\beta_\nu J^\nu \Delta\mu + P^i \Delta\beta_i - \Delta\Sigma^0 \beta_0 \int_0^{P^0} c_s^2(e) de \quad , \quad P_{\alpha=0,i=1.3} \equiv T_{\alpha\beta} d\Sigma^\beta$$

Crooks theorem becomes

$$\frac{\mathcal{P}\left\{ P_\mu|_\tau \rightarrow P_\mu|_{\tau+\Delta\tau} \right\}}{\mathcal{P}\left\{ P_\mu|_{\tau+\Delta\tau} \rightarrow P_\mu|_\tau \right\}} \sim \exp[\ln \mathcal{Z}|_{\tau+\Delta\tau} - \ln \mathcal{Z}|_\tau] \quad , \quad \Delta\tau = \beta_\mu \frac{\Delta^3 \Sigma^\mu}{\Delta^3 \phi_{i=1,2,3}}$$

Cumulant expansion: Partition function is Gaussian! $\ln \mathcal{Z} \simeq \ln \mathcal{Z}|_0 -$

$$- \left.\frac{\partial^2 \ln \mathcal{Z}}{\partial \beta_\mu \partial \beta_\nu}\right|_0 \ln \prod_{\Sigma(x),\Sigma(x')} \exp\left[-\frac{1}{2} \left\langle \Delta T_{\mu\nu}(\Sigma(x')) \right\rangle C^{\mu\nu\alpha\beta}(\Sigma(x),\Sigma(x')) \left\langle \Delta T_{\alpha\beta}(\Sigma(x)) \right\rangle \right.$$

A covariantization of $\left\langle E^2 \right\rangle - \left\langle E \right\rangle^2 \equiv C_V T \Rightarrow\Rightarrow C_{\alpha\beta\mu\nu} \sim \left.\frac{\partial \ln \mathcal{Z}}{\partial \beta_\mu \partial \beta_\nu}\right|_0 F(\Sigma)_{\alpha\beta}$ and Ward identity imposed on width $C_{\alpha\beta\gamma\nu}$ manifestly diffeo invariant Crooks theorem means

$$\ln \mathcal{Z}|_{\tau+\Delta\tau} - \ln \mathcal{Z}|_{\tau} \sim \exp\left[-\Delta_\mu \beta_\nu C^{\mu\nu\alpha\zeta} \Delta_\alpha \beta_\zeta \right] \quad , \quad \Delta_\mu O \equiv \frac{\Delta O(x^\mu)}{\Delta x^\mu}$$

An possibly diffeomorphism invariant alternative to gradient expansion which isn't!

fluctuation-dissipation relation From Kramers-Konig relations

$$\mathrm{Im}\left[\tilde{\mathcal{G}}^{\mu\nu,\alpha\beta}(\omega,k)\right] = -\frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty}\frac{\mathrm{Re}\left[\tilde{\mathcal{G}}^{\mu\nu,\alpha\beta}(\omega,k)\right]}{\omega'-\omega}d\omega'$$

$$\mathrm{Re}\left[\tilde{\mathcal{G}}^{\mu\nu,\alpha\beta}(\omega,k)\right] = \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty}\frac{\mathrm{Im}\left[\tilde{\mathcal{G}}^{\mu\nu,\alpha\beta}(\omega,k)\right]}{\omega'-\omega}d\omega'$$

Direct consequence of causality, relate the real and imaginary part of the response function in momentum space But non-local in frequency, generally invalidates gradient expansion! (inherently breaks fluctuation-dissipation)

Apply on the linear response function of energy-momentum tensor

$$T_{\mu\nu}(\Sigma) = \int \mathcal{G}^{\mu\nu,\alpha\beta}(\Sigma_0' - \Sigma_0)\delta T_{\alpha\beta}(\Sigma_0')d\Sigma_0'$$

$$\tilde{\mathcal{G}}^{\mu\nu\alpha\beta} = \frac{1}{2i}\left(\frac{\tilde{G}^{\alpha\beta\mu\nu}(\Sigma_0, k)}{\tilde{G}^{\alpha\beta\mu\nu}(-i\epsilon\Sigma_0, k)} - 1\right)$$

And $G_{\alpha\beta\gamma\mu}$ is what comes from the Ward identity! These equations together should do it!

Only in terms of $T_{\mu\nu}, J_\mu, \Sigma_\mu$ "observables" and a "gauge" reditemSecond law imposed via fluctuation dissipation (redundances, fluctuations of observables)

Fluctuations in non-ideal hydrodynamics not well understood

Intimately related to entropy current, double counting of DoFs
Could alter fluctuation-dissipation expectation, "fluctuations help dissipate", in analogy to Gauge theory

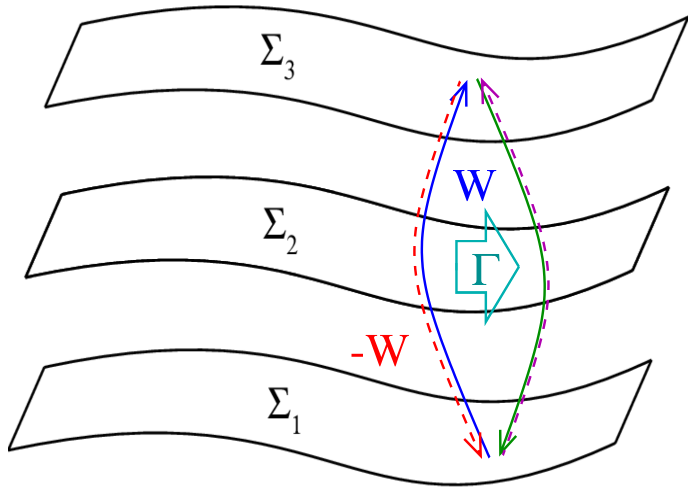
Approximate local equilibrium not understood in Gibbsian picture
My proposal: applicability of fluctuation-dissipation

Need a covariant description purely in terms of observable quantities
Ergodicity works in ideal hydro, Crooks theorem/K-K beyond it?

Could be relevant for hydro in small systems

A non-relativistic limit? (Brazil nut effect) all depends on time dilation,so a bit at a loss! But...

entropic gravity? Ted Jacobson, gr-qc/9504004 derived GR from the area entropy relation! conjectured gravity is "thermal" rather than "quantum"



Perhaps techniques shown here can be used to build a fluctuation-hydro like entropic gravity, whose fluctuations preserve covariance under locally Lorentz diffeomorphisms! . Connection to Unruh effect and Bose-Marletto-Vedral experiments , see 2210.08586,2201.10457,2405.08192 (different talk!)

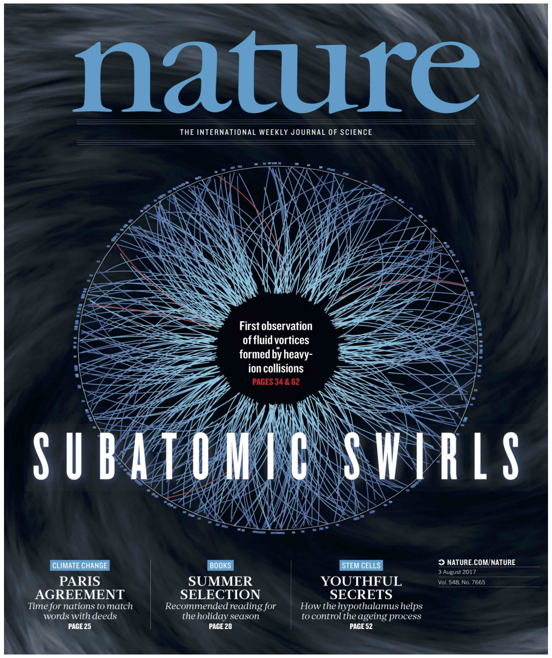
"the universe is governed by Crooks","the biggest nut goes on top",... ,
towards a theory of everything with plenty of empirical support!

PS: Onto spin hydrodynamics

STAR
collaboration
1701.06657

NATURE
August 2017

Polarization by vorticity
in heavy ion collisions



Could give new talk about this, but will mention hydro with spin not developed and a lot of conceptual debates.

- "at what order in Gradient" are spin-vorticity interactions? Causality constraints ,"minimum viscosity", "same order as fluctuations" (microstates).
  Spin hydrodynamics is transfer of micro to macro DoFs

- Transport description inherently "non-local" (violation of ensemble average/molecular chaos)

- Pseudogauge! Spin part of angular momentum not uniquely defined!

- "trivial" in a sense: Let $\Phi^{\alpha\beta\gamma}$ be fully antisymmetric

$$T_{\mu\nu} \to T'_{\mu\nu} + \frac{1}{2}\partial_\lambda \left(\Phi^{\lambda,\mu\nu} + \Phi^{\mu,\nu\lambda} + \Phi^{\nu,\mu\lambda}\right) \quad , \quad \partial^\mu T_{\mu\nu} = \partial^\mu T'_{\mu\nu} = 0$$

- Can move around spin and angular momentum
- Can symmetrize $T_{\mu\nu}$ (good for gravity, bad for equilibrium spin-orbit)
- For particles $\vec{S} = \sum_i \vec{S}_i$ but remember, spin violation of molecolar chaos
- Not clear if dynamics should depend on it! Most approaches pseudo-gauge covariant but Entropy usually does, hence fluctuations!
- Spin 1: Pseudogauge $\to$ Gauge symmetry "ghosts"? GT,1810.12468

Pseudo-gauge symmetries physical interpretation: T.Brauner, 1910.12224

$$x^\mu \to x^\mu + \epsilon \zeta^\mu(x) \quad , \quad \psi_a \to \psi_a + \epsilon \psi'_a \quad , \quad \mathcal{S} \to \mathcal{S}$$

For particles field redefinition "observable", but what about for fluctuating fields?
Entropy depends on pseudogauge as spin-orbit interactions mix entropyless vortices with entropyful spin microstates

Previous picture offers a way out! Pseudo-gauge transformations could be exactly the sort of equations that produce redundancies!
$\ln \mathcal{Z}|_{class}$ not invariant but full $\ln \mathcal{Z}$ should be! Spin $\leftrightarrow$ fluctuation, need equivalent of DSE equations!

Basic idea: Define ensemble via $\ln \mathcal{Z}$ and "gauge constraints" so that pseudo-gauge transformations "move aorund" the ensemble

How to see it: Grossi,Floerchinger, 2102.11098 (PRD) Let us define a J co-moving with u_μ and use the "exact" (before coarse-graining) partition function to build

$$\Gamma(\phi) = \mathrm{Sup_J}\left(\int J(x)\phi(x) - i \ln \mathcal{Z}[J]\right)$$

$u_\mu \to u'_\mu$ non-inertial and does not change $\langle T_{\mu\nu}\rangle$, so one can define

$$J_{\mu\nu\gamma} = \frac{1}{\sqrt{g}} \frac{\delta \ln \mathcal{Z}[J']}{\delta \Gamma^{\alpha\nu\gamma}} \quad , \quad D_\mu J^{\mu\nu\gamma} = 0$$

Setting the gauge at the level of the microscopic approximately thermalized partition function equivalent adding auxiliary field $D_\mu M_{\alpha\beta} = 0$ to

$$\mathcal{Z}[J_{\alpha\beta\gamma}] = \int \mathcal{D}\phi \mathcal{D} M_{\alpha\beta} \exp\left[\int det[M] d^4x \mathcal{L}\left(\phi, \partial_\mu + \Gamma ...\right) + \int d\Sigma^\gamma M^{\alpha\beta} J_{\alpha\beta\gamma}\right]$$

Anisotropy, transport and statistical mechanics Anisotropic hydrodynamics justified within transport via improved relaxation time

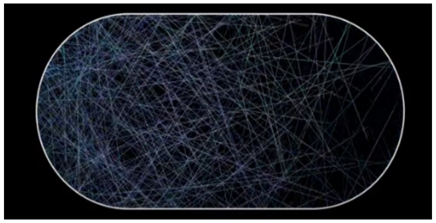
$$f(x,p) = f_{eq}\left(1+\phi(x,p)\right) \rightarrow f_{eq}\left(1+\phi(x,p)+a_\mu(x)p^\mu\right)$$

Problem: Boltzmann is an approximation where $f(x,p)$ represents an infinity of particles . Fundamentally, hydrodynamics comes from Kubo

$$\eta = \lim_{k\rightarrow 0}\frac{1}{k}\mathrm{Im}\int dx \left\langle \hat{T}_{xy}(x)\hat{T}_{xy}(y)\right\rangle \exp\left[ik(x-y)\right]$$

Usually semiclassical approximation yields Boltzmann equation than relaxation time, which guarantees the Kinchin condition to be fulfilled. Above demonstration reliable only in that limit

The basic problem with f(x,p)

 $=$  ?

Let's solve the simplest transport equation possible: Free particles

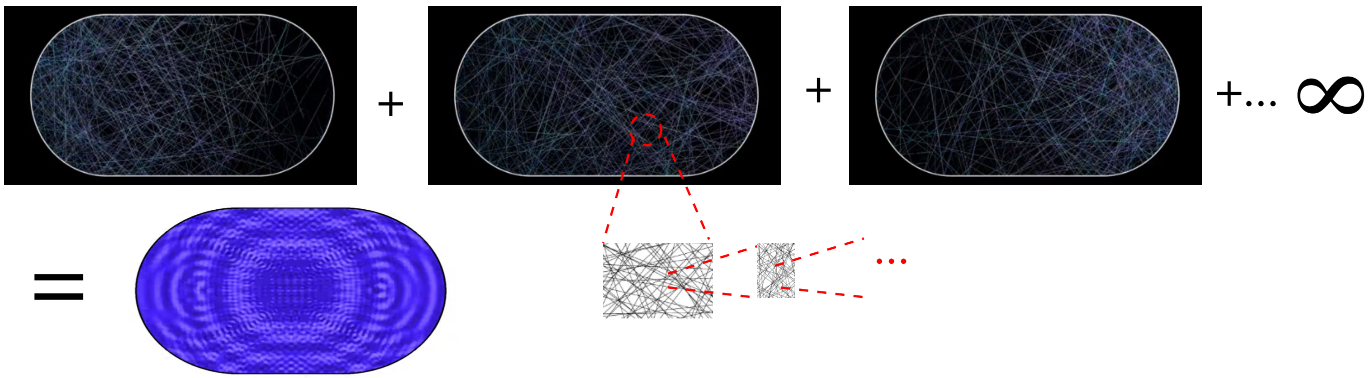
$$\frac{p^\mu}{m}\partial_\mu f(x,p) = 0 \rightarrow f(x,p) = f\left(x_0 + \frac{p}{m}t, p\right)$$

obvious solution is just to propagate

What is weird is that "hydro-like" solution possible too (eg vortices)!

$$f(x,p) \sim \exp\left[-\beta_\mu p^\mu\right] \quad , \quad \partial_\mu \beta_\nu + \partial_\nu \beta_\mu = 0$$

But obviously unphysical, no force! What's up?



This paradox is resolved by remembering that $f(x,p)$ is defined in an ensemble average limit where the number of particles is not just “large” but uncountable . curvature from continuity!

BUt this suggests Boltzmann equation <u>disconnected</u> from $N_{dof} \leq \infty$!

In Anisotropic hydro β_μ not Killing vector . So no reason to assume ensemble average/thermal fluctuations sampled fairly close to equilibrium! Boltzmann equilibrium and Gibbs-type thermal equilibrium could be very different. lets work with the latter

Vlasov and Boltzmann in a classical world

Villani , `https://www.youtube.com/watch?v=ZRPT1Hzze44`

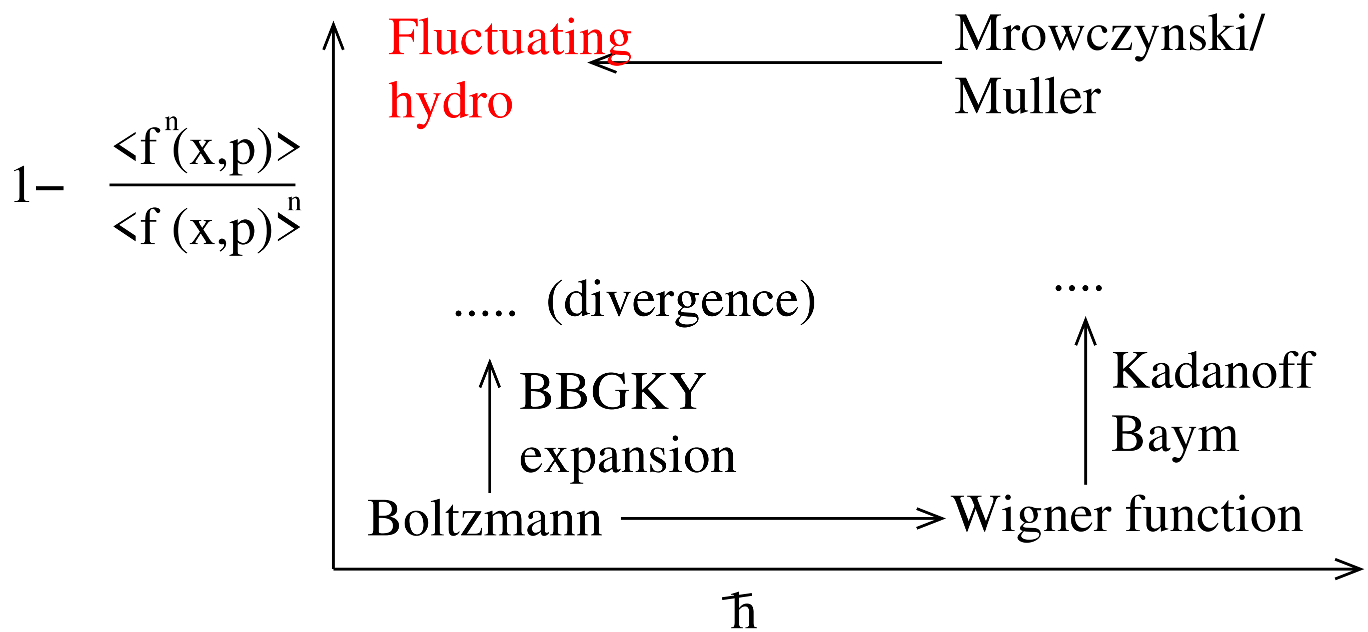
Vlasov equation contains all classical correlations, instability-ridden, "filaments", cascade in scales.
$N_{DOF} \rightarrow \infty$ invalidates KAM theorem stability

Boltzmann equation "Semi-Classical UV-completion" ov Vlasov equation, first term in BBGK hyerarchy, written in terms of Wigner functions.

Infinitely unstable jerks on infinitely small scales Random scattering

But if number of particles $N \ll \infty$ Correlations important! .

Boltzmann equation,BBGKY and limits



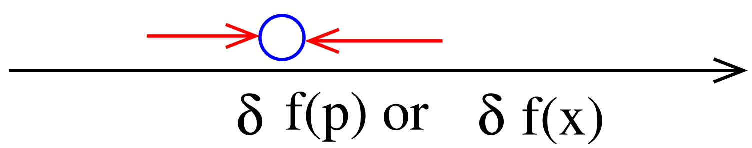
Boltzmann equation emerges as a double limit from microscopic correlations, $\hbar \to 0$ Relaxing the latter limit would destroy statistical independence CHSH relations , so probably not relevant (phases "chaotic"). But fluctuating hydro "non-perturbative" in correlations

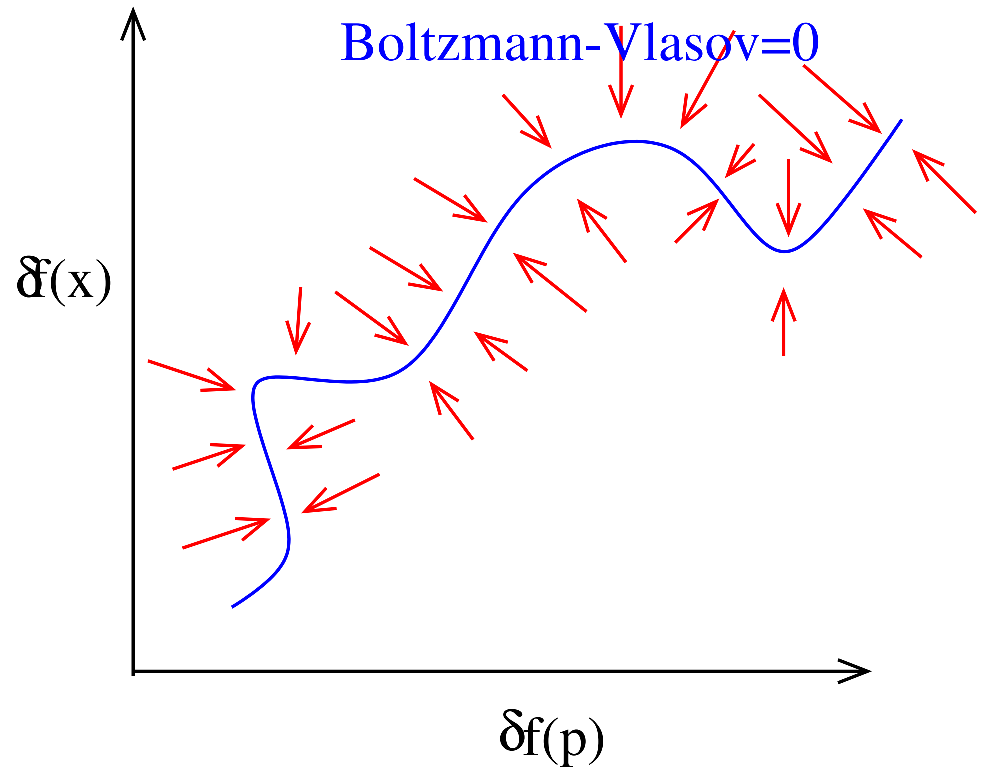
Finite number of particles: $f(x,p)$ not a function but a functional ($\mathcal{F}(f(x,p)) \underbrace{\rightarrow}_{Boltzmann} \delta\left(f' - f(x,p)\right)$), incorporating continuum of functions and all correlations. Perhaps solvable!

$$\frac{p^\mu}{\Lambda}\frac{\partial}{\partial x^\mu} f(x,p) = \left\langle \underbrace{\hat{C}[\tilde{W}(\tilde{f}_1, \tilde{f}_2)] - g\frac{p^\mu}{\Lambda}\hat{F}^{\mu\nu}[\tilde{f}_1, \tilde{f}_2]\frac{\delta}{\delta \tilde{f}_{1,2}}\tilde{W}\left(\tilde{f}_1, \tilde{f}_2\right)}_{How \quad many \quad A-B=0?} \right\rangle$$

Wigner functional to $\mathcal{O}\left(h^0\right)$. What is the effect? If only Boltzmann term not much!

If Both Vlasov and Boltzmann terms, redundancy-ridden!





One can deform $f(x,p)$ by $\delta f(x)$ or $\delta f(p)$ so that $\hat{C} - \hat{W}$ cancels. In ensemble average deformation makes no sense, but away from it it does!

Discretize $x, p \to$ random matrix problem!

$$\dot{f}_{ij} - \left[\frac{\vec{p}_k}{\Lambda}.\Delta_k\right] f_{ij} = \left\langle \hat{\Omega} \right\rangle$$

$$\hat{\Omega} \sim d\left[f'_{i_1 j_1}\right]\left[\mathcal{W}_{i_1 j_1 i j}\left(\mathcal{C}_{j j_1}\left(f_{ij} f'_{i_1 j_1} - f_{i j_1} f'_{i_1 j}\right) - \mathcal{V}^{\mu}_{i i_1} f_{ij} f'_{i_1 j_1} \frac{\Delta f_{ij}}{\Delta p^{\mu}}\right)\right]$$

- Theorems of random matrix theory can be used to prove limit very different from RTA!
- can be tested numerically with a lattice Boltzmann algorithm
- connects to Zubarev Gibbs-Duhem relation
  $\ln \mathcal{Z} = \ln \left[\prod_{i=1}^{N} \exp\left(\Delta^3 \Sigma_\mu \left(\beta_\nu T^{\mu\nu} - \mu_i J^\mu\right)\right)\right]$

SPARE SLIDES