Quantum free-streaming in cosmological space-time

Prof. F.Becattini & D.Roselli

Based on:

- F. Becattini and D. Roselli, Class. Quant. Grav. 40 (2023) no.17, 175007.
- F. Becattini and D. Roselli, [arXiv:2403.08661 [gr-qc]]

$$\hat{\rho}=\frac{1}{Z}\exp\left(-\int_{\Sigma} d\Sigma_{\mu}\hat{T}^{\mu\nu}\beta_{\nu}\right)$$

Introduction

Introduction:Cosmological Free-streaming

Expanding universe →

Flat Robertson-Walker metric

$$ds^2 = dt^2 - a^2(t)\, d\gamma^2\,; \qquad ds^2 = a^2(\eta)\, ds^2_{flat}$$

$$\eta(t) = \int_{t_0}^{t} \frac{dt'}{a(t')}$$

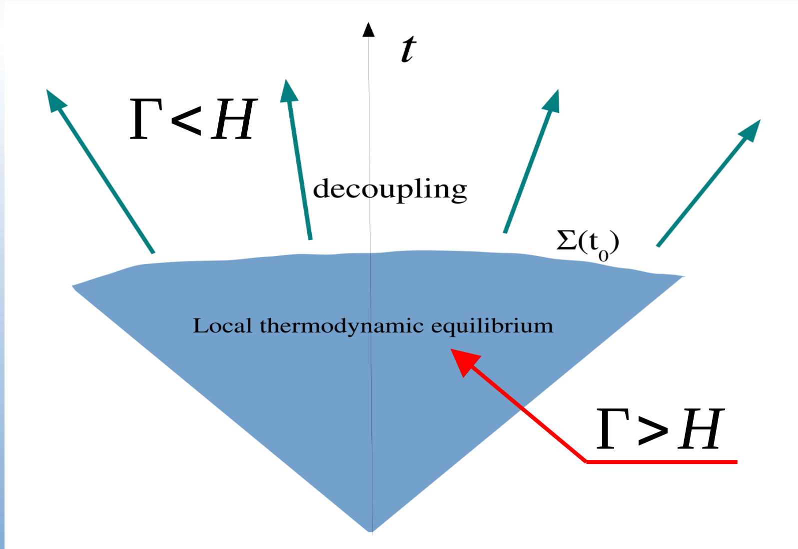
Conformal time

Universe expands with rate

$$H = \frac{\dot{a}(t)}{a(t)} = \frac{a'(\eta)}{a^2(\eta)}$$

Interacting fluid.
Rate of interaction Γ

★ Decoupling from a thermal bath → Free evolution



Basic physical picture:

1. Interactive system (particle of fluid)
2. Interaction length smaller that mean free-path. Divergence from equilibrium
3. Interaction cease, “decoupling” and free particles

Approximation:

Decoupling is instantaneous

Decoupling time

$$\Gamma(t_0) = H(t_0)$$

Introduction: classical free-streaming

One can calculate the energy density and the pressure after the decoupling: → Using Kinetic theory

$$\varepsilon(t)=\frac{1}{(2\pi)^3 a^4(t)}\int dk^3\sqrt{k^2+m^2a^2(t)}\,f(k,t_0)$$

$$p(t)=\frac{1}{(2\pi)^3 a^4(t)}\int dk^3\frac{k^2}{3\sqrt{k^2+m^2a^2(t)}}\,f(k,t_0)$$

$\sqrt{k^2+m^2a^2(t)} > 0$

Distribution function at the decoupling, can be classic or quantum

$m=0$
$p=\frac{\varepsilon}{3}$ $p,\varepsilon\propto a^{-4}$

$m>0$
$p\propto a^{-5}$ $\varepsilon\propto a^{-3}$ $p\simeq 0$

The pressure is always positive and smaller than the energy density

The Quantum field theory approach

The quantum problem: QFT in CST

To describe free-streaming in a full quantum theory

- Define LTE using stat.mech (I.C) → New
- Matter described by STEM
- Describe free evolution using Klein-Gordon eq.

In RW metric lot of literature

Field described by stress-tensor

$$T_{\mu\nu}=\nabla_\mu\Phi\nabla_\nu\Phi-\frac{1}{2}g_{\mu\nu}\left(g^{\rho\sigma}\nabla_\rho\Phi\nabla_\sigma\Phi-\frac{1}{2}m^2\Phi^2\right)+\xi\left(G_{\mu\nu}+g_{\mu\nu}g^{\rho\sigma}\nabla_\rho\nabla_\sigma-\nabla_\mu\nabla_\nu\right)\Phi^2$$

homogeneity+isotrpy

$$T_{\mu\nu}=(\varepsilon+p)u^\mu u^\nu-pg_{\mu\nu}$$

$$\varepsilon=\langle T_{00}\rangle \quad p=\frac{1}{3}\sum_{j=1}^{3}\langle T_{jj}\rangle$$

After decopuling free-evolution

$$\left(g^{\mu\nu}\nabla_\mu\nabla_\nu+m^2-\xi R\right)\Phi=0$$

Expansion in normal modes

$$\Phi=\frac{1}{(2\pi)^{3/2}}\int dk^3\left(v_k\hat{a}_k e^{i\vec{k}\cdot\vec{x}}+c.c\right)$$

$$v''_k+\left(k^2+m^2a^2-(1-6\xi)\frac{a''}{a}\right)v_k=0$$

Equation for the modes

$$\hat{a}_k\Phi_0=0$$

Def. The vacuum

The quantum problem: LTE in CST

The field is not in the vacuum!
At decoupling is in LTE

Calculated at the decoupling. Independent from time

State of LTE:

State that maximize entropy with constraint of fixed energy density

$$S = -Tr\left(\hat{\rho}\ln\hat{\rho}\right)$$

MAX

$$\hat{\rho} = \frac{1}{Z}\exp\left(-\int_{\Sigma(0)} d\Sigma_\mu T^{\mu\nu}\beta_\nu\right)$$

$$\beta_\nu = u_\nu / T_{dec} \qquad d\Sigma(0) = a^3(0)\,dxdydz$$

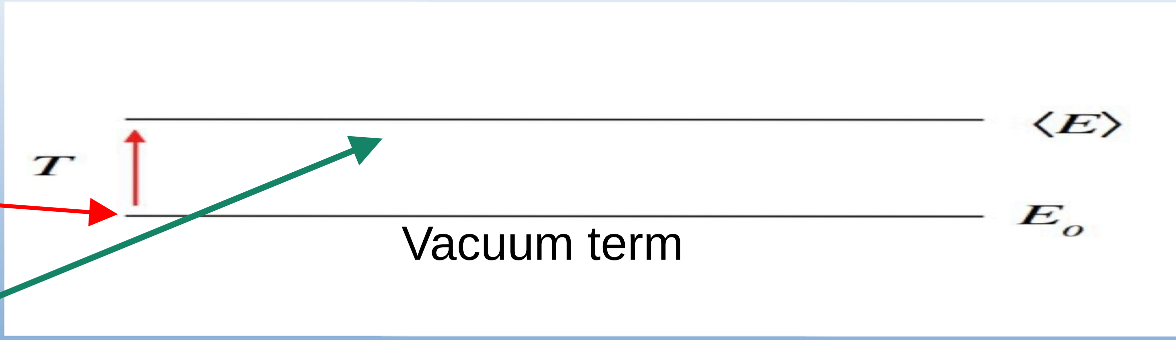
Expectation value

$$\langle T_{\mu\nu}\rangle = Tr\left(\hat{\rho}\,T_{\mu\nu}\right)$$

$$\nabla_\mu\left(\langle T^{\mu\nu}\rangle - \langle 0_{t_0}|T^{\mu\nu}|0_{t_0}\rangle\right) = 0$$

$$\lim_{T_{dec}\to 0}\langle T^{\mu\nu}\rangle - \langle 0_{t_0}|T^{\mu\nu}|0_{t_0}\rangle = 0$$

Measure excitations induced by the thermal bath



The quantum problem: the choice of the vacuum

To single out the excitations Produced by the thermal bath → We must subtract the vacuum term

In a general curved spacetime: →

- Energy is not conserved
- Less symmetric than Minkowski
- Different possible coordinate systems

→ The choice of the vacuum is arbitrary

We choose the vacuum as: → State that minimize the "energy" at the decoupling →

Effective Hamiltonian

$$H(0)=\int dx^3 T_{00}(0,\vec{x})$$

$$v_k(0)=\frac{1}{\sqrt{2\omega_k(0)}} \qquad v'_k(0)=\frac{-i}{2v_k(0)}+(1-6\xi)H_0 v_k(0)$$

$$\omega_k(\eta)=\sqrt{k^2+m^2a^2(\eta)+6\xi(1-6\xi)\frac{a'^2(\eta)}{a^2(\eta)}}$$

Choice fix I.C

The quantum problem: Quantum free-stream

Plugging the field expansion and calculating the expectation value we find:

Energy density & pressure

$$\varepsilon(\eta)=\frac{1}{(2\pi)^3}\int dk^3\,\omega_k(\eta)K_k(\eta)\underline{n_B(0,k)}$$

$$p(\eta)=\frac{1}{(2\pi)^3}\int dk^3\,\omega_k(\eta)\Gamma_k(\eta)\underline{n_B(0,k)}$$

Classical results are recovered for:

$$a\rightarrow 1$$
$$a'=a''=0$$

Where:

$$K_k(\eta)=\frac{1}{\omega_k(\eta)}\left(|v'_k(\eta)|^2+\omega_k^2(\eta)|v_k(\eta)|^2-2(1-6\xi)\frac{a'}{a}\Re\left(v'_k(\eta)v_k(\eta)\right)\right)$$

$$\Gamma_k(\eta)=\frac{1}{\omega_k(\eta)}\left((1-4\xi)|v'_k(\eta)|^2-\frac{\gamma_k(\eta)}{3}|v_k(\eta)|^2-2(1-6\xi)\frac{a'}{a}\Re\left(v'_k(\eta)v_k(\eta)\right)\right)$$

$$\omega_k^2(\eta)=k^2+m^2a^2(\eta)+6\xi(1-6\xi)\frac{a'^2(\eta)}{a^2(\eta)}$$

$$\gamma_k(\eta)=(12\xi-1)k^2+3(4\xi-1)m^2a^2(\eta)+3(1-6\xi)\frac{a'^2(\eta)}{a^2(\eta)}-12\xi(1-6\xi)\frac{a''(\eta)}{a(\eta)}$$

The quantum problem: Solution

The full quantum problem require to solve the following coupled system

$$R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R=8\pi G Tr\left(\hat{\rho}T_{\mu\nu}\right)$$
$$\left(g^{\mu\nu}\nabla_\mu\nabla_\nu+m^2-\xi R\right)\Phi=0$$

- Expansion of the universe modify energy density and pressure
- Quantum effects modify the scale factor

Can be studied only numerically!!

To weight the impact of quantum corrections:

- We choose an expansion model: fix a
- We solve the mode equation
- We study the behaviour of energy density and pressure

Only field dynamic

$$v''_k+\left(k^2+m^2a^2-(1-6\xi)\frac{a''}{a}\right)v_k=0$$
$$v_k(0)=\frac{1}{\sqrt{2\omega_k(0)}}\qquad v'_k(0)=\frac{-i}{2v_k(0)}$$

Solving the differential equation

Solutions: Massive field

Differential equation

$$v''_k+\left(k^2+m^2a^2-(1-6\xi)\frac{a''}{a}\right)v_k=0$$

$\rightarrow \Omega_k^2(\eta)$

Slow expansion

$$P_k\equiv\frac{\Omega'_k}{\Omega_k^2}\ll 1$$

$$P_k=H\frac{m^2-\frac{1-6\xi}{a^2}\left(\frac{a'''}{a'}-\frac{a''}{a}\right)}{\left(m^2+\frac{k^2}{a^2}-\frac{1-6\xi}{a^2}\frac{a''}{a}\right)^{3/2}}\ll 1$$

We have studied:

$a(t)=\left(\frac{t}{t_0}\right)^{\sigma}$

- $0<\sigma\leq 1$ → Decelerated expansion
- $\sigma>1$ → Accelerated expansion

$a(t)=e^{H(t-t_0)}, H=conts.$ → Accelerated expansion

- If the expansion is decelerated always $\lim_{t\to\infty} P_k=0$
- If the expansion is accelerated is possible $\lim_{t\to\infty} P_k\geq 1$

Quantum effects never survive at late times

Quantum effects dominate at late times

Whenever expansion is NOT slow Quantum effects dominate!

$T_{dec}/m \quad H/m$

Solutions: De Sitter universe

De Sitter Universe:
Exponentially accelerated expansion

$$a(t)=e^{H(t-t_0)}, H=conts.$$

$$\frac{d^2 v_k(y)}{dy^2}+\left(\frac{k^2}{H^2}+\frac{m^2/H^2-2(1-6\xi)}{y^2}\right)v_k(y)=0$$

$$y=1-H\eta, y(t_0)=1, y(\infty)=0$$

Two different regimes throughout the evolution:

$m>H$ Or $m\leq H$

We can exactly solve the equations. For example for $m>H$

Def:

$$\mu=\frac{1}{2}\sqrt{4m^2/H^2+48\xi-9}\simeq m/H$$

$$v_k(y)=C_k\sqrt{\frac{\pi}{4H}}e^{-\mu\pi/2}\sqrt{y}H^{(1)}_{i\mu}\left(\frac{ky}{H}\right)+D_k\sqrt{\frac{\pi}{4H}}e^{\mu\pi/2}\sqrt{y}H^{(2)}_{i\mu}\left(\frac{ky}{H}\right)$$

C_k, D_k Fixed by initial conditions and such that: $|C_k|^2-|D_k|^2=1$

Solutions: De Sitter universe, plot m>H

$m > H$

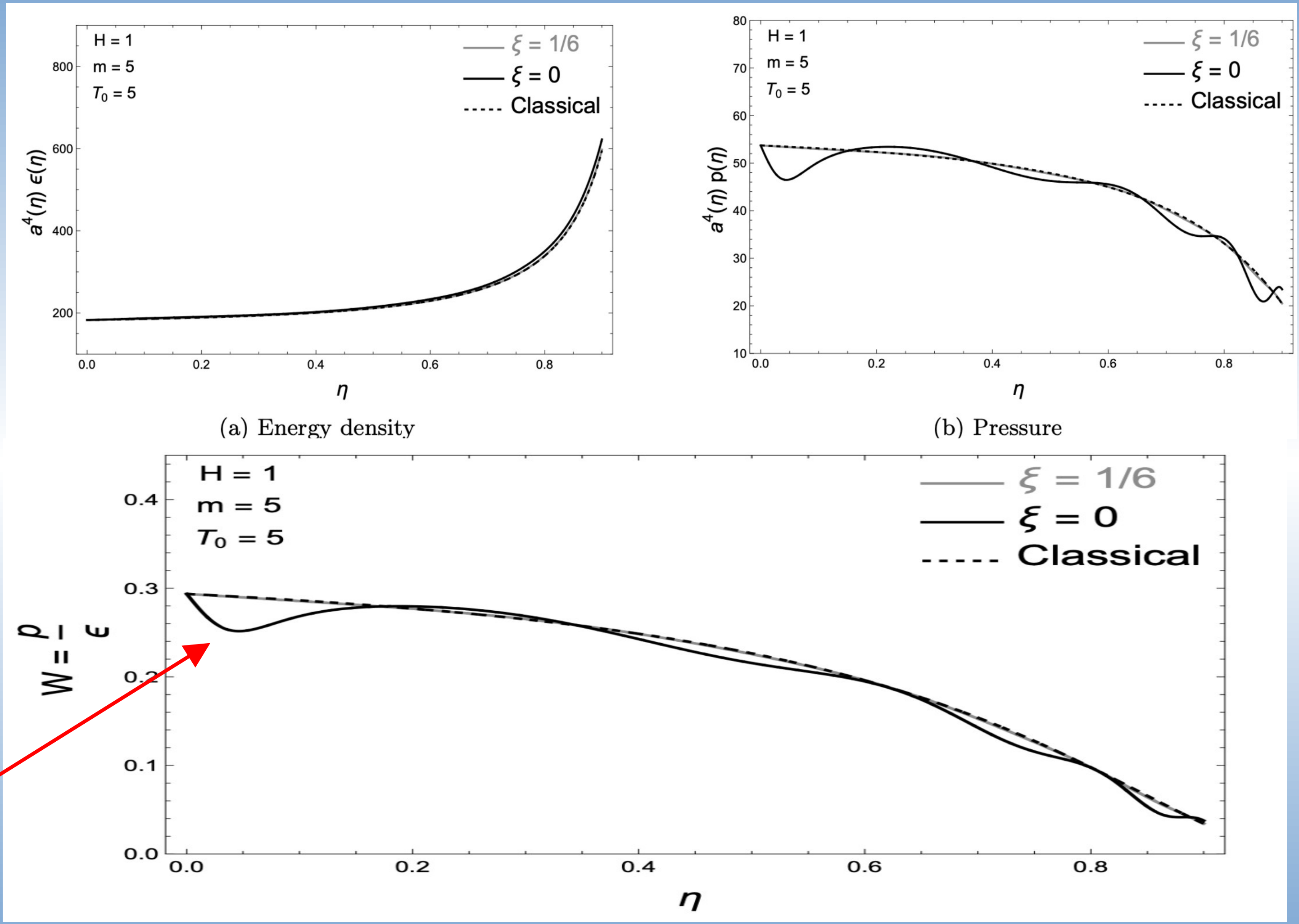
Quantum field effects are suppressed

Plots are for $T_{dec} = m$

For $T_{dec} > m$

Classic relativistic gas

Only small effects near decoupling



(a) Energy density

(b) Pressure

Solutions: De Sitter universe, solution m<H

Solution of the equation: $v_k(y)=C_k\sqrt{\frac{\pi}{4H}}\sqrt{y}\,H_\nu^{(2)}\left(\frac{ky}{H}\right)+D_k\sqrt{\frac{\pi}{4H}}\sqrt{y}\,H_\nu^{(1)}\left(\frac{ky}{H}\right)$

Asymptotic

$t\rightarrow\infty\Rightarrow y\rightarrow 0$

$$\nu=\frac{1}{2}\sqrt{9-48\xi+4m^2/H^2}$$

$$1/2\leq\nu\leq 3/2$$

We got for every coupling: $p\propto\varepsilon\propto a^{2\nu-3}$

$m/H\ll 1$ → ν → $1/2\ for\ \xi=1/6$; $3/2\ for\ \xi=0$

At late times the pressure is comparable with the energy density even for the massive case → $\lim_{t\rightarrow\infty} p<0$

For the minimal case the equation of state is the same of a cosmological constant: $p=-\varepsilon$

Solutions: De Sitter universe, plot m<H

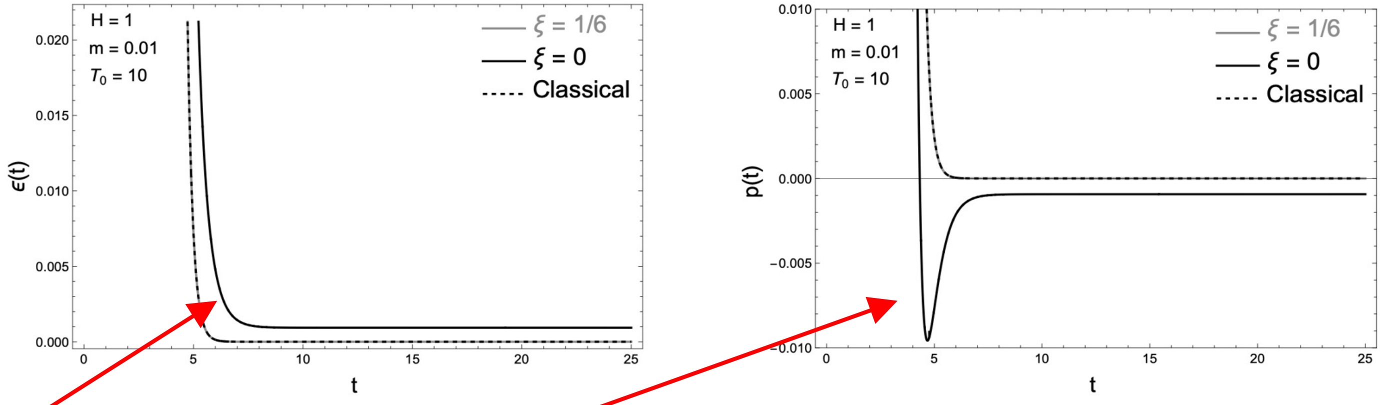
$m<H$

Quantum field effects dominate and persist

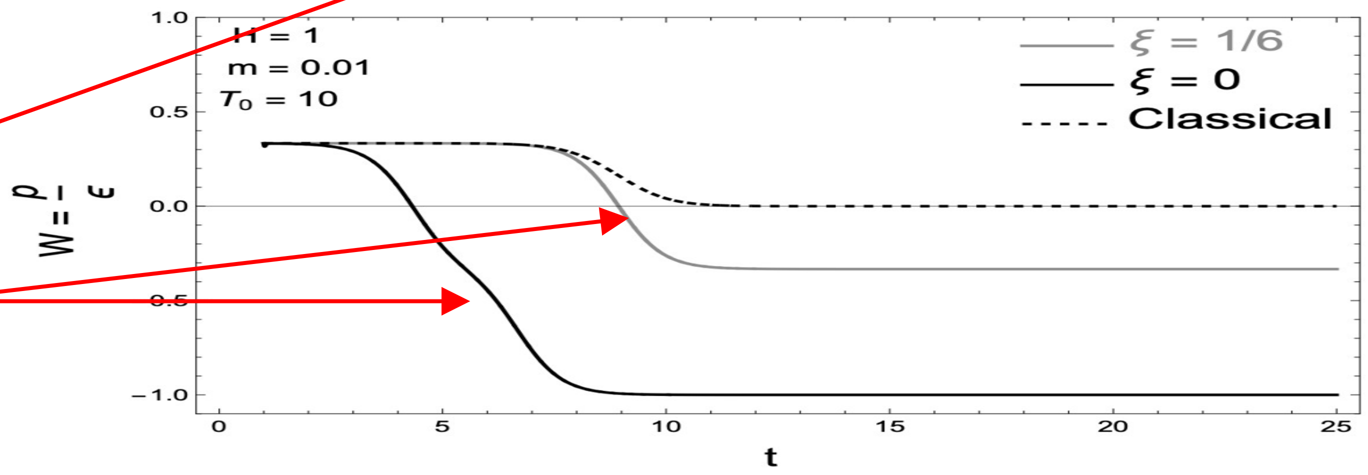
Plots are for $T_{dec}>m$

Quantum effects:

- Energy density enhanced (GPP)
- Pressure goes towards negative values
- Asymptotic pressure is negative



(a) Energy density (b) Pressure



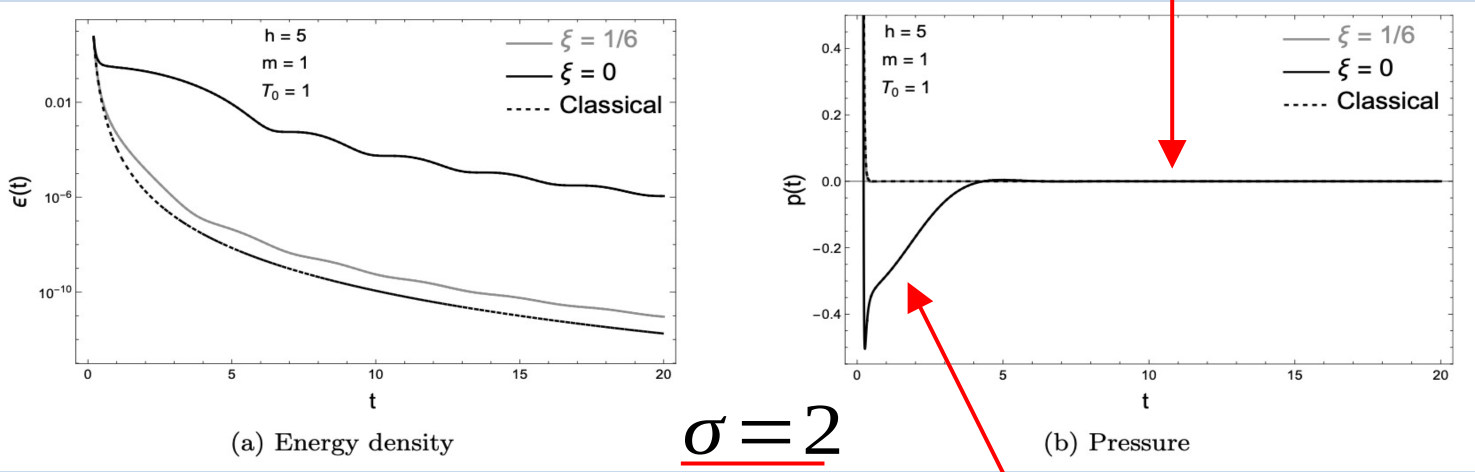
Solutions: Power law expansion

Possible effects at the decoupling

True for every σ

$$h=\frac{\sigma-1}{t_0}>T_{dec}>m$$

At late times classical expressions



(a) Energy density

$\sigma=2$

(b) Pressure

The energy density is enhanced by quantum corrections

Quantum effects only near decoupling

Solutions: Massless field m=0

Massless equation

$$v''_k + (k^2 - (1-6\xi)a''/a)v_k = 0$$

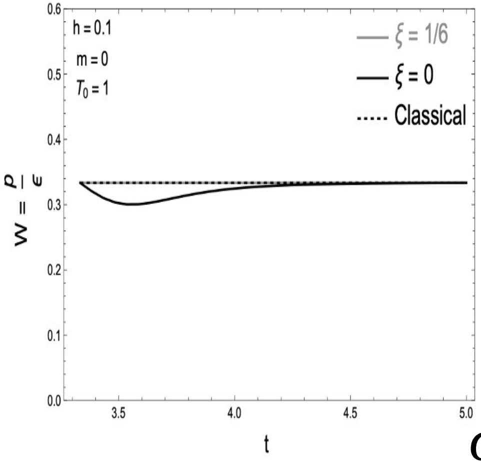
The massless case Is different from $m \to 0$

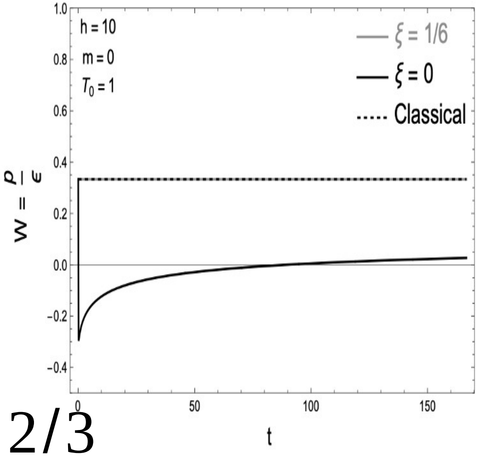
For $\xi = 1/6$ → $v''_k + k^2 v_k = 0$ Same equation of flat space-time → No effects!

For the minimal case $\xi = 0$

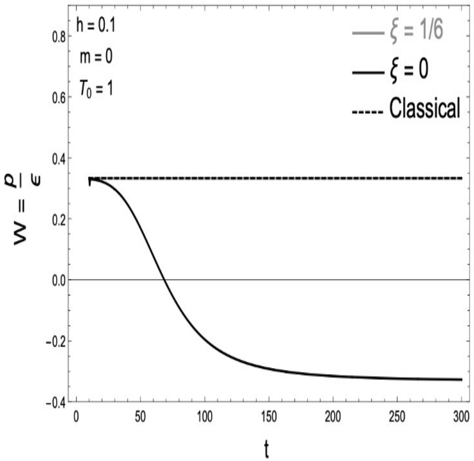
Decelerated Expansion is always adiabatic at late times, possible effects at decoupling

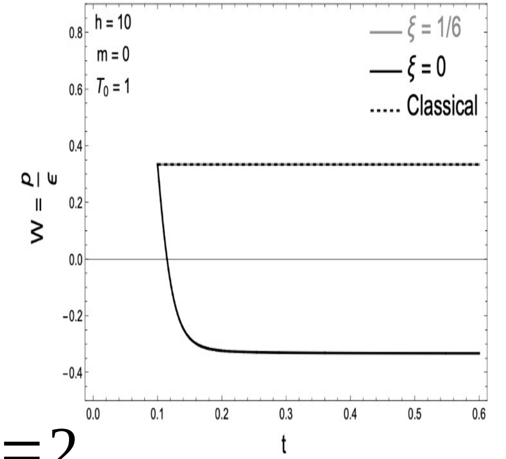
Accelerated expansion is always non-adaibatic at late times, quantum effects dominate





$\sigma = 2/3$





$\sigma = 2$

Summary

Quantum field effects can strongly modify the free-streaming propreties

Condition for Q.E

$$\frac{a'}{a} \geq \frac{mc^2}{\hbar}, \frac{K_B T_{dec}}{\hbar}$$

Depends on:
- Mass of the field
- Decoupling Temperature
- Rate of expansion
- Acceleration/deceleration

- Pressure tends toward negative values
- Energy density is enhanced
- Pressure and energy density are comparable
- Field mimics cosmological constant

Open Question

Can this corrections alter the dynamic of the Universe?

We only solved half of the problem!

Outlook and current works

Current work: Dissipative effects

The formalism is suitable to describe dissipative effects

$$\widehat{\rho} = (1/Z)\exp\left[-\int_{\Sigma(\tau)} \mathrm{d}\Sigma_\mu \widehat{T}^{\mu\nu}\beta_\nu + \int_\Theta \mathrm{d}\Theta \widehat{T}^{\mu\nu}\nabla_\mu\beta_\nu\right]$$

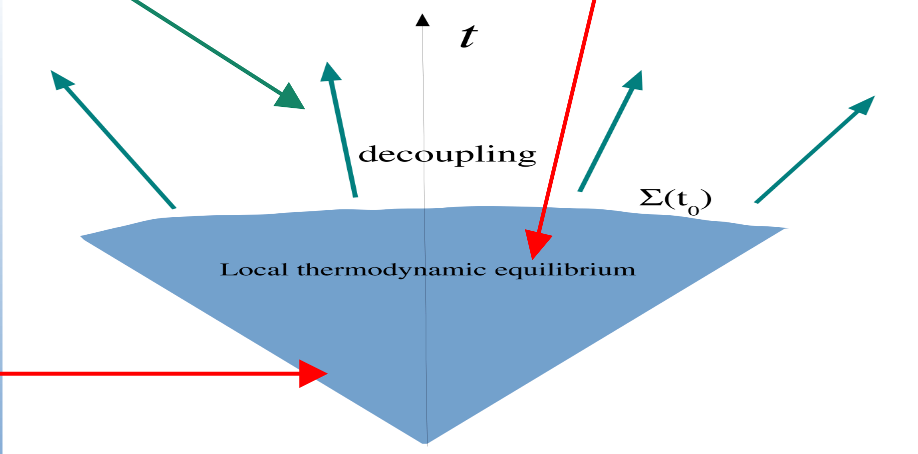
Local equilibrium term

$$\widehat{\rho} \simeq \frac{1}{Z}\exp\left[-\frac{a(t_0)\widehat{\mathcal{H}}(t_0)}{T(t_0)} + \frac{1}{a(t_0)T(t_0)}\int_{t_{eq}}^{t_0} \mathrm{d}t a^3(t)\dot{a}(t)\widehat{\mathcal{T}}(t)\right]$$

- Fluid still coupled
- Non-reversible effects

Entropy production

$$\nabla_\mu s^\mu \geq 0$$



Thank you for your attention!