Classification of Irreducible Harish-Chandra Modules for Map-Extended Witt Algebras

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Abstract

This poster classifies irreducible modules for map-extended Witt algebras with finite-dimensional weight spaces. These modules are either uniformly bounded or highest-weight modules. We further prove that all such modules are single-point evaluation modules ($n \geq 2$).

Preliminaries

Let $\mathfrak{g}$ be a Lie algebra with Cartan subalgebra $\mathfrak{h}$.

- A $\mathfrak{g}$-module V is a weight module if $V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$, where $V_\lambda = \{v \in V \mid h.v = \lambda(h)v \text{ for all } h \in \mathfrak{h}\}$.
- The weights of V are the elements of $\mathrm{Supp}(V) = \{\lambda \in \mathfrak{h}^* : V_\lambda \neq 0\}$, and corresponding V_λ are the weight spaces.
- A Harish-Chandra module is a weight module with finite-dimensional weight spaces.
- A $\mathfrak{g}$-module V is called uniformly bounded if it is a Harish-Chandra module and the dimensions of all its weight spaces are uniformly bounded.
- The Lie algebra $\mathfrak{g} \otimes B$ is called a map algebra associated with $\mathfrak{g}$, with the following bracket operation:

$$[x \otimes a, y \otimes b] = [x, y] \otimes ab, \quad x, y \in \mathfrak{g},\ a, b \in B.$$

- A representation (V, ρ) of $\mathfrak{g} \otimes B$ is called a single-point generalized evaluation module if $\rho : \mathfrak{g} \otimes B \to \mathrm{End}(V)$ factors through $\mathfrak{g} \otimes (B/\mathfrak{m}^k)$ for some maximal ideal $\mathfrak{m} \subset B$ and $k \in \mathbb{N}$.
- If $k = 1$, V is called a single-point evaluation module.

Overview

- Representations of $\mathfrak{g} \otimes \mathbb{C}[t]$ and $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ were studied in great detail by Chari and Pressley, where $\mathfrak{g}$ is a finite-dimensional simple algebra.
- The classification of finite-dimensional irreducible representations of $\mathfrak{g} \otimes A_n$ was done by Rao in 2001.
- Michael Lau provided a complete classification of finite-dimensional irreducible representations of $\mathfrak{g} \otimes B$ in 2015.
- Over time, many researchers have contributed to classifying irreducible Harish-Chandra modules for various categories of Lie algebras and superalgebras.

Virasoro algebra

- Consider $A_1 = \mathbb{C}[t_1^{\pm 1}]$. $W_1 = \mathrm{Der}(A_1)$ is the Lie algebra of polynomial vector fields on the circle, with basis $\{t_1^m d_1 : m \in \mathbb{Z}\}$ and bracket:

$$[t_1^m d_1, t_1^n d_1] = (n - m)t_1^{m+n} d_1 \text{ for all } m, n \in \mathbb{Z}.$$

- The Virasoro algebra $\mathrm{Vir} = W_1 \oplus \mathbb{C}C$ has the bracket:

$$[t_1^m d_1, t_1^n d_1] = (n - m)t_1^{m+n} d_1 + \delta_{m,-n}\tfrac{m^3 - m}{12} C,$$

$$[x_n, C] = 0, \quad \text{for all } m, n \in \mathbb{Z}.$$

- Vir= $\bigoplus_{m \in \mathbb{Z}} (\mathrm{Vir})_m$.
- $(\mathrm{Vir})_0 = \mathrm{Vir}^0 = \mathbb{C}t^0 d_1 \oplus \mathbb{C}C$ (Cartan).
- $(\mathrm{Vir})_m = \mathbb{C}t_1^m d_1$ for $m \neq 0$.
- $\mathrm{Vir} = \mathrm{Vir}^- \oplus \mathrm{Vir}^0 \oplus \mathrm{Vir}^+$ (standard triangular decomposition), where $\mathrm{Vir}^\pm = \bigoplus_{\pm m \in \mathbb{N}} (\mathrm{Vir})_m$.
- A Vir-module V is called a highest-weight module if it is a weight module and there exists a non-zero weight vector v such that $\mathrm{Vir}^+.v = 0$ and $U(\mathrm{Vir}).v = V$.

Classification (Virasoro algebra)

- Define the class of intermediate modules $V_{\alpha,\beta}$ for Vir with two parameters $\alpha, \beta \in \mathbb{C}$. As a vector space $V_{\alpha,\beta} = \bigoplus_{n \in \mathbb{Z}} \mathbb{C}v_n$ and Vir action on $V_{\alpha,\beta}$ is given by

$$t_1^n d_1.v_k = (\alpha + k + n\beta)v_{k+n},\ C.v_k = 0\ \forall n, k \in \mathbb{Z}.$$

- **Lemma** (Kac, Raina, *Adv. Ser. Math. Phys.*, 1987):
  - The Vir-module $V_{\alpha,\beta} \simeq V_{\alpha+m,\beta}$ for all $m \in \mathbb{Z}$.
  - The Vir-module $V_{\alpha,\beta}$ is irreducible if and only if $\alpha \notin \mathbb{Z}$ or $\beta \notin \{0, 1\}$.
  - $V_{0,0}$ has a unique trivial proper submodule $\mathbb{C}v_0$ and $V'_{0,0} = V_{0,0}/\mathbb{C}v_0$.
  - $V_{0,1}$ has a unique non-zero proper submodule $V'_{0,1} = \bigoplus_{i \neq 0} \mathbb{C}v_i$.
  - $V'_{\alpha,0} \simeq V'_{\alpha,1}$ for all $\alpha \in \mathbb{C}$.

- **Conjecture** (Kac, 1982):
  - Any irreducible Harish-Chandra module over the Virasoro algebra is a highest-weight module, a lowest-weight module or a module of the intermediate series.
- **Proof of Kac's Conjecture**: (Mathieu *Inventiones Mathematicae*, 1992).

Classification (Map Virasoro algebra)

- Vir(B):=Vir$\otimes B$ is a Lie algebra (Map Virasoro algebra) with the brackets $[x \otimes b_1, y \otimes b_2] = [x, y] \otimes b_1 b_2\ \forall x, y \in$ Vir and $b_1, b_2 \in B$.
- $\mathrm{Vir}(B) = \mathrm{Vir}(B)^- \oplus \mathrm{Vir}(B)^0 \oplus \mathrm{Vir}(B)^+$ (standard triangular decomposition), where

$$\mathrm{Vir}(B)^\pm = \mathrm{Vir}^\pm \otimes B \text{ and } \mathrm{Vir}(B)^0 = \mathrm{Vir}^0 \otimes B.$$

- Let $\mathbb{C}_\psi$ be the one dimensional representation of $\mathrm{Vir}(B)^0$, where $\psi \in \mathrm{Hom}(\mathrm{Vir}(B)^0, \mathbb{C})$. Make $\mathbb{C}_\psi$ as a $\mathrm{Vir}(B)^0 \oplus \mathrm{Vir}(B)^+$-module with a trivial action of $\mathrm{Vir}(B)^+$on it. Then consider the Verma module

$$M(\psi) := \mathrm{Ind}_{\mathrm{Vir}(B)^0 \oplus \mathrm{Vir}(B)^+}^{\mathrm{Vir}(B)} \mathbb{C}_\psi.$$

- Let $V(\psi)$ be the unique irreducible quotient of $M(\psi)$.
- Let $B = \mathbb{C}[t, t^{-1}]$.
- **Theorem** (Guo, Lu, Zhao, *Forum Math.*, 2011):
  - Let V be an irreducible Harish-Chandra $\mathrm{Vir} \otimes \mathbb{C}[t, t^{-1}]$-module. Then V is a uniformly bounded single-point evaluation module or a highest or lowest weight module. Further, if V is a highest (lowest)-weight module, it is a tensor product of finitely many irreducible single-point generalised evaluation highest (lowest)-weight modules.
- **Theorem** (Savage, *Transformation Groups*, 2012):
  - Any irreducible Harish-Chandra $\mathrm{Vir}(B)$-module is a single-point evaluation module, a highest-weight module or a lowest-weight module. Further, if V is a highest (lowest)-weight module, it is a tensor product of finitely many irreducible single-point generalised evaluation highest(lowest)-weight modules.

Witt algebra

- $A_n := \mathbb{C}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$.
- Define $t^{\mathbf{m}} := t_1^{m_1} \cdots t_n^{m_n} \in A_n$ for $\mathbf{m} = (m_1, \ldots, m_n) \in \mathbb{Z}^n$ and $d_i = t_i \frac{d}{dt_i}$ for $1 \leq i \leq n$.
- $W_n := \mathrm{Der}(A_n)$ (Witt algebra), the set $\{t^{\mathbf{m}} d_i : \mathbf{m} \in \mathbb{Z}^n, 1 \leq i \leq n\}$ forms a $\mathbb{C}$-basis for W_n and the bracket in W_n is given by

$$[t^{\mathbf{m}} d_i, t^{\mathbf{k}} d_j] = k_i t^{\mathbf{m}+\mathbf{k}} d_j - m_j t^{\mathbf{m}+\mathbf{k}} d_i.$$

- Unlike the case $n = 1$, W_n for $n \geq 2$ is centrally closed.

Representations of Witt algebra

- Larsson-Shen modules (or modules of tensor fields on a torus): For $n \geq 2$, let

$$T(U, \gamma) := t^\gamma A_n \otimes U,$$

where U is a finite-dimensional irreducible $\mathfrak{gl}_n(\mathbb{C})$ representation and $\gamma \in \mathbb{C}^n$, which has the following W_n-module structure:

$$t^{\mathbf{m}} d_i.(t^{\gamma+\mathbf{s}} \otimes u) = (\gamma_i + s_i)t^{\gamma+\mathbf{s}+\mathbf{m}} \otimes u + \textstyle\sum_{j=1}^n m_j t^{\gamma+\mathbf{s}+\mathbf{m}} \otimes E_{ji} u,$$

where $\mathbf{m}, \mathbf{s} \in \mathbb{Z}^n$, $u \in U$, $1 \leq i \leq n$, and E_{ji} is the $n \times n$ matrix with 1 in the (j, i)th position and 0 elsewhere.
- Let G be a subgroup of $\mathbb{Z}^n$ and $\boldsymbol{\beta} \in \mathbb{Z}^n - \{0\}$ such that $\mathbb{Z}^n = G \oplus \mathbb{Z}\boldsymbol{\beta}$.
- Subalgebras of W_n:

$$(W_n)_G^0 = \oplus_{\alpha \in G} (W_n)_\alpha,$$

$$(W_n)_G^- = \oplus_{\alpha \in G, k \in \mathbb{N}} (W_n)_{\alpha - k\beta},$$

$$(W_n)_G^+ = \oplus_{\alpha \in G, k \in \mathbb{N}} (W_n)_{\alpha + k\beta}.$$

- Triangular decomposition:

$$W_n = (W_n)_G^- \oplus (W_n)_G^0 \oplus (W_n)_G^+.$$

- Let X be a simple weight module over $(W_n)_G^0$. Setting $(W_n)_G^+ X = 0$, we get a module over $(W_n)_G^0 \oplus (W_n)_G^+$.
- The generalized Verma module is defined as:

$$M(G, \beta, X) = \mathrm{Ind}_{(W_n)_G^0 \oplus (W_n)_G^+}^{W_n} X,$$

with a unique simple quotient $L(G, \beta, X)$.

Classification (Witt algebra)

- **Conjecture** (Rao, 2004): Any non-trivial irreducible Harish-Chandra module over the Witt algebra is either a highest-weight module or a module of tensor fields on a torus and their quotient.
- **Theorem** (Guo, Liu, Zhao, Ark. Mat., 2014)
  - Let V be a simple Harish-Chandra $W_n \ltimes A_n$-module. Then V is either uniformly bounded or highest weight module. Moreover, if V is uniformly bounded, then
    - If $t^{\mathbf{0}}$ acts as an identity, then A_n acts associatively on V and $V \cong T(U, \gamma)$, where U is an irreducible finite dimensional $\mathfrak{gl}_n$-module and $\gamma \in \mathbb{C}^n$.
    - If $t^{\mathbf{0}}$ acts as a zero then $A_n.V = 0$ and hence V is an irreducible W_n-module.
- **Proof of Rao's Conjecture** (Billig, Futorny, *CRELLE*, 2016).

Classification (map Witt algebra)

- Let S be any finite-dimensional abelian Lie algebra over $\mathbb{C}$.
- The Witt algebra W_n acts on $S \otimes A_n$ by derivations:

$$[t^{\mathbf{r}} d_i, s \otimes t^{\mathbf{m}}] = m_i s \otimes t^{\mathbf{m}+\mathbf{r}}, \text{ for all } 1 \leq i \leq n, \mathbf{m}, \mathbf{r} \in \mathbb{Z}^n \text{ and } s \in S.$$

The emerging Lie algebra $\mathfrak{L}_{S,n} := W_n \ltimes (S \otimes A_n)$ is called an extended Witt algebra.
- The Lie algebra $\mathfrak{L}_{S,n}(B) := (W_n \ltimes (S \otimes A_n)) \otimes B$ is called a map extended Witt algebra.
- $H := (\bigoplus_{i=1}^n \mathbb{C}d_i) \oplus \mathbb{C}$ be an abelian subalgebra of $\mathfrak{L}_{S,n}(B)$ which plays a role of a Cartan subalgebra for $\mathfrak{L}_{S,n}(B)$.
- When dim $S = 1$, $\mathfrak{L}_{S,n} = \mathfrak{L}$.
- Let G be a subgroup of $\mathbb{Z}^n$ and $\boldsymbol{\beta} \in \mathbb{Z}^n - \{0\}$ such that $\mathbb{Z}^n = G \oplus \mathbb{Z}\boldsymbol{\beta}$.
- Subalgebras of $\mathfrak{L}_{S,n}(B)$:

$$\mathfrak{L}_{S,n}^-(B) = \oplus_{d \in \mathbb{N}} (\mathfrak{L}_{S,n}(B))_{G - d\beta},$$

$$\mathfrak{L}_{S,n}^+(B) = \oplus_{d \in \mathbb{N}} (\mathfrak{L}_{S,n}(B))_{G + d\beta},$$

$$(\mathfrak{L}_{S,n}(B))_G = \oplus_{\mathbf{r} \in G} (\mathfrak{L}_{S,n}(B))_{\mathbf{r}}.$$

- Triangular decomposition:

$$\mathfrak{L}_{S,n}^-(B) \oplus (\mathfrak{L}_{S,n}(B))_G \oplus \mathfrak{L}_{S,n}^+(B),$$

- An $\mathfrak{L}_{S,n}(B)$-module V is a highest-weight module if it is a weight module with a non-zero weight vector $v \in V$ such that $\mathfrak{L}_{S,n}^+(B) \cdot v = 0$ and $U(\mathfrak{L}_{S,n}(B)) \cdot v = V$.
- Let Y be a simple weight module over $(\mathfrak{L}_{S,n}(B))_G$. Setting $\mathfrak{L}_{S,n}^+(B)Y = 0$, we get a module over $(\mathfrak{L}_{S,n}(B))_G) \oplus \mathfrak{L}_{S,n}^+(B)$.
- The generalized Verma module is defined as:

$$M_{\mathfrak{L}_{S,n}(B)}(G, \beta, Y) = \mathrm{Ind}_{(\mathfrak{L}_{S,n}(B))_G \oplus \mathfrak{L}_{S,n}^+(B)}^{\mathfrak{L}_{S,n}(B))} Y$$

- $L_{\mathfrak{L}_{S,n}(B)}(G, \beta, Y)$ a unique simple quotient.

Main Theorem
(Sharma, Chakraborty, —, Rao, *J. Algebra*, 2024)

Let V be a non-trivial irreducible $\mathfrak{L}_{S,n}(B)$-module with finite-dimensional weight spaces. Then V is either a uniformly bounded module or a highest-weight module. Further,

- If V is uniformly bounded, then V can be considered as an irreducible uniformly bounded module for $\mathfrak{L}(B)$ or $W_n(B)$ ($n \geq 1$). Moreover, considered as an $\mathfrak{L}(B)$ or $W_n(B)$-module, V is a single-point evaluation module.
- If V is a highest-weight module, then V can be considered as an irreducible highest-weight module for $\mathfrak{L}(B)$ or $W_n(B)$ ($n \geq 2$). Moreover, considered as an $\mathfrak{L}(B)$ or $W_n(B)$-module, V is a single-point evaluation module.

References

[1] S. S. Sharma, P. Chakraborty, R. K. Pandey, and S. Eswara Rao. (2024): Representations of map extended Witt algebras. *J. Algebra*. 639: 327–353.