
V&V

Expressive curves

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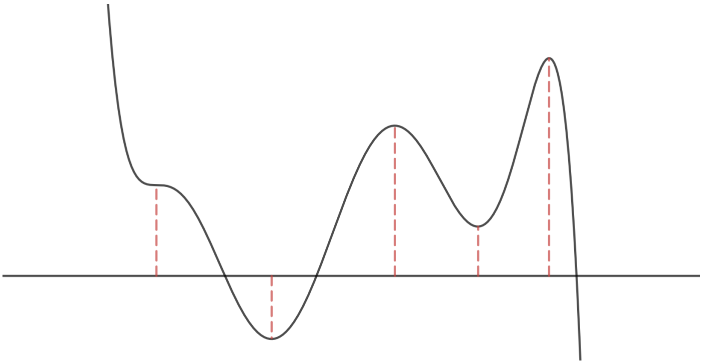
S. Fomin and E. Shustin, Expressive curves, *Comm. Amer. Math. Soc.* **3** (2023).

S. Fomin, P. Pylyavskyy, E. Shustin, and D. Thurston, Morsifications and mutations, *J. Lond. Math. Soc.* **105** (2022).

Rolle's Theorem

Theorem [M. Rolle, 1690]

A real polynomial has a critical point between each pair of real roots.

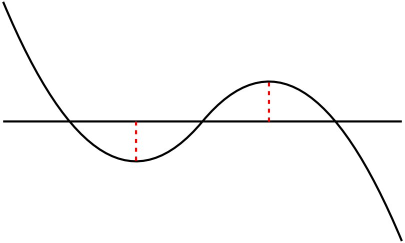


A. L. Cauchy [1823] extended this theorem to differentiable functions.

Rolle's Theorem (strong version)

Theorem

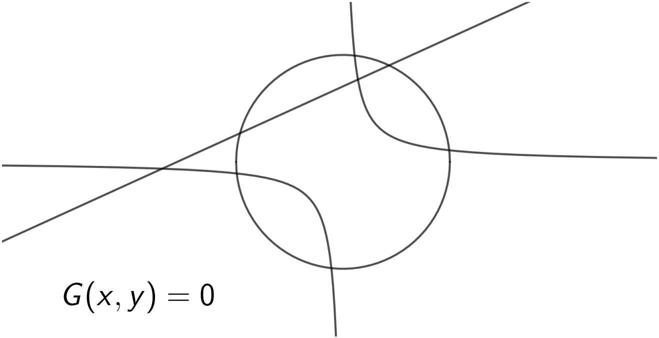
Let $g(x) \in \mathbb{R}[x]$ be a polynomial whose roots are real and distinct. Then g has exactly one critical point between each pair of consecutive roots, and no other critical points (even over $\mathbb{C}$).



Rolle's theorem (as proved by Cauchy) is an analytic statement. The above version relies on algebra.

What are the multivariate analogues of these results?

Rolle's Theorem in 2D



$G(x, y) = 0$

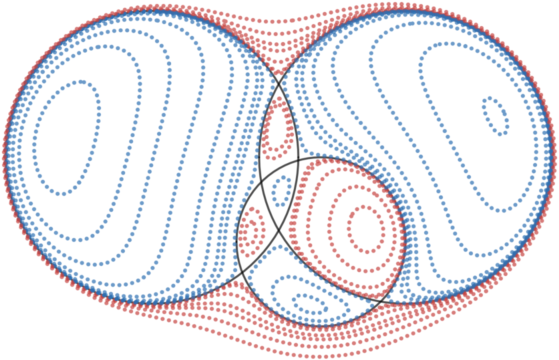
Where are the critical points of a bivariate polynomial G located?

There is a saddle at each self-intersection (a hyperbolic node).
There is at least one extremum inside each bounded region.

When is the above list of critical points complete?

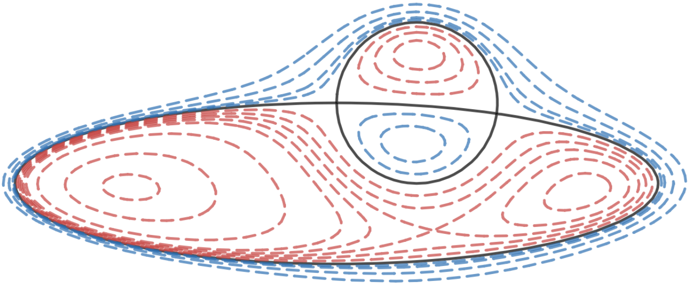
Expressive curves: informal definition

A real nodal algebraic curve $C = \{G(x, y) = 0\}$ (or the polynomial G) is called expressive if G has the smallest number of (complex) critical points allowed by the topology of the set of real points of C.



This notion can be viewed as a bivariate counterpart of the notion of a real-rooted univariate polynomial.

Example of a non-expressive curve



Our main result is a criterion for deciding whether a given curve is expressive. To state this criterion, we will need some preparations.

Expressive curves: formal definition

$G(x,y) \in \mathbb{R}[x,y] \subset \mathbb{C}[x,y]$	polynomial with real coefficients
$C = \{(x,y) \in \mathbb{C}^2 \mid G(x,y) = 0\}$	affine plane algebraic curve
$C_\mathbb{R} = \{(x,y) \in \mathbb{R}^2 \mid G(x,y) = 0\}$	set of real points of C

Definition

Polynomial G (resp., curve C) is called expressive if

- all critical points of G are real;
- at each critical point, G has a nondegenerate Hessian;
- each bounded connected component of $\mathbb{R}^2 \setminus C_\mathbb{R}$ contains exactly one critical point of G;
- each unbounded component of $\mathbb{R}^2 \setminus C_\mathbb{R}$ contains no critical points;
- $C_\mathbb{R}$ is connected, and contains infinitely many points.

L^∞-regular curves

x, y, z	projective coordinates in $\mathbb{P}^2$
$L^\infty = \{z = 0\}$	line at infinity
$\mathbb{C}^2 = \mathbb{P}^2 \setminus L^\infty$	affine complex plane
$F(x, y, z)$	homogeneous polynomial over $\mathbb{C}$
$C = \{F(x, y, z) = 0\} \subset \mathbb{P}^2$	complex projective curve

Definition

C is called L^∞-regular if, for any $p \in \{\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = 0\} \cap L^\infty \subset C \cap L^\infty$, we have

$$(*) \qquad (\{\frac{\partial F}{\partial x} = 0\} \cdot \{\frac{\partial F}{\partial y} = 0\})_p = \mu(C, p) + (C \cdot L^\infty)_p - 1.$$

An affine curve is L^∞-regular if its projective closure is L^∞-regular.

In general, LHS $\geq$ RHS in $(*)$. In order to have LHS$>$RHS, C must exhibit some rather exotic behavior at p. All expressive curves of degree ≤ 4 are L^∞-regular.

Polynomial and trigonometric curves

Proposition

Let $C \subset \mathbb{C}^2$ be a real plane curve. The following are equivalent:

- C admits a real polynomial parametrization $t \mapsto (P(t), Q(t))$;
- C is a real rational curve with a unique local branch at infinity.

We then say that C is a (real) polynomial curve.

Proposition

Let $C \subset \mathbb{C}^2$ be a real plane curve. The following are equivalent:

- C admits a real trigonometric parametrization
$$t \mapsto (P(\cos t, \sin t), Q(\cos t, \sin t));$$
- C is a real rational curve with an infinite real point set and with two complex conjugate local branches at infinity.

We then say that C is a (real) trigonometric curve.

Expressivity criterion

Theorem [SF–E. Shustin]

Let $C \subset \mathbb{C}^2$ be a reduced real algebraic curve, with all irreducible components real. The following are equivalent:

- C is expressive and L^∞-regular;
- each component of C is either trigonometric or polynomial, all singular points of C in the affine plane are hyperbolic nodes, and the set of real points of C in the affine plane is connected.

In particular, any polynomial or trigonometric curve all of whose singular points (away from infinity) are real hyperbolic nodes is both expressive and L^∞-regular.

An expressive curve can have complicated singularities at infinity. All critical points that escaped to infinity must concentrate in a small number of locations.

Expressivity criterion

Theorem [SF–E. Shustin]

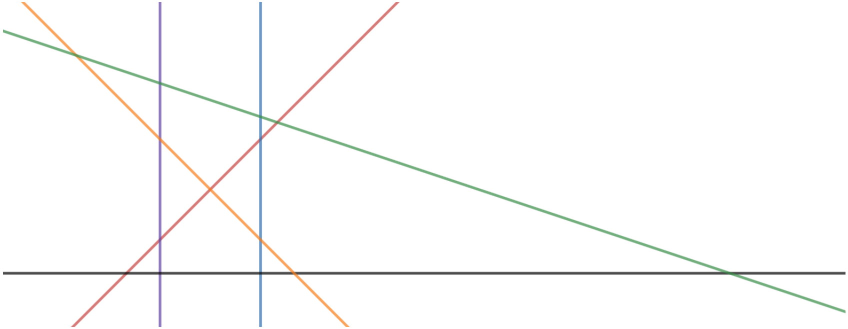
Let $C \subset \mathbb{C}^2$ be a reduced real algebraic curve, with all irreducible components real. The following are equivalent:

- C is expressive and L^∞-regular;
- each component of C is either trigonometric or polynomial, all singular points of C in the affine plane are hyperbolic nodes, and the set of real points of C in the affine plane is connected.

Expressivity is a rather restrictive property. All components must be either polynomial or trigonometric. All their points of intersection must be real hyperbolic nodes.

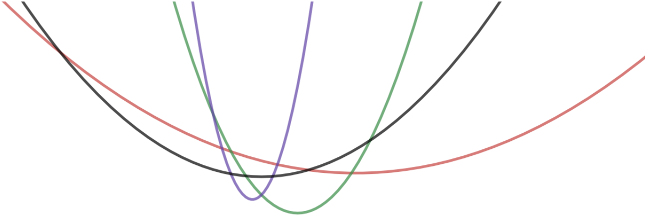
Still, the “zoo” of expressive curves is surprisingly rich.

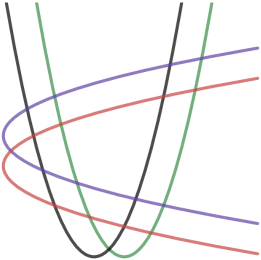
Example I: Line arrangements

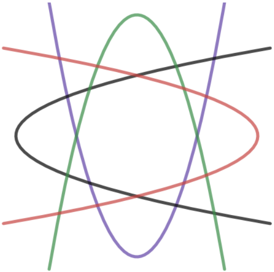


Any nodal connected real line arrangement is an expressive curve.

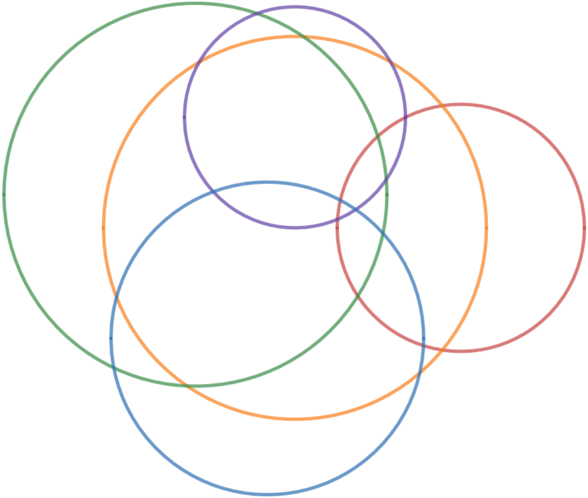
Example II: Arrangements of parabolas



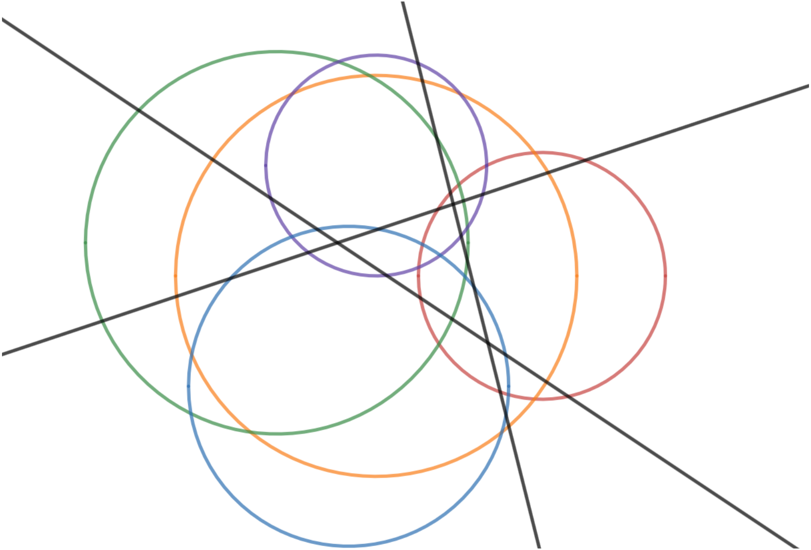




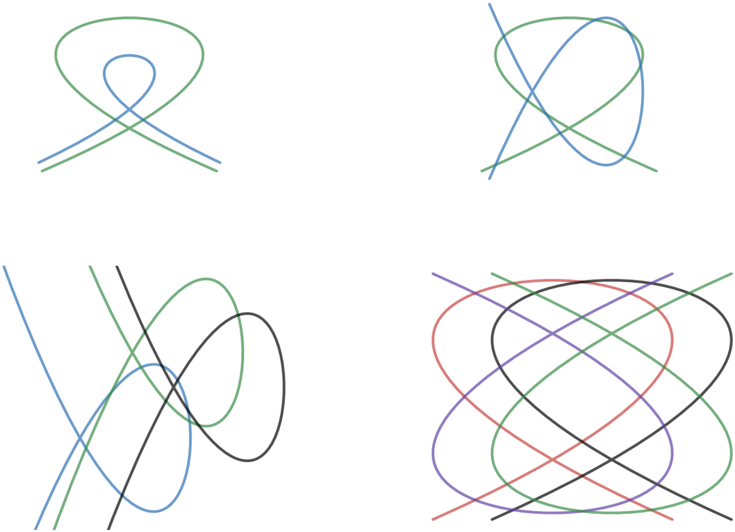
Example III: Circle arrangements



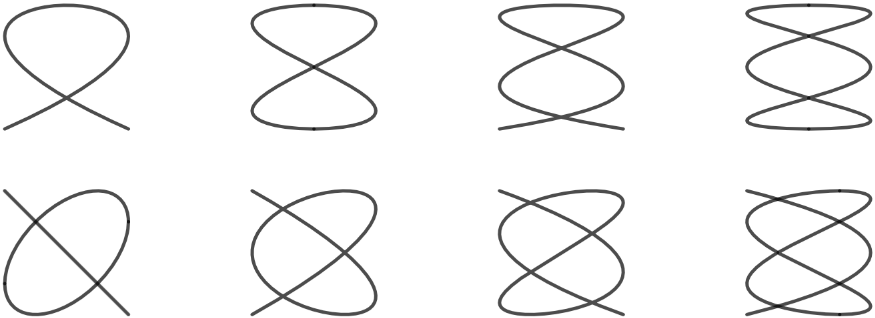
Example IV: Arrangements of lines and circles



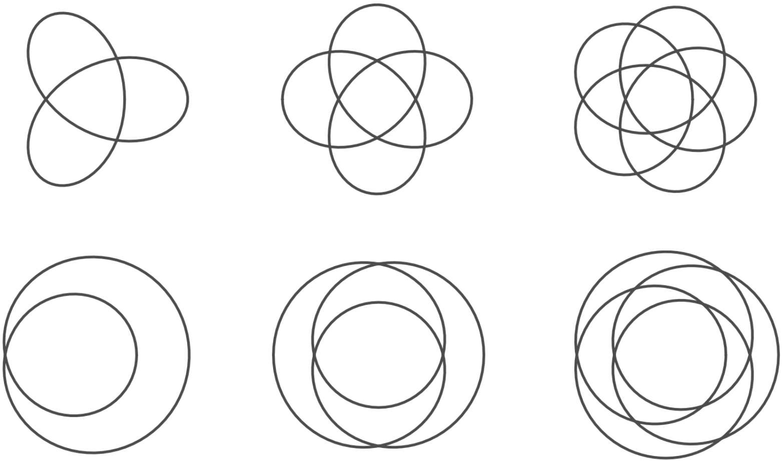
Example V: Arrangements of nodal cubics



Example VI: Lissajous-Chebyshev curves



Example VII: Hypotrochoids and epitrochoids



Local theory

The theory of expressive curves is a "global" counterpart of the "local" theory of morsifications, developed in the 70s by V. Arnold, N. A'Campo, and S. Guseĭn-Zade.

This theory studies isolated singularities of algebraic curves in the complex affine plane. They are viewed up to topological equivalence, i.e., up to homeomorphisms of a neighborhood of the singular point.

The link $L(C, z)$ of an isolated singular point z on a plane curve C is defined by intersecting C with a small sphere centered at z.

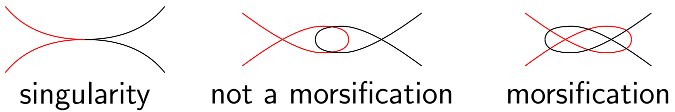
The link $L(C, z)$ completely determines—and is determined by—the local topology of C at z.

How can we compute these links?

Real singularities and their morsifications

Any complex plane curve singularity has at least one real form, including a totally real form (with all local branches real).

A (real) morsification of a real singularity is a real nodal deformation that has the maximal possible number of real hyperbolic nodes.



For a morsification $\{f_t(x,y)=0\}$, all critical points of f_t that lie near the original singular point z are real, with nondegenerate Hessian. The locations of these critical points are analogous to the case of expressive curves. In particular, all saddles are at the zero level.

Existence of morsifications

Theorem (N. A'Campo–S. Guseĭn-Zade, 1974)

Any totally real plane curve singularity possesses a real morsification.

Conjecture

Any real plane curve singularity possesses a real morsification.

Real forms and their morsifications

A typical complex singularity has several distinct real forms.

Real forms of the homogeneous singularity of degree 4

Four real lines:

$$x^3y - xy^3 = xy(x-y)(x+y) = 0.$$

Two real and two complex conjugate lines:

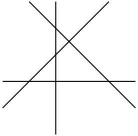
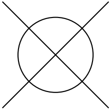
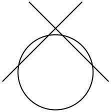
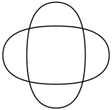
$$x^4 - y^4 = (x-y)(x+y)(x-iy)(x+iy) = 0.$$

Two pairs of complex conjugate lines:

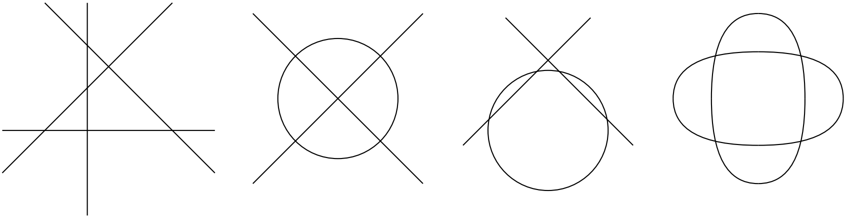
$$(x^2+4y^2)(4x^2+y^2) = (x-2iy)(x+2iy)(2x-iy)(2x+iy) = 0.$$

Each real form has its own morsifications (conjecturally at least one).

Morsifications of different real forms of a singularity

$x^3y-xy^3=0$	$xy(x-y+t)(x+y-2t)=0$	
$x^4-y^4=0$	$(x^2-y^2)(x^2+y^2-t^2)=0$	
	$(x^2-(y-1.2t)^2)(x^2+y^2-t^2)=0$	
$(x^2+4y^2)(4x^2+y^2)=0$	$(x^2+4y^2-t^2)(4x^2+y^2-t^2)=0$	

Equivalence between morsifications (or expressive curves)



Q How can one tell that all these morsifications arose from (different real forms of) the same complex singularity?

A Compute the link of the singularity from each morsification.

Q Viewing them as expressive curves, what do they have in common?

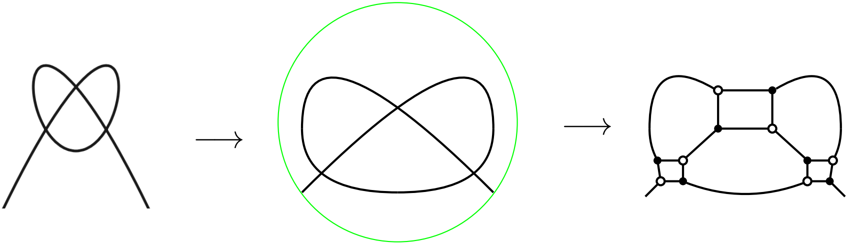
A All these curves have the same profile at infinity, hence the same link at infinity.

Next: computing these links.

Curve/germ → divide → plabic graph

A nodal curve in the real affine plane defines a divide in a disk.
In the local version, the divide comes from a morsification.

A divide, in turn, gives rise to a plabic (planar bicolored) graph:

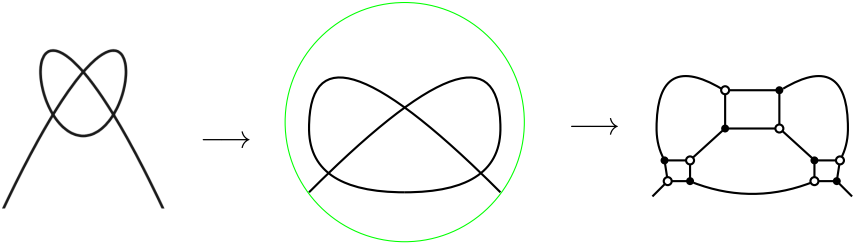


curve

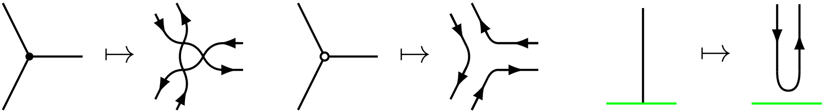
divide

plabic graph

Curve/germ → divide → plabic graph → link



We then use a construction [T. Kawamura + FPST] that associates a canonical (transverse) link in S^3 to any plabic graph.



Related constructions: W. Gibson–M. Ishikawa, D. Thurston, V. Shende–D. Treumann–H. Williams–E. Zaslow.

Link of a curve/germ

A divide (coming from a curve or a germ) yields a link:

$$\text{curve/germ} \to \text{divide} \to \text{plabic graph} \to \text{link.}$$

Theorem [FPST + N. A'Campo 1999]

In the setting of morsifications, the above construction produces a link isotopic to the link of the underlying singularity.

Thus, the divide of a morsification determines, in explicit combinatorial terms, the topological type of a singularity.

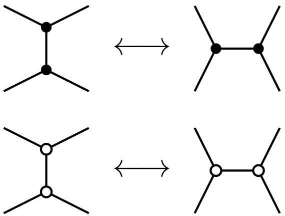
Theorem [S. F. and E. Shustin, 2024+]

In the setting of expressive curves, under mild technical assumptions, the above construction produces a link isotopic to the link of the curve at infinity.

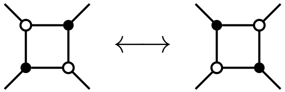
Move equivalence of plabic graphs

Two plabic graphs are called move equivalent if they can be obtained from each other via repeated application of the following moves:

flip moves



square move

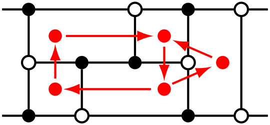


Theorem [FPST]

Move equivalent plabic graphs have isotopic links.

Plabic graph → quiver

Any plabic graph defines a quiver:



Plabic graph → quiver

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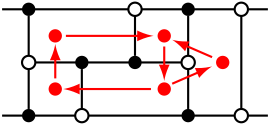
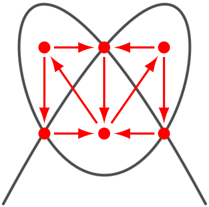
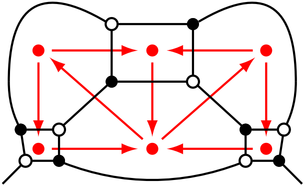


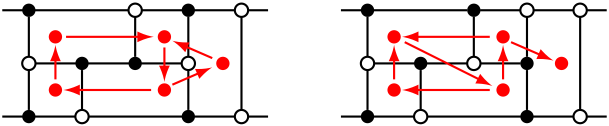
In the case of morsifications, we recover the (oriented version of) A'Campo-Guseĭn-Zade diagrams.





Topological equivalence *vs.* mutation equivalence

Local moves on plabic graphs translate into quiver mutations:



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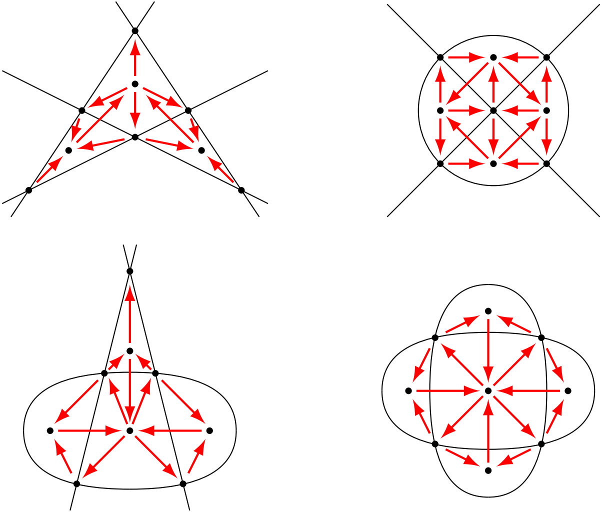
isotopy of links	$\Longleftarrow$	move equivalence of plabic graphs	$\Longrightarrow$	mutation equivalence of quivers

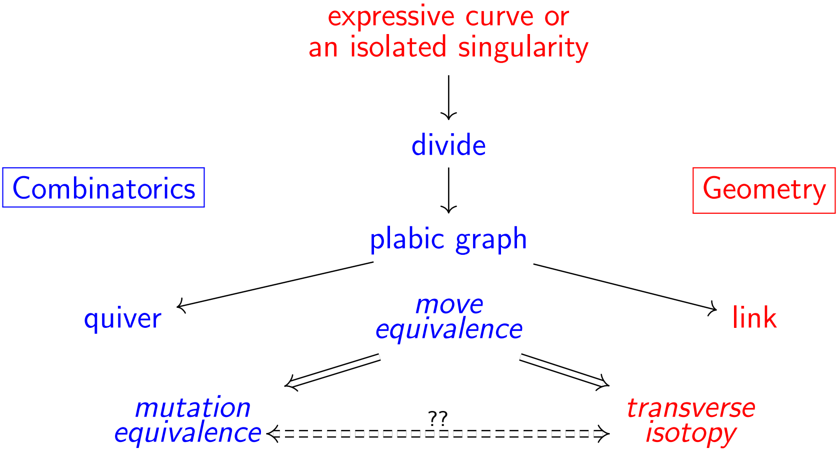
Conjecture

Quivers coming from two morsifications (resp., two expressive curves) are mutation equivalent if and only if the corresponding links are (transverse) isotopic.

Comparing morsifications of the same complex singularity, or expressive curves with the same profile at infinity

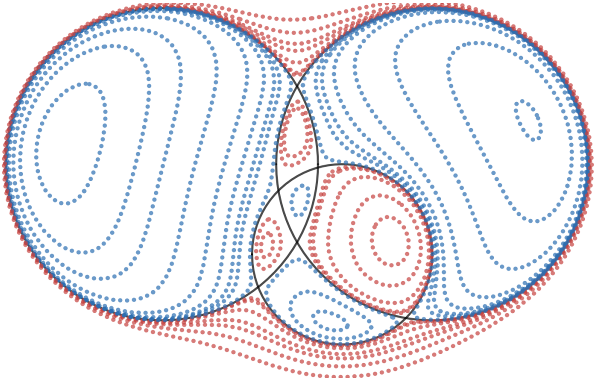


From plane curves to cluster algebras?



Is there a **cluster algebra** behind every equivalence class of plane curve singularities (resp., expressive curves)?

The end



CONGRATULATIONS, V&V!