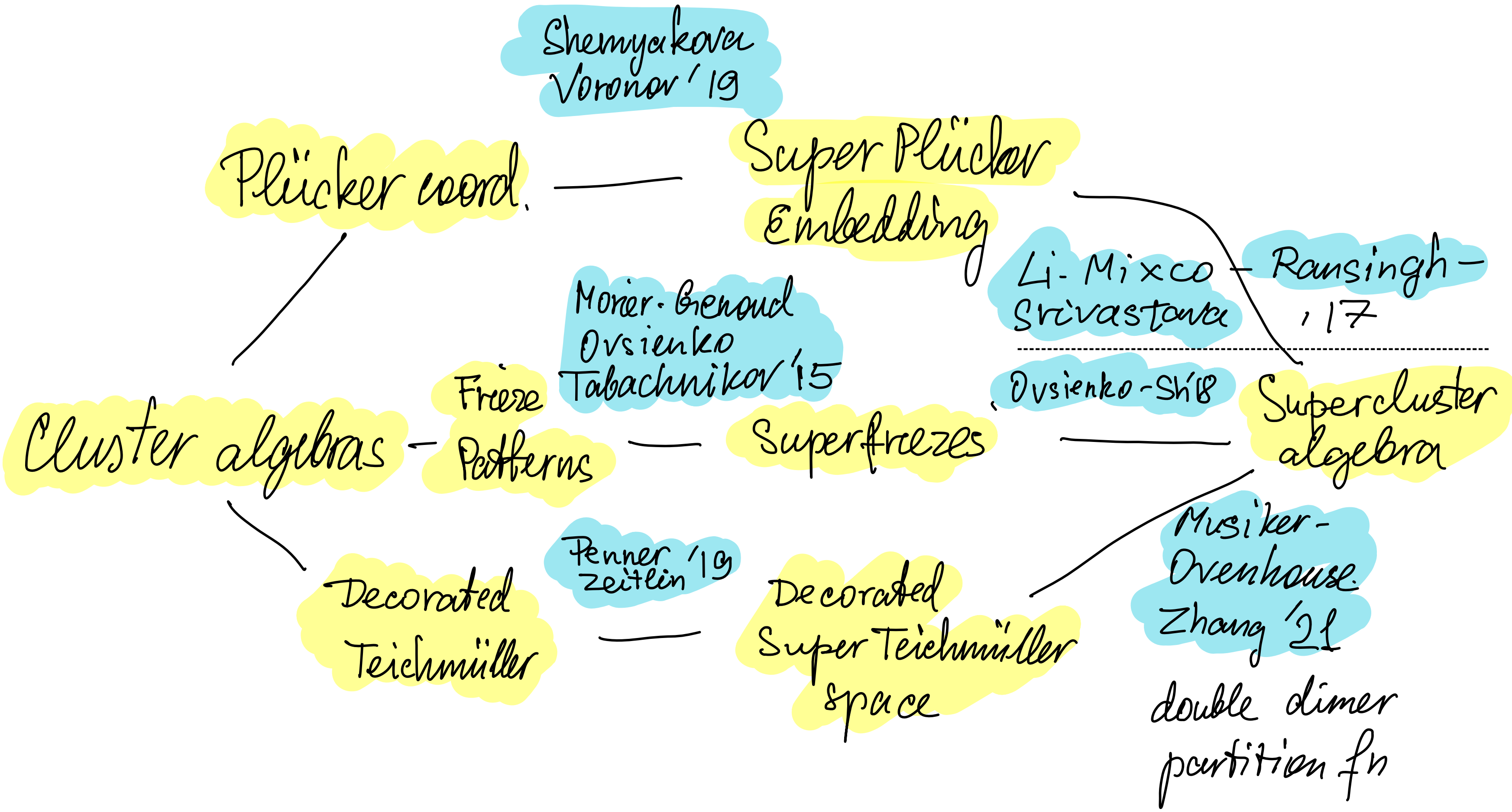
Cluster Superalgebras in progress

Ovenhouse, Ovsienko, Sh.



Volodya, Norway
Summer '24

Has nothing
to do with
this talk.

As I thought...

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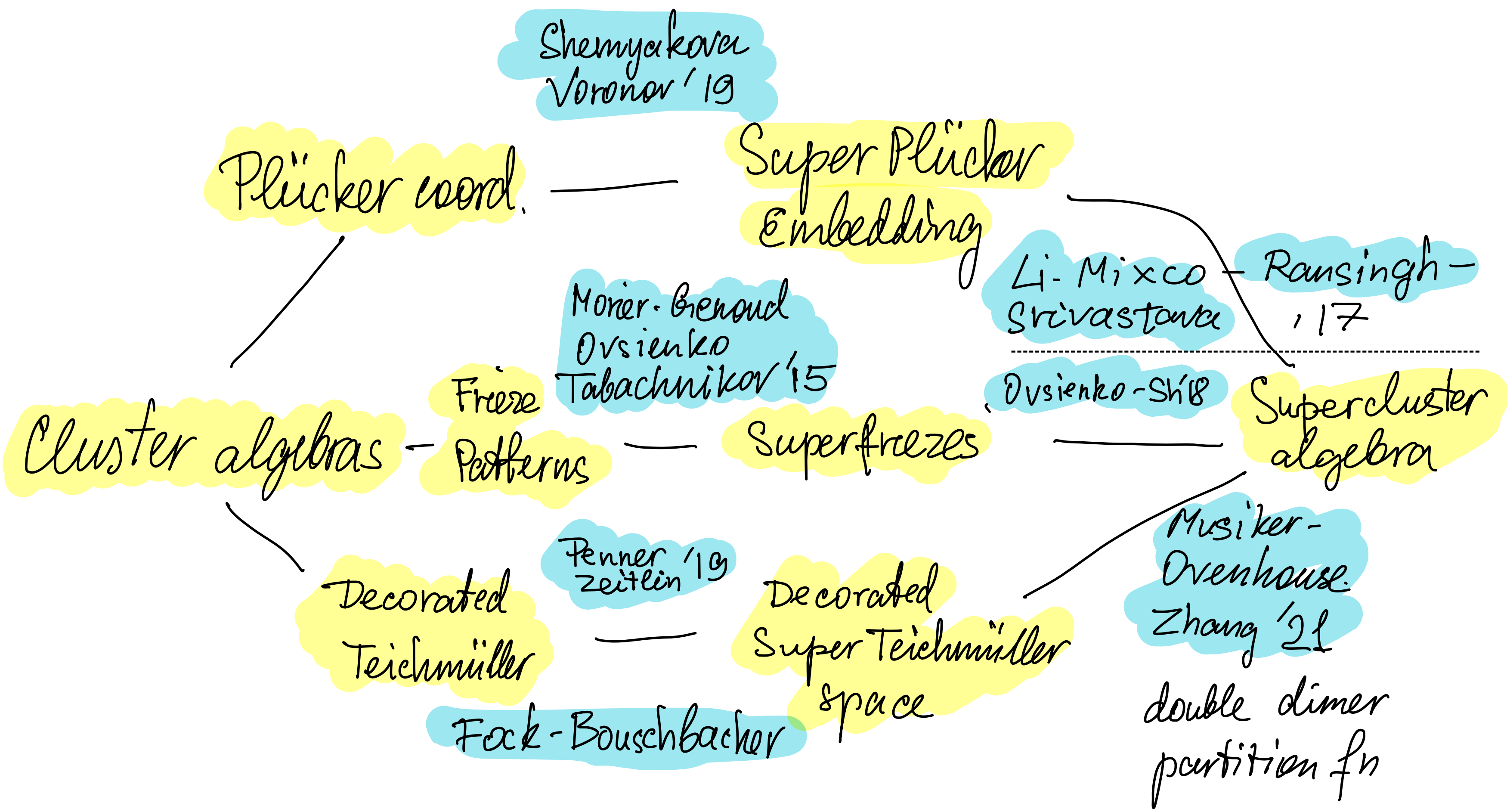
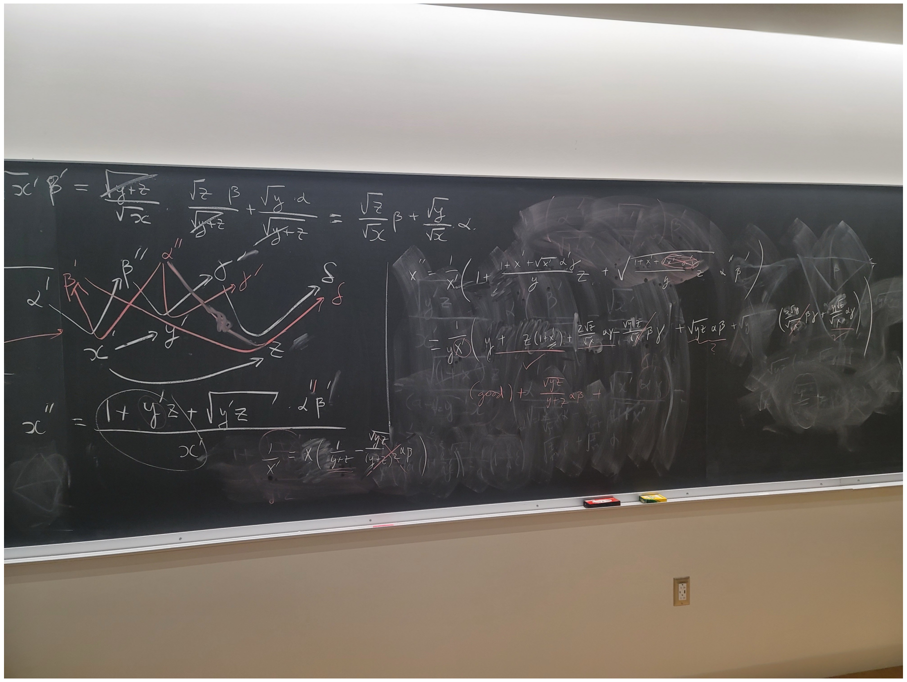




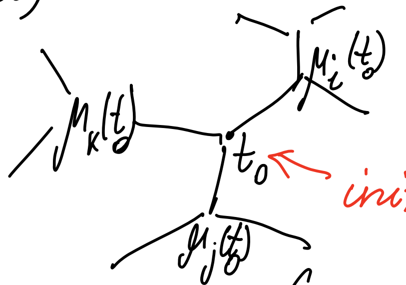
Working with Nick
in Yale

Working with Valya
in Reims.

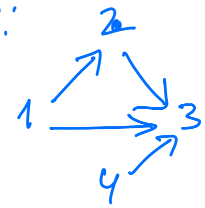
Cluster algebra

- Seed t := (cluster variables $x_1(t) \ldots x_n(t)$, directed graph $Q(t)$ called quiver)

Seeds are organized into graph



Example: $Q(t)$:



- mutation μ_k in direction k = transformation from seed t to seed t' (involution)

$$t \xrightarrow{\mu_k} t'$$

$$\text{seed } t' = \left(\begin{matrix} \text{cluster variables} \\ x_1(t') \ldots x_n(t') \end{matrix} , \begin{matrix} \text{quiver} \\ Q(t') = \mu_k(Q(t)) \end{matrix} \right)$$

$Q(t')$ depends on $Q(t)$

$x_j(t') = x_j(t)$ if $j \neq k$

$$x_k(t) \cdot x_k(t') = \prod_{i \to k} x_i(t) + \prod_{k \to l} x_l(t)$$

Properties of cluster mutations

Def Presymplectic two form (closed, maybe degenerate)

$$\omega(t) := \sum_{\text{edge of } Q:\ x_i(t) \to x_j(t)} d\log x_i(t) \wedge d\log x_j(t)$$

- $\omega(t') = \omega(t)$

Denote by t_0 initial cluster.

Laurent property: For any cluster t and any $1 \le j \le n$:
$x_j(t)$ is a Laurent polynomial
of $x_1(t_0), \dots, x_n(t_0)$.

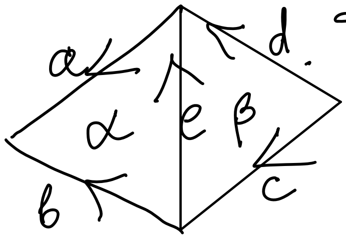
Goal: we want to extend classical construction to super case.

- Variables of two types: commutative ("even") x_i : $x_i x_j = x_j x_i$ + anticommutative ("odd") θ_i : $\theta_i \theta_j = -\theta_j \theta_i$

  $x_i \theta_j = \theta_j x_i$

- Each vertex of superquiver ~~carries~~ is either odd or even
- Subquiver Q_{even} of Q generated by even vertices
- If all $\theta_i = 0$ then mutation of even variables is determined in classical way by Q_{even}.
- Mutation rule for quiver Q_{even} is classical
- Mutations preserve some 2-form (to be discussed)
- Even variables satisfy some Laurent property (to be discussed)

Main example: (Penner - Zeitlin)



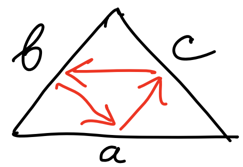
Surface is triangulated in ideal triangles

- Seed = triangulation
- Sides are even variables
- Δs with oriented bndry intervals are odd variables.

Even variables (=sides) are denoted by latin letters

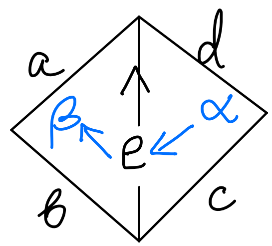
Odd variables (= Δ's) are denoted by greek letters.

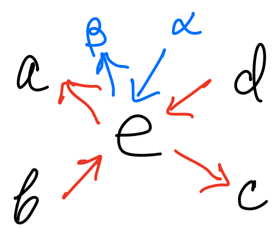
Quiver: **even-even edges** follows the standard (adjacency) cluster agreement in every Δ:



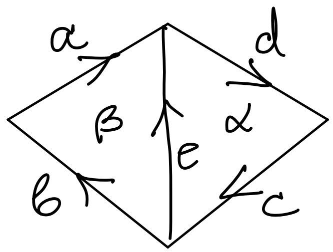
Quiver: **odd-even edges** every even variable has two odd neighbors

Quiver: no **odd-odd edges**

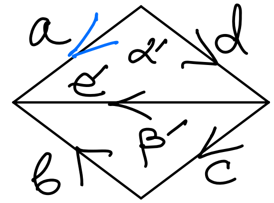




Super Mutation rule: flip



μ_e = anticlockwise rotation of $\vec{e}$



even variable $\quad ee' = \underbrace{bd + ac}_{\text{body}} + \underbrace{\sqrt{abcd}\,\alpha\beta}_{\text{soul}}$

odd variables: $\quad \alpha' = \dfrac{\sqrt{bd}\,\alpha - \sqrt{ad}\,\beta}{\sqrt{bd+ac}}$

$$\beta' = \frac{\sqrt{bd}\,\beta + \sqrt{ac}\,\alpha}{\sqrt{bd+ac}}$$

Note:

$$\boxed{\mu_e^8 = \mathrm{Id}}$$

$$\boxed{\alpha'\beta' = \alpha\beta}$$

Thm (P.Z) (2 form)

two form: $\hat{\omega}(t) = \sum_{\text{even } a\to b} \frac{da}{a} \wedge \frac{db}{b} + \sum_{\text{odd } \alpha} (d\alpha)^2$ is preserved.

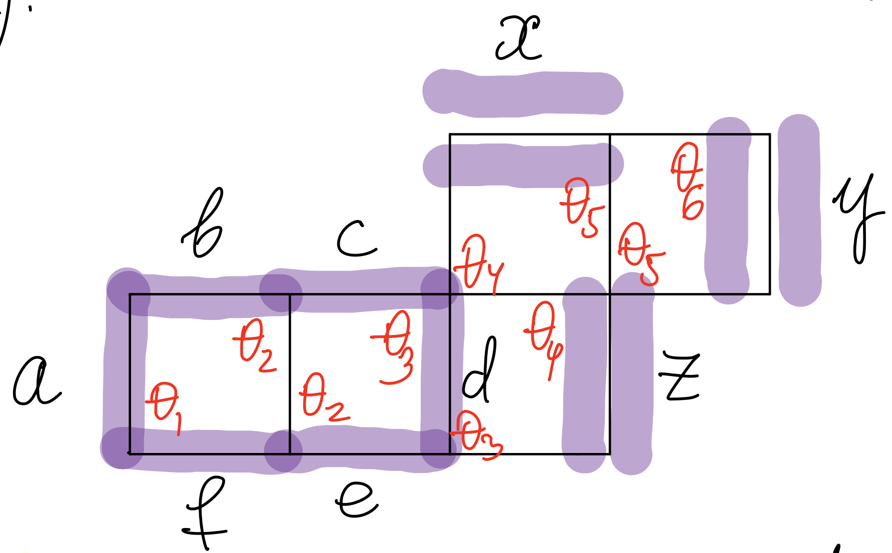
Laurent property

Musiker – Ovenhouse – Zhang for n-gon

Even cluster variables are expressed in terms of initial cluster t_0 as partition function of double dimer covering of snake graphs (alt., T-paths)

e.g.

weight $= xyz\sqrt{abcdef}\,\theta_1\theta_3$

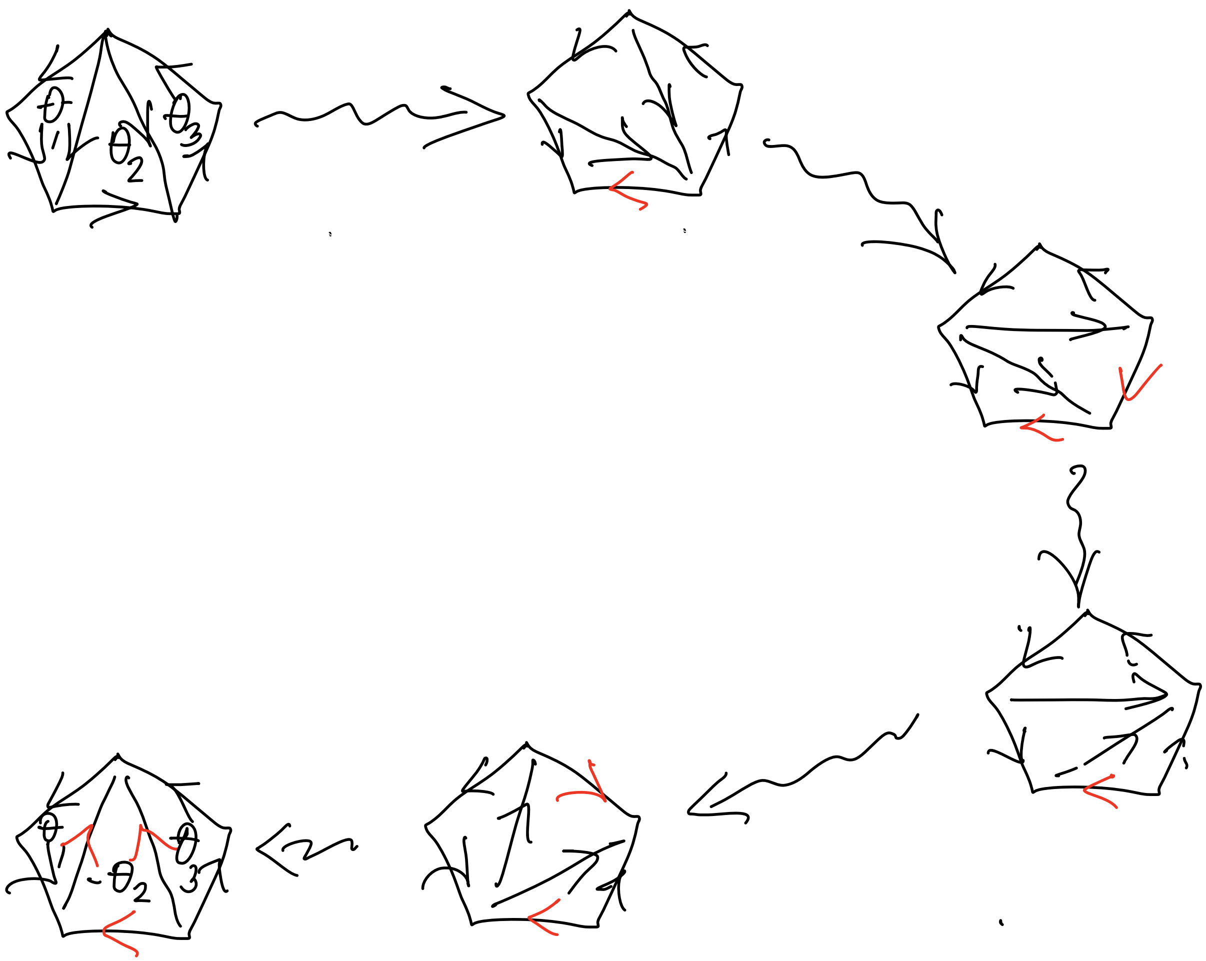


$$x_i(t) = \frac{1}{\text{cross}} \sum_{\substack{\text{double}\\ \text{dimer}\\ \text{cover}}} \text{weight}$$

monomial in initial cluster variables.

Cor. $x_j(t) \in \mathbb{Q}\left[x_i(t_0)^{\pm 1/2} \mid \theta_\ell(t_0)\right]$

Pentagon relation.



Problem: Extend Penner-Zeitlin construction to more general quivers.

Suggested generalization

Superquiver contains even and odd vertices

- every even vertex is connected with exactly one incomming odd vertex and exactly one outgoing

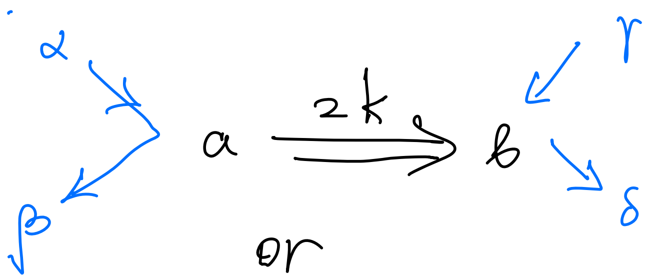
$$\begin{tikzcd} \beta & & \alpha \arrow[dl] \\ & a \arrow[ul] & \end{tikzcd}$$

- Every pair of even a and b connected by odd number of arrows $a \xRightarrow{2k+1} b$ has exactly 3 odd neighbors

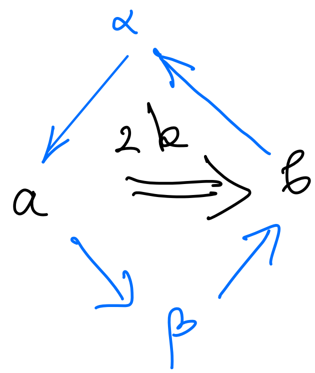
$$\begin{tikzcd} \alpha & & \beta \arrow[dl] \arrow[dr] & & \gamma \\ & a \arrow[ul] \arrow[rr, Rightarrow, "2k+1"] & & b \arrow[ur] & \end{tikzcd}$$

orientation is not fixed like in the Fig

- Every pair of even x_a, x_b connected with even number of arrows (or no arrows) has even number of odd neighbors (2 or 4)



or

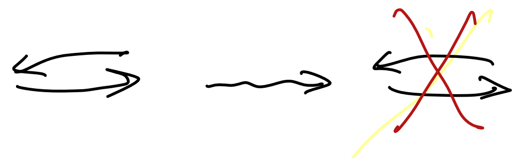


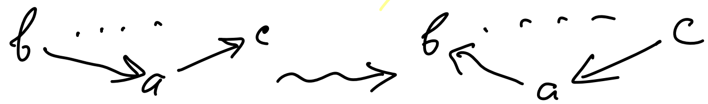
Mutation: quiver mutation

even quiver mutates as in cluster theory

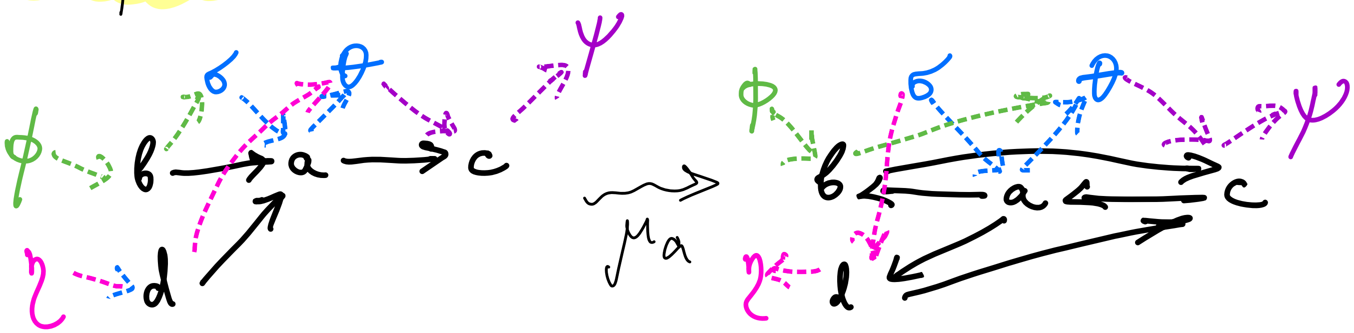
three steps:

(a) 

(b) 

(c) 

odd quiver



Mutation of cluster variables

Denote for vertex

$$\begin{tikzcd} & \sigma \arrow[d] & \theta & c_1 \\ b_1 \arrow[r] & a \arrow[ur] \arrow[urr] \arrow[drr] & & \\ b_2 \arrow[ur] & & & c_2 \end{tikzcd}$$

by $\Pi_{in} = \prod_{\text{even } b \to a} b$

$\Pi_{out} = \prod_{\text{even } a \to c} c$

$$a a' = \underbrace{\Pi_{in} + \Pi_{out}}_{\text{body}} + \underbrace{\sqrt{\Pi_{in}\Pi_{out}}\,\sigma\theta}_{\text{soul}}$$

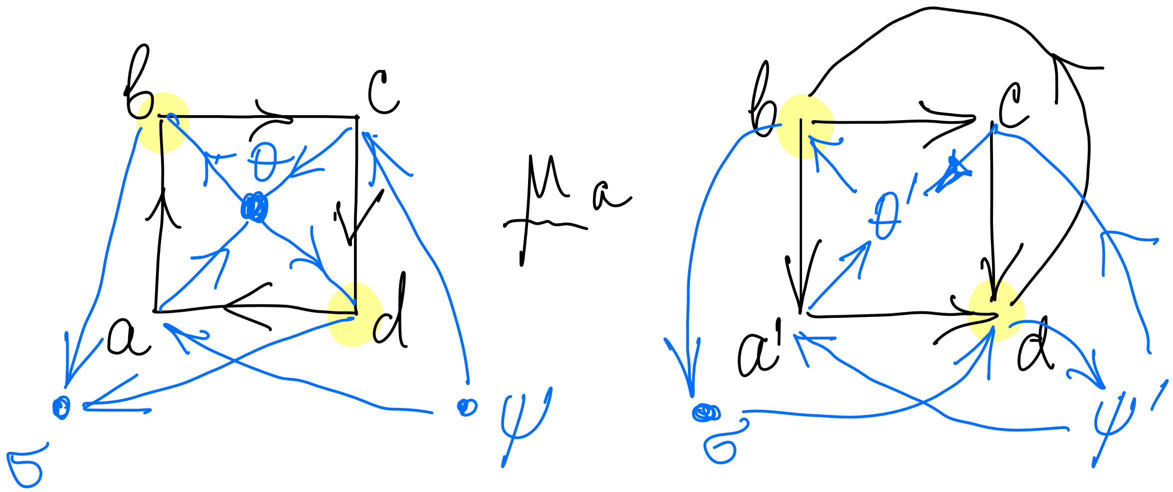
$$\sigma' = \frac{\sqrt{\Pi_{in}}\,\sigma - \sqrt{\Pi_{out}}\,\theta}{\sqrt{\Pi_{in}+\Pi_{out}}}$$

$$\theta' = \frac{\sqrt{\Pi_{in}}\,\theta + \sqrt{\Pi_{out}}\,\sigma}{\sqrt{\Pi_{in}+\Pi_{out}}}$$

Obs. Removing soul we restore the classical formula for cluster mutation.

<u>Proposition</u>. If Q is a superquiver then $\mu_k(Q)$ is a superquiver.

e.g.



not from triangulation because θ has 4 neighbors

* **Claim.** 2 form

$$\omega = \sum_{\text{even } a \to b} \frac{da}{a} \wedge \frac{db}{b} + \sum_{\text{odd } \alpha} (d\alpha)^2$$ is preserved under mutations

* **Claim.** $x_i(t) \in \mathbb{Q}[x_j(t_0)^{\pm\frac{1}{2}} \mid \theta_k(t_0)]$.

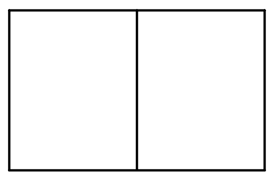
(modification of)

Idea (use caterpillar lemma by Fomin Zelevinsky)

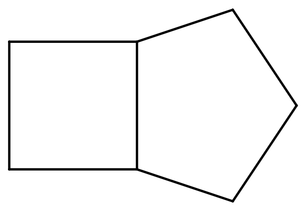
If $\mu_i \mu_j \mu_i$ has Laurent property, and $(\mu_i)^d$ has Laurent property for $d = 1, \dots, 7$ then any chain of supercluster mutations has Laurent property.

Warning: Not every even quiver allows superquiver extension.

e.g.

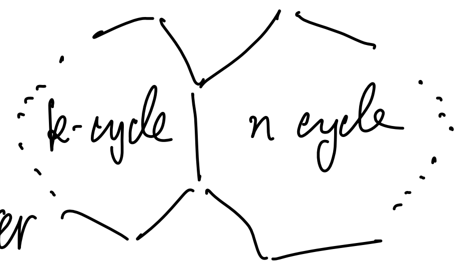


and

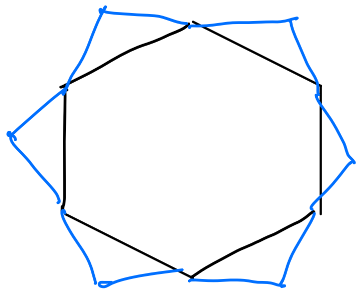


do not allow

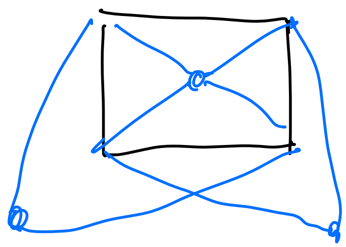
In general, k-cycle and n-cycle sharing exactly one common edge do not allow superquiver extension if $k, n > 3$.

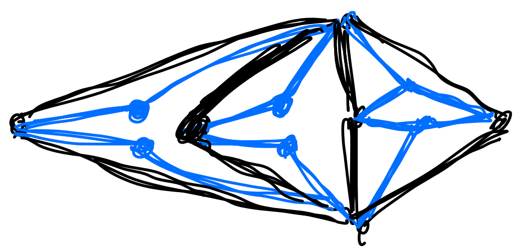


Examples of superquivers.

(a)  n-cycle

(b) Quiver of triangulation (Penner-Zeitlin)

(c) 

(d) ... 

(e) Generalized Penner-Zeitlin

V = an n dimensional manifold

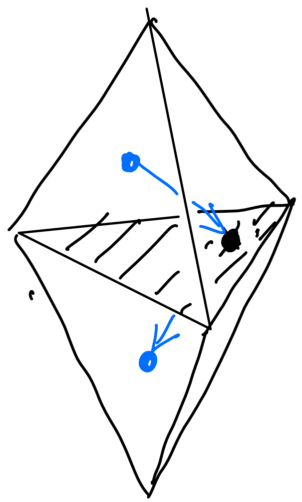
Σ = a nice decomposition of V into simplices (no selffoldings)

Quiver: • $(n-1)$-faces of Σ = even vertices

- even edges between faces of one simplex (any orientation)

• n-simplices are odd vertices

- odd-even edges connects face with its neighbor simplex (any allowed orientation)



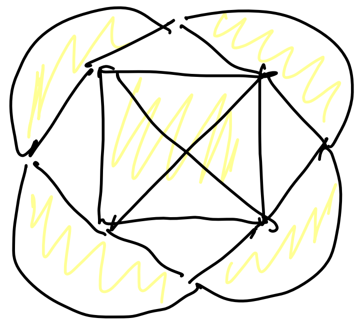
Rmk. The class of generalized Penner Zeitlin quivers is not invariant under mutations.

(d) clique generalization:

even quiver = union of cliques (= complete graphs)

each vertex belong to exactly two cliques

e.g.



odd variables = cliques

<u>Rmrk</u> The class of clique quivers is invariant under mutations.

<u>Question</u> Is there a geom. meaniningful construction of clique quivers?

<u>Question</u>: characterize quivers allowing superquiver extension ?

Happy

birthday!