Positively graded (super) Lie algebras

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Ovsienko+Fock=5!-meeting,

Olегонище, October 21 – 25, 2024

Definition (Lie superalgebra)

A vector space $\mathfrak{g} = \mathfrak{g}_{\underline{0}} \oplus \mathfrak{g}_{\underline{1}}$ – a direct sum $\mathfrak{g}_{\underline{0}}$ of (even) and $\mathfrak{g}_{\underline{1}}$ (odd) parts with a super bracket $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, satisfying

$$
\begin{gathered}
[\mathfrak{g}_{\underline{i}}, \mathfrak{g}_{\underline{j}}] \subset \mathfrak{g}_{\underline{i+j}\,(mod\,2)}, \underline{i}, \underline{j} \in \{0,1\}, \\
[x,y] = -(-1)^{\underline{ij}}[y,x], x \in \mathfrak{g}_{\underline{i}}, y \in \mathfrak{g}_{\underline{j}}, \underline{i}, \underline{j} \in \{0,1\}, \\
(-1)^{\underline{ik}}[[x,y],z] + (-1)^{\underline{kj}}[[y,z],x] + (-1)^{\underline{ji}}[[z,x],y] = 0, \\
x \in \mathfrak{g}_{\underline{i}}, y \in \mathfrak{g}_{\underline{j}}, z \in \mathfrak{g}_{\underline{k}}, \underline{i}, \underline{j}, \underline{k} \in \{0,1\},
\end{gathered} \tag{1}
$$

is called a Lie superalgebra

1) if $e \in \mathfrak{g}_{\underline{0}}$, then $[e,e] = 0$.
2) if $f \in \mathfrak{g}_{\underline{1}}$, then $\quad -3\,[[f,f],f] = 0$.

Hence there are only 3 one-generated Lie superalgebras:

$$\langle e \rangle, \langle f \rangle, \langle f, [f,f] \rangle.$$

$\mathbb{N}$-graded Lie superalgebras

Definition

Lie superalgebra $\mathfrak{g}$ is called $\mathbb{N}$-graded, if

$$\mathfrak{g} = \oplus_{i \in \mathbb{N}} \mathfrak{g}_i, \ [\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}, \quad i, j \in \mathbb{N}, i, j > 0.$$

We assume also that

$$\mathfrak{g}_i = \mathfrak{g}_{\underline{0},i} \oplus \mathfrak{g}_{\underline{1},i}, i \in \mathbb{N}, \qquad \mathfrak{g}_{\underline{0}} = \oplus_{i \in \mathbb{N}} \mathfrak{g}_{\underline{0},i}, \mathfrak{g}_{\underline{1}} = \oplus_{i \in \mathbb{N}} \mathfrak{g}_{\underline{1},i}$$

Definition

A grading $\mathfrak{g} = \oplus_{i=1}^{+\infty} \mathfrak{g}_i$ is called **natural**, if

$$[\mathfrak{g}_1, \mathfrak{g}_i] = \mathfrak{g}_{i+1}, i \geq 1.$$

Let be $\mathfrak{g} = \mathcal{L}(m)$ a free Lie algebra of m generators $a_1, \dots, a_m$. Define by $\mathfrak{g}_k = \langle [a_{i_1}, [a_{i_2}, [\dots, \dots]], a_{i_k}] \rangle$ the linear span of all k-words. This grading is natural.

The ideals $\mathfrak{g}^m$ of the lower central series of a Lie algebra $\mathfrak{g}$

$$\mathfrak{g}^1 = \mathfrak{g},\ \mathfrak{g}^{m+1} = [\mathfrak{g}, \mathfrak{g}^m],$$

form a decreasing filtration of the Lie algebra $\mathfrak{g}$. Consider the associated graded Lie algebra

$$\mathrm{gr}\mathfrak{g} = \oplus_{i=1}^{+\infty} \mathfrak{g}^i / \mathfrak{g}^{i+1}.$$

Proposition

A Lie algebra $\mathfrak{g}$ is called naturally graduable if and only if

$$\mathfrak{g} \cong \mathrm{gr}\mathfrak{g}.$$

Narrow Lie (super)algebras after Shalev and Zelmanov

Definition (Zelmanov, Shalev)

A $\mathbb{N}$-graded Lie (super)algebra $\mathfrak{g} = \oplus_{i=1}^{+\infty}\mathfrak{g}_i$ is called a Lie (super)algebra of finite width, if $\exists C \geq 0$ such that

$$\dim \mathfrak{g}_i \leq C, \forall i \in \mathbb{N}.$$

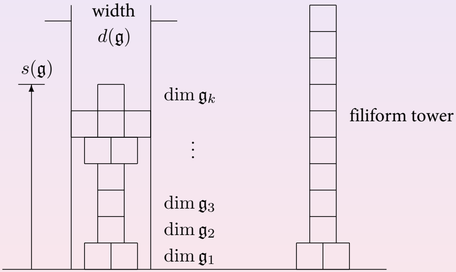
The **width** $d(\mathfrak{g})$ of $\mathfrak{g} = \oplus_{i=1}^{+\infty}\mathfrak{g}_i$ is

$$d(\mathfrak{g}) = \max_{i \in \mathbb{N}} \dim \mathfrak{g}_i.$$

Definition (Narrow Lie (super)algebra)

A $\mathbb{N}$-graded Lie algebra $\mathfrak{g} = \oplus_{i=1}^{+\infty}\mathfrak{g}_i$ is called **narrow** if $d(\mathfrak{g}) \leq 2$.

The width of a $\mathbb{N}$-graded Lie algebra

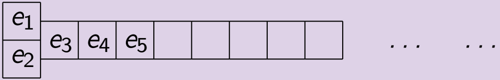


The narrowest naturally graded Lie algebra

Theorem (M. Vergne, 1970)

Let $\mathfrak{g} = \oplus_{i=1}^{+\infty} \mathfrak{g}_i$ *be a naturally graded Lie algebra such that*

$$\dim \mathfrak{g}_1 = 2, \quad \dim \mathfrak{g}_i = 1, i \geq 2.$$



Then $\mathfrak{g}$ *is isomorphic to* $\mathfrak{m}_0$, *where* $\mathfrak{m}_0$ *is defined by its infinite basis* $e_1, e_2, e_3, e_4, \dots$ *and relations*

$$[e_1, e_i] = e_{i+1}, i \geq 2, \ [e_i, e_k] = 0, \ i,k \neq 1.$$

Lie algebras of width one

Let $\mathfrak{g} = \oplus_{i=1}^{+\infty} \mathfrak{g}_i$ be a $\mathbb{N}$-graded Lie algebra such that

$$\dim \mathfrak{g}_i = 1, i \geq 1.$$

e_1	e_2	e_3	e_4				...	...

1) $\mathfrak{m}_0$ admits this grading but it is **not natural**!!

$$[e_1, e_i] = e_{i+1}, i \geq 2, \ [e_i, e_k] = 0, \ i,k \neq 1.$$

2) the positive part W^+ of the Witt algebra

$$[e_i, e_j] = (j - i)e_{i+j}, i,j \geq 1.$$

It is easy to see that

$$grW^+ \cong gr\mathfrak{m}_0 \cong \mathfrak{m}_0.$$

Toward of classification of graded Lie algebras of width one

Theorem (Fialowski 1983, Shalev and Zelmanov 1997)

Let $\mathfrak{g} = \oplus_{i=1}^{+\infty} \mathfrak{g}_i$ *be an infinite-dimensional Lie algebra of width* 1

$$[\mathfrak{g}_1, \mathfrak{g}_i] = \mathfrak{g}_{i+1}, i \geq 2.$$

Then $\mathfrak{g}$ *is isomorphic to one of the following three Lie algebras*

$$\mathfrak{m}_0, \ \mathfrak{m}_2, \ W^+,$$

where W^+ *is positive part of the Witt algebra, and* $\mathfrak{m}_2$ *is defined by the following commutation relations*

$$[e_1, e_i] = e_{i+1}, i \geq 2, \ [e_2, e_i] = e_{i+2}, i \geq 3.$$

Arnold's A-algebras

Arnold studied narrow algebras, although these were not Lie algebras, but associative, commutative graded algebras of width one. Arnold called such algebras A-algebras and expressed the number of non-isomorphic A-algebras with three generators using continued fractions.

V.I. Arnold, *A-graded algebras and continued fractions*, Communications on Pure and Applied Mathematics, **42**:7 (1989), 993–1000.

V.I. Arnold, *Higher-dimensional continued fractions*, Regul. Chaotic Dyn., **3**:3 (1998), 10–17.

Benoist's classification

Benoist classified Lie algebras алгебры Ли $\mathfrak{a}_r$ defined by

1) two generators e_1 and e_2;
2) two relations

$$[e_2, e_3] = e_5,\ [e_2, e_5] = re_7,$$

where r is a scalar parameter and $e_i, i \geq 3$ are defined recursively by the relation $e_{i+1} = [e_1, e_i]$.

Theorem (Benoist, 1992)

If $r \neq \frac{9}{10}, 1$, then $\mathfrak{a}_r$ is finite dimensional.
1) Let $r = \frac{9}{10}$, then $\mathfrak{a}_r \cong W^+$,
2) Let $r = 1$, then $\mathfrak{a}_r \cong \mathfrak{m}_2$.
3) Let $r \neq 0, \frac{9}{10}, 1, 2, 3$, then $\mathfrak{a}_r$ is a 11-dimensional graded filiform Lie algebra.

Why $\frac{9}{10}$?

We need a cyclic basis of W^+. So rescale our standard basis

$$e'_n = 6(n-2)!e_n, n \geq 1.$$

Then

$$[e'_1, e'_k] = e'_{k+1}, k \geq 1, [e'_2, e'_3] = 6^2 \cdot [e_2, e_3] = 36 \cdot e_5 = e'_5.$$

And also

$$[e'_2, e'_5] = 6^2 \cdot 3! \cdot [e_2, e_5] = 6^3 \cdot (5-2) \cdot e_7 = \frac{6^3 \cdot 3}{6 \cdot 5!} \cdot e'_7 = \frac{9}{10} \cdot e'_7$$

What was the aim of Benoist's classification?

In 1992, **Benoit gave a negative answer to Milnor's 1977 question (conjecture)**, who asked whether a simply connected nilpotent group always admits a left-invariant affine structure?

Benoit gave an example of a compact 11-dimensional nilpotent Lie group **which does not admit any such complete affine structure.** For his counterexample, Benoist used the 11-dimensional **filiform** graded algebra $\mathfrak{a}_{-2}$, considering its generic deformation $\mathfrak{a}_{-2,1,t}$.

The hard part of his proof: showing that the Lie algebra $\mathfrak{a}_{-2,1,t}$ **has no exact representations of the dimension** 12.

Cartan matrices and 2-generated graded Lie algebras

Consider a Cartan 2×2-matrix $A = (a_{ij})$.

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}. \qquad (2)$$

Then one can define 2-generated nilpotent Lie algebra by two relations

$$ad(e_1)^{-a_{12}+1}(e_2) = 0, \; ad(e_2)^{-a_{21}+1}(e_1) = 0.$$

In the case of degenerate Cartan matrices

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$$

we get finite-dimensional nilpotent subalgebras of simple Lie algebras A_2, C_2, G_2.

Two infinite-dimensional 2-generated graded Lie algebras

For two generalized Cartan matrices

$$A_1 = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}, \; A_2 = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}$$

we get two infinite dimensional graded Lie algebras

- $\mathfrak{n}_1$ (nilpotent part of the affine Kac-Moody algebra $A_1^{(1)}$), two generators e_1, e_2 and two relations

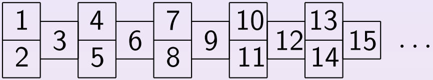
$$ad^3 e_1(e_2) = 0, \; ad^3 e_2(e_1) = 0;$$

- $\mathfrak{n}_2$ (nilpotent part of the affine Kac-Moody algebra $A_2^{(2)}$), two generators e_1, e_2 and two relations

$$ad^5 e_1(e_2)=0, \; ad^2 e_2(e_1)=0.$$

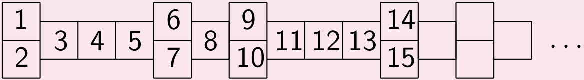
Natural grading of $\mathfrak{n}_1$ and $\mathfrak{n}_2$

$\mathfrak{n}_1$:



$$
\begin{array}{cccc}
e_1 & & e_4 = [e_1, [e_1, e_2]] & \\
 & e_3 = [e_1, e_2] & & \dots \\
e_2 & & e_5 = [e_2, [e_1, e_2]] & \\
1 & 2 & 3 & \dots
\end{array}
$$

$\mathfrak{n}_2$:



Naturally graded Lie algebras of width 3/2

Theorem (M., 2017, Doklady Mathematics 2018, Sbornik Math 2019)

Let $\mathfrak{g} = \oplus_{i=1}^{+\infty}\mathfrak{g}_i$ *be a complex positively graded Lie algebra such that*

$$[\mathfrak{g}_1, \mathfrak{g}_i] = \mathfrak{g}_{i+1}, \ \dim \mathfrak{g}_i + \dim \mathfrak{g}_{i+1} \leq 3, \forall i \in \mathbb{N}.$$

Then $\mathfrak{g} = \oplus_{i=1}^{+\infty}\mathfrak{g}_i$ *is isomorphic to one and only one Lie algebra from the list*

$$\mathfrak{m}_0, \mathfrak{n}_1, \ \mathfrak{n}_2, \mathfrak{n}_2^3, \left\{\mathfrak{m}_0^R \mid R \subset \{3, 5, 7, 9, \dots\}\right\}.$$

Ramond and Neveu-Schwarz superalgebras

KORTEWEG—DE VRIES SUPEREQUATION AS AN EULER EQUATION

V. Yu. Ovsienko and B. A. Khesin UDC 517.9

It is known that the Korteweg—de Vries (KdV) equation is associated with the Virasoro algebra (see [2; 3]). In [6] (see also [5; 7]) the Korteweg—de Vries superequation (sKdV) was proposed, corresponding to the simplest superanalogues of the Virasoro algebra, i.e., the Neveu—Schwarz and the Ramond superalgebras. The present note concerns one geometric aspect of this connection. Its goal is to show that (s)KdV is the Euler equation on the corresponding groups, i.e., the equation of the geodesics of some one-sidely invariant metrics.

1. Recall the well-known definitions from mechanics (see [1]). Let $\mathfrak{G}$ be a Lie (super)algebra. The (right-)invariant metric on the corresponding group is uniquely defined by symmetric operator $A:\mathfrak{G}\to\mathfrak{G}^*$, which is called the inertia operator of an extended rigid body. It is given by the conveyance over the group of (right) shifts of the scalar product on $\mathfrak{G}$:

$$(\xi, \eta) = \langle A\xi, \eta\rangle, \quad \text{where} \quad \xi, \eta \in \mathfrak{G}.$$

Ramond and Neveu-Schwarz superalgebras

Basis of the even part consits of $L_i, i \in \mathbb{Z}$, and basis of the odd part consists of G_r and the central element C:
$[L_i, C] = 0, [G_r, C] = 0.$

$$
\begin{aligned}
[L_i, L_j] &= (i-j)L_{i+j} + \frac{i^3 - i}{12}\delta_{i+j,0}C, \\
[L_i, G_r] &= \left(\frac{i}{2} - r\right) G_{r+i}, \\
\{G_r, G_s\} &= 2L_{r+s} + \frac{1}{3}\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0}C.
\end{aligned} \tag{3}
$$

- the Ramond superalgebra R odd elements G_s have integer values s $\dots, G_{-2}, G_{-1}, G_0, G_1, G_2, \dots$
- in the case of the Neveu-Schwarz superalgebra NS the basic G_s have **half integer** downscripts $\dots, G_{-\frac{3}{2}}, G_{-\frac{1}{2}}, G_{\frac{1}{2}}, G_{\frac{3}{2}}, \dots$

Positive parts of Ramond and Neveu-Schwarz algebras

- $R^+ = R^+_{\underline{0}} \oplus R^+_{\underline{1}} = \langle L_1, L_2, L_3, \dots \rangle \oplus \langle G_1, G_2, G_3, \dots \rangle$
- $NS^+ = NS^+_{\underline{0}} \oplus NS^+_{\underline{1}} = \langle L_1, L_2, L_3, \dots \rangle \oplus \langle G_{\frac{1}{2}}, G_{\frac{3}{2}}, G_{\frac{5}{2}}, \dots \rangle$

with relations for $i, j, r, s > 0$,

$$\begin{gathered} [L_i, L_j] = (i-j)L_{i+j}, \\ [L_i, G_r] = \left(\frac{i}{2} - r\right) G_{r+i}, \\ \{G_r, G_s\} = 2L_{r+s}. \end{gathered} \tag{4}$$

1) Even parts $R^+_{\underline{0}} = NS^+_{\underline{0}} = W^+$
2) Odd parts $R^+_{\underline{1}}$ и $NS^+_{\underline{1}}$ are narrow W^+-modules.

The positive part R^+ of the Ramond algebra R

naturally graded Lie superalgebra of width two

$$\begin{array}{ccccccccccc} R^+ & = & \langle L_1, G_1 \rangle & \oplus & \langle L_2, G_2 \rangle & \oplus & \langle L_3, G_3 \rangle & \oplus & \langle L_4, G_4 \rangle & \oplus & \dots \\ & & \| & & \| & & \| & & \| & & \\ & & R_1^+ & & R_2^+ & & R_3^+ & & R_4^+ & & \end{array}$$

R^+ is generated by **one even** L_1 and by **one odd** G_1:

$$\{G_1, G_1\} = 2L_2,\ [L_1, L_2] = -L_3,$$

$$[L_1, L_i] = (1-i)L_{i+1}, i \geq 2,\ [L_1, G_s] = \left(\frac{1}{2} - s\right) G_{s+1}, s \geq 1.$$

R^+:

L_1	L_2	L_3	L_4	$\dots$	L_n	$\dots$
G_1	G_2	G_3	G_4		G_n	

The positive part NS^+ of the Neveu-Schwarz algebra NS

$\mathbb{N}$-**graded Lie superalgebra** of width one

$$
\begin{array}{ccccccccccccc}
NS^+ & = & \langle G_{\frac{1}{2}} \rangle & \oplus & \langle L_1 \rangle & \oplus & \langle G_{\frac{3}{2}} \rangle & \oplus & \langle L_2 \rangle & \oplus & \langle G_{\frac{5}{2}} \rangle & \oplus & \dots \\
 & & \| & & \| & & \| & & \| & & \| & & \\
 & & NS^+_1 & & NS^+_2 & & NS^+_3 & & NS^+_4 & & NS^+_5 & &
\end{array}
$$

NS^+ is generated by **two odd generators** $G_{\frac{1}{2}}$ and $G_{\frac{3}{2}}$:

$$
\left\{ G_{\frac{1}{2}}, G_{\frac{1}{2}} \right\} = 2L_1, \ \left\{ G_{\frac{1}{2}}, G_{\frac{3}{2}} \right\} = 2L_2,
$$

$$
[L_1, L_i] = (1-i)L_{i+1}, i \geq 2, \ [L_1, G_{\frac{2s+1}{2}}] = -sG_{\frac{2s+3}{2}}, s \geq 0.
$$

NS^+:

$G_{\frac{1}{2}}$	L_1	$G_{\frac{3}{2}}$	L_2	$G_{\frac{5}{2}}$	L_3	$\dots$

$\mathbb{N}$-graded Lie superalgebra $\mathfrak{M}_2$ of width two

$\mathfrak{M}_2$:

e_1	e_2	e_3	e_4	$\dots$	e_n	$\dots$
f_1	f_2	f_3	f_4		f_n	

1) its even part is $\mathfrak{m}_2$

$$[e_1, e_i] = e_{i+1},\ i \geq 2,\ \ [e_2, e_i] = e_{i+2},\ i \geq 3.$$

2) $\mathfrak{m}_2$-module of odd part is cyclic

$$[e_1, f_i] = f_{i+1},\ i \geq 1,\ \ [e_2, f_i] = f_{i+2},\ i \geq 2,$$
$$[e_2, f_1] = 0,\ [e_k, f_1] = -f_{k+1}, k \geq 3,$$

3) odd product $\{,\}$:

$$\{f_1, f_1\} = e_2, \{f_1, f_m\} = \frac{1}{2}e_{m+1}, m \geq 2.$$

Super analogue N1 of Benoist's classification

Consider a Lie superalgebra $\mathfrak{R}_x$ generated by one even f_1 and one odd e_1 with two relations

$$[e_1,[e_1,[f_1,f_1]]] = x\,[[f_1,[e_1,[e_1,f_1]]],$$
$$[e_1,[e_1,[e_1,f_1]]] = [[f_1,f_1],[e_1,f_1]]$$

where x denotes a parameter

Theorem (M. + Pokrovskiy, 2024, Doklady Mathematics)

1) $x = \frac{3}{8} \implies \mathfrak{R}_x \cong R^+$;
2) $x = \frac{1}{2} \implies \mathfrak{R}_x \cong \mathfrak{M}_2$;

If $x \neq \frac{3}{8}, \pm\frac{1}{2}, 0, \frac{1}{6}$, *then* $\dim \mathfrak{R}_x = 16$.
Its even part $(\mathfrak{R}_x)_{\underline{0}}$ *is 8-dimensional and isomorphic to the truncated Lie algebra* $\mathfrak{a}_r/\mathfrak{a}_r^8$ *from Benoit's list with* $r = \frac{1}{2}\frac{60x^2-20x+3}{(2x+1)(6x-1)}$.

Super analogue N2 of Benoist's classification

Consider a Lie superalgebra $\mathfrak{N}S_x$ generated by two odd elements f_1 and f_3 with two relations

$$ad(f_1)^3(f_3) = [f_1,[f_1,[f_1,f_3]]] = -[f_3,f_3],$$
$$\left[ad(f_1)^2(f_3), ad(f_1)^2(f_3)\right] = y \cdot ad(f_1)^7(f_3),$$

where y denotes a parameter

Theorem (M. + Pokrovskiy, 2024, Doklady Mathematics)

1) $y = -\frac{1}{6} \implies \mathfrak{N}S_y \cong NS^+$;
2) $y = 0 \implies \mathfrak{N}S_y \cong \mathfrak{M}_2'$;

If $y \neq -\frac{1}{6}, \pm 1, 0, -\frac{1}{2}$, *then* $\dim \mathfrak{N}S_y = 14$.
Its even part $(\mathfrak{R}_x)_{\underline{0}}$ *is 7-dimensional and isomorphic to the truncated Lie algebra* $\mathfrak{a}_r/\mathfrak{a}_r^7$ *from Benoit's list with* $r = \frac{3y^2+3y+1}{(1-y)(2y+1)(1+y)}$.

$\mathbb{N}$-graded Lie superalgebra $\mathfrak{M}'_2$ of width one

$\mathfrak{M}'_2$: | f_1 | e_2 | f_3 | e_4 | f_5 | e_6 | f_7 | $\dots$

1) its even part is $\mathfrak{m}_2$

$$[e_2, e_{2i}] = e_{2(i+1)},\ i \geq 2,\ \ [e_4, e_{2j}] = e_{2(j+2)},\ j \geq 3.$$

2) $\mathfrak{m}_2$-module of odd part is cyclic

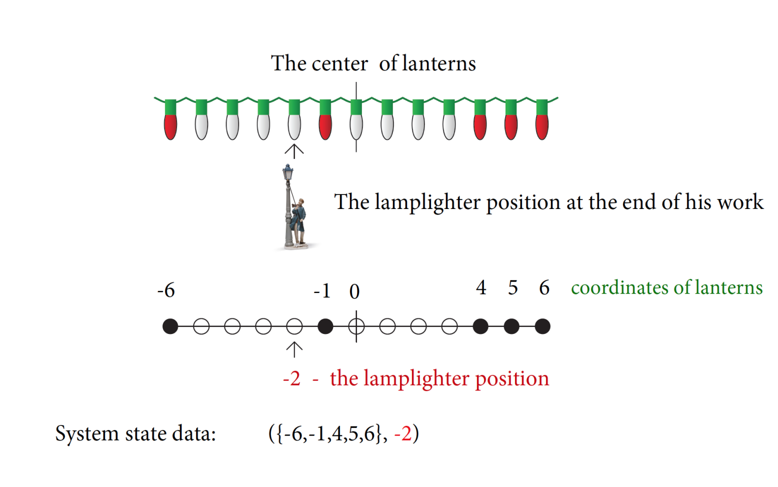
$$[e_2, f_1] = 0, \quad [e_2, f_{2j+1}] = f_{2j+3},\ j \geq 1,$$
$$[e_4, f_1] = f_5,\ [e_4, f_{2j+1}] = f_{2j+5}, j \geq 1,\ [e_{2k}, f_1] = -f_{2k+1}, k \geq 3,$$

3) odd product $\{,\}$

$$\{f_1, f_1\} = e_2, \{f_1, f_{2m+1}\} = \frac{1}{2}e_{2m+2}, m \geq 1,$$
$$\{f_3, f_3\} = e_6, \{f_3, f_{2k+1}\} = \frac{1}{2}e_{2k+4},$$

Lamplighters group – Jim Cannon

A chapter by J.Taback in "Office Hours with a Geometric Group Theorist"Edited by Matt Clay & Dan Margalit



Lamplighter's group multiplication

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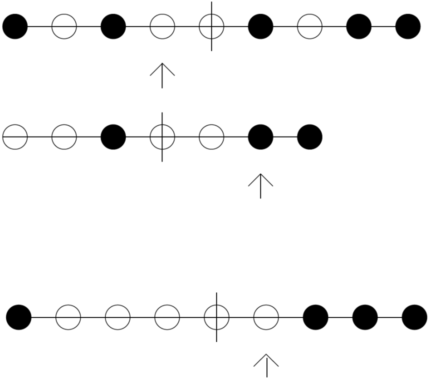
Result



As a set it is $\mathbb{Z} \times \bigoplus_{n\in\mathbb{Z}} \mathbb{Z}_2$ with the multiplication

$$(k, \{a_n\}) \star (l, \{b_m\}) = (k+l, \{c_n\}), \quad c_n = a_n + b_{n-k}, n \in \mathbb{Z}.$$

It is more convenient to consider the Laurent polynomials $\sum_{n\in\mathbb{Z}} a_n x^n, a_n \in \mathbb{Z}_2$:

$$\left(k, \sum_{n\in\mathbb{Z}} a_n x^n\right) \star \left(l, \sum_{n\in\mathbb{Z}} b_n x^n\right) = \left(k+l, \sum_{n\in\mathbb{Z}} a_n x^n + x^k \sum_{n\in\mathbb{Z}} b_n x^n\right).$$

The integer lamplighter group $L(\mathbb{Z})$ and matrices

Consider 2×2-matrices

$$\begin{pmatrix} x^k & p(x) \\ 0 & 1 \end{pmatrix}, k \in \mathbb{Z},\ p(x) \in \mathbb{Z}_2[x, x^{-1}].$$

With the standard matrix multiplication.

The integer lamplighter group $L(\mathbb{Z})$, defined by Sergey Ivanov and Roman Mikhailov (2021) – they considered Laurent polynomials $p(t)$ with integer coefficients $p(x) \in \mathbb{Z}[x, x^{-1}]$.

The lamplighter Lie algebra

Definition (Ivanov, Mikhailov, Zaikovskii 2021)

The rational lamplighter Lie algebra is defined as the semidirect product $\mathfrak{l} = \mathbb{Q}t \ltimes \mathbb{Q}[x]$ of one-dimensional $\mathbb{Q}t$ and infinite-dimensional abelian $\mathbb{Q}[x]$ with relations

$$[t, p(x)] = xp(x), \quad p(x) \in \mathbb{Q}[x].$$

Proposition

The lamplighter Lie algebra $\mathfrak{l} = \mathbb{Q}t \ltimes \mathbb{Q}[x]$ *is isomorphic to* $\mathfrak{m}_0$.

$$\begin{gathered} t \to e_1, \\ 1 \to e_2, \\ x \to e_3, \\ \dots, \\ x^n \to e_{n+2}, \\ \dots \end{gathered}$$

The Lamplighter from "Little Prince" by Antoine de Saint-Exupery

Le petit prince et un allumeur(lamplighter)

Le petit prince et un allumeur(lamplighter)

La cinquième planète était très curieuse. C'était la plus petite de toutes. Il y avait là juste assez de place pour loger un réverbère et un allumeur de réverbères. Le petit prince ne parvenait pas à s'expliquer à quoi pouvaient servir, quelque part dans le ciel, sur une planète sans maison, ni population, un réverbère et un allumeur de réverbères. Cependant il se dit en lui-même:

Peut-être bien que cet homme est absurde. Cependant il est moins absurde que le roi, que le vaniteux, que le businessman et que le buveur. Au moins son travail a-t-il un sens. Quand il allume son réverbère, c'est comme s'il faisait naître une étoile de plus, ou une fleur. Quand il éteint son réverbère, ça endort la fleur ou l'étoile. C'est une occupation très jolie. C'est véritablement **utile puisque c'est joli**.

The little prince and the lamplighter

The fifth planet was very strange. It was the smallest of all. There was just enough room on it for a street lamp and a lamplighter. The little prince was not able to reach any explanation of the use of a street lamp and a lamplighter, somewhere in the heavens, on a planet which had no people, and not one house. But he said to himself, nevertheless:

"It may well be that this man is absurd. But he is not so absurd as the king, the conceited man, the businessman, and the tippler. For at least his work has some meaning. When he lights his street lamp, it is as if he brought one more star to life, or one flower. When he puts out his lamp, he sends the flower, or the star, to sleep. That is a beautiful occupation. And since it is beautiful, it is truly useful."

Dear Valia and Volodia,

I wish you to light thousands of stars!!!