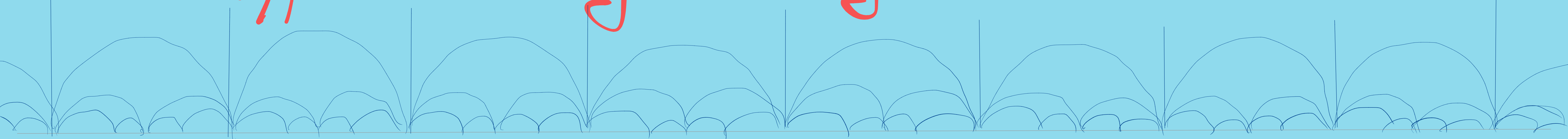
Hyperbolic geometry of friezes



Anna Felikson (Durham)

Pavel Tumarkin (Durham)

arXiv:2410.13511v1

21-25 October 2024, New trends in Geometry, Combinatorics and Mathematical Physics

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1 3 2 2 1 4 2 1 3 2 2 1 4 2 1 3 2 2 1 4 2 1 2 2

1 2 5 3 1 3 7 1 2 5 3 1 3 7 1 2 5 3 1 3 7 1 2 3

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1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1

On the occasion of Vladimir Fock's and Valentin Ovsienko's 60th birthdays.

Frieze patterns (Coxeter '1971)

…		1		1		1		1		1		1		1		1		…	
	…		1		3		2		2		1		4		2		1		…
…		1		2		5		3		1		3		7		1		…	
	…		1		3		7		1		2		5		3		1		…
…		2		1		4		2		1		3		2		2		…	
	…		1		1		1		1		1		1		1		1		…

- table (of finite width) of integers satisfying unimodular rule:

diamond

	b	
a		d
	c	

$$ad - bc = 1$$

Question: Is it true that every matrix $A \in SL_2(\mathbb{Z}_+)$ appears in some frieze?

← Question by Alain Valette (Neuchâtel)

Q1 Does every $A \in SL_2(\mathbb{Z}_+)$ appear?

Recall from Conway-Coxeter:

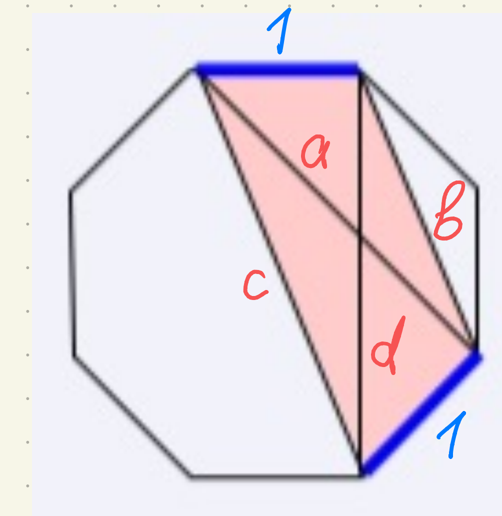
- Finite width friezes $\leftrightarrow$ triangulated polygons
- Entry in the frieze $\leftrightarrow$ diagonal in the polygon P
- Diamond in the frieze $\leftrightarrow$ quadrilateral inside P

- sides of P $\longleftrightarrow$ 1's (bdy)
- arcs of triangulation $\longleftrightarrow$ 1's (internal)
-

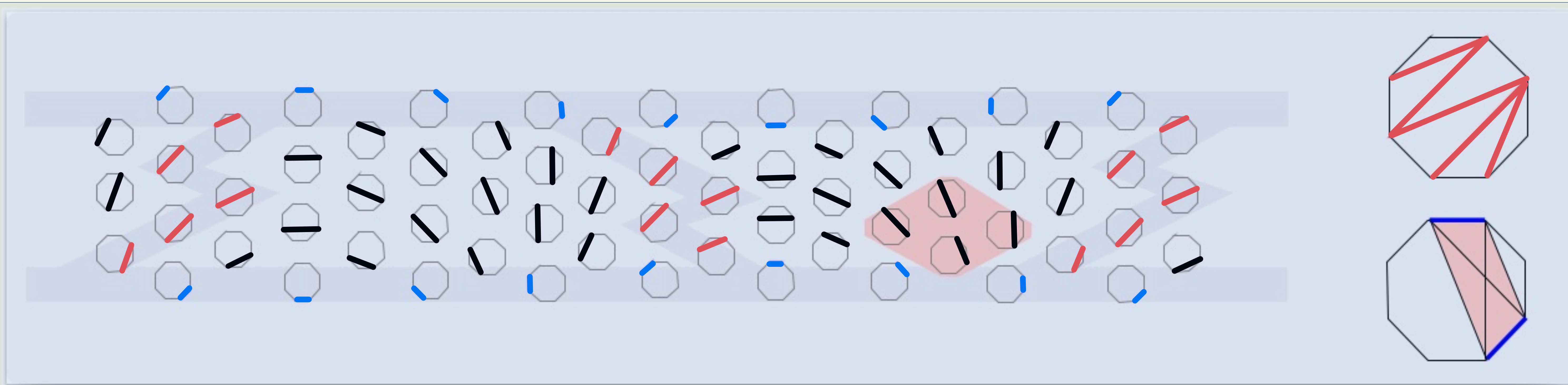
$$\begin{matrix} & b & \\ a & & d \\ & c & \end{matrix}$$

$ad - bc = 1$

$\longleftrightarrow$



diagonals a, d
opposite sides b, c



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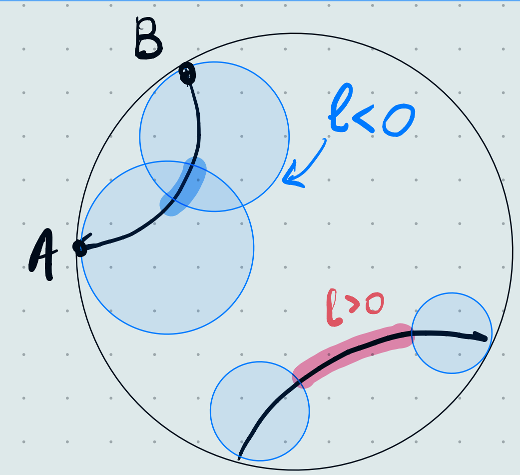
and from Penner:

- Entry in the frieze $=$ λ-length of the diagonal

λ-length:

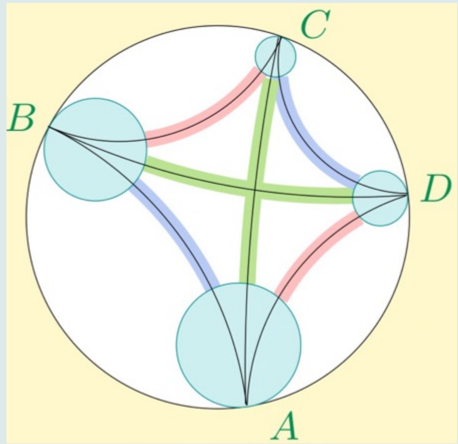
- For $A, B \in \partial \mathbb{H}^2$, $d(A,B) = \infty$
- Choose horocycles h_A and h_B at A and B
- $\ell :=$ signed distance $d(h_A, h_B)$



- $\lambda_{AB} := e^{\ell/2}$ λ-length

Ptolemy relation:

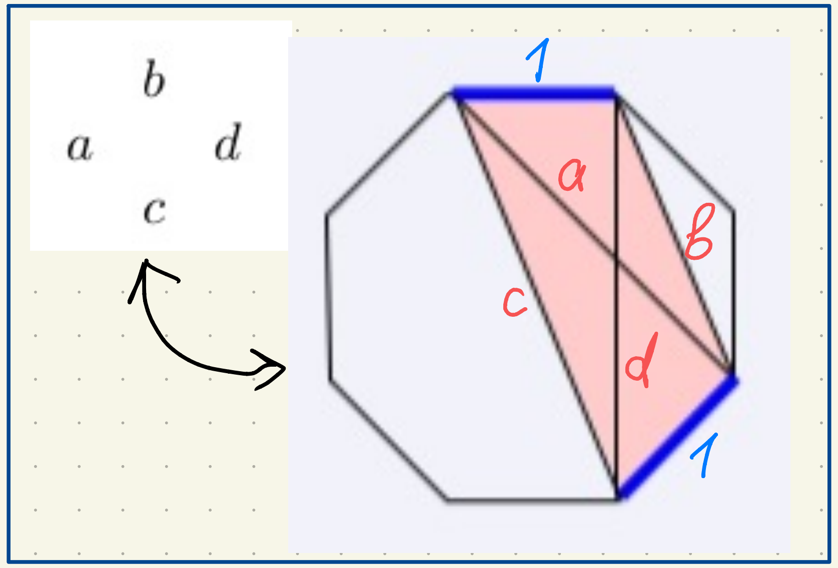
- $\lambda_{AC}\lambda_{BD} = \lambda_{AB}\lambda_{CD} + \lambda_{BC}\lambda_{BD}$



To get

b
a d
c

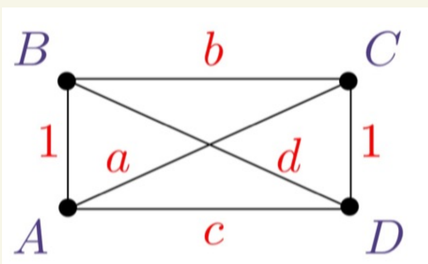
Need: to find a triangulated ideal polygon $P \subset \mathbb{H}^2$ with λ-lengths of sides and arcs of triangulation $= 1$ and a quadrilateral with diagonals a, d and sides $b, 1, c, 1$ in P



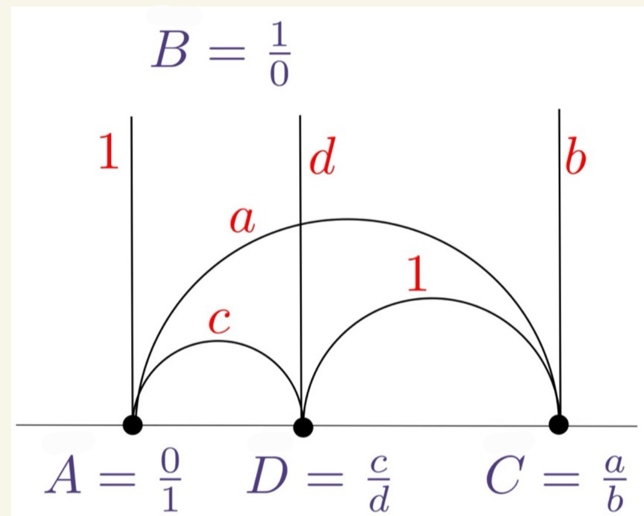
Q1 Does every $A \in SL_2(\mathbb{Z}_+)$ appear?

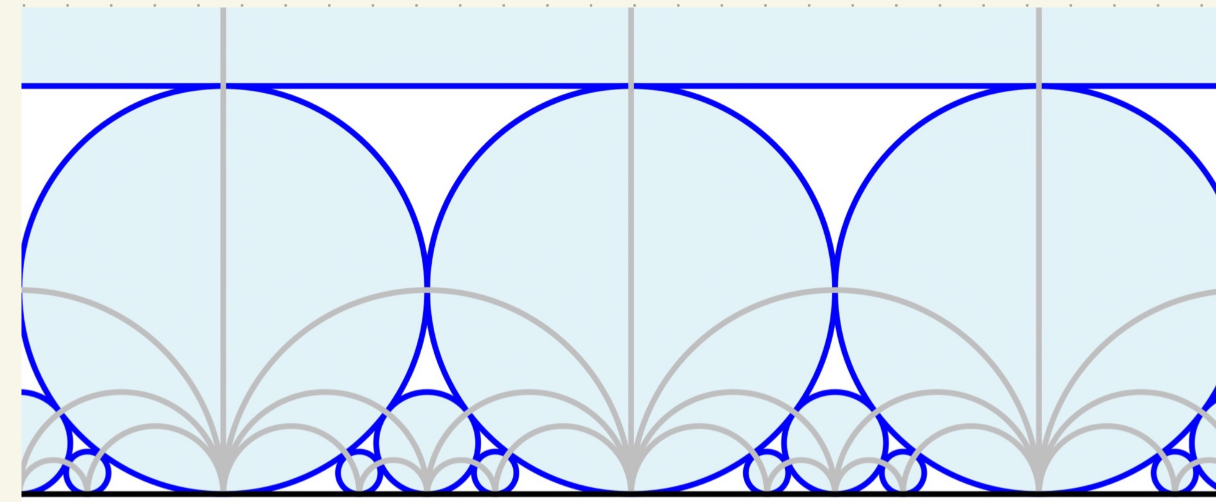
To construct $\mathcal{P}$, consider Farey graph together with Ford circles:

Need:



Take:





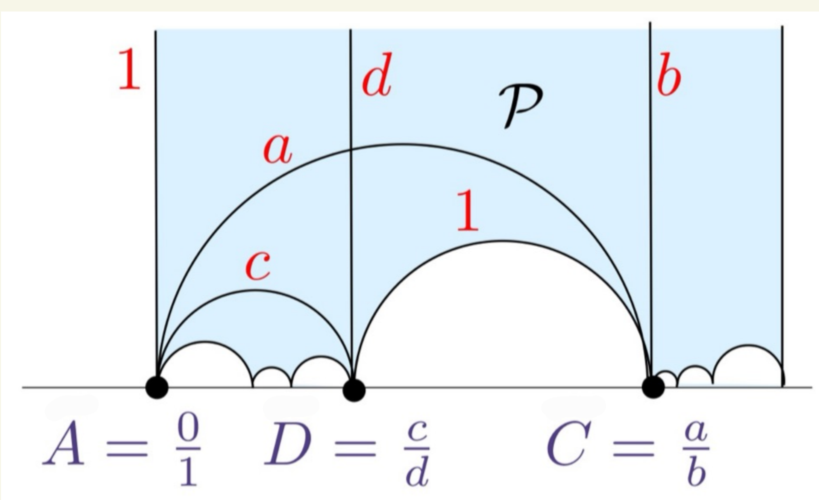
$\mathcal{F}$: vertices $\leftrightarrow$ points in $\mathbb{Q}$

arcs $\frac{p}{q}, \frac{r}{s}$ $\leftrightarrow$ $|ps-rq|=1$

reduced

Then

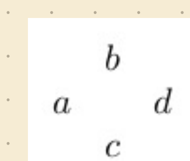
- λ-length of arcs of $\mathcal{F}$ $= 1$
- $\lambda\left(\frac{p}{q}, \frac{r}{s}\right) = |ps-rq|$



$\mathcal{P} = \bigcup$ (triangles in $\mathcal{F}$ intersecting ABCD)

YES!

To get



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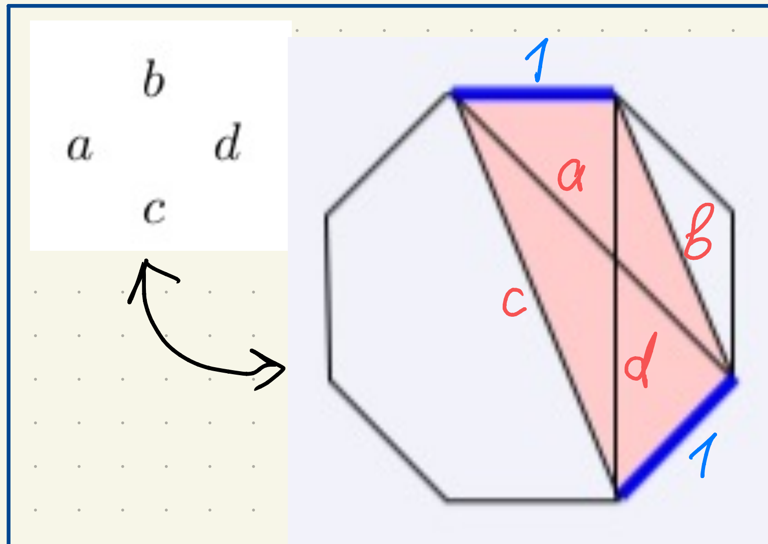
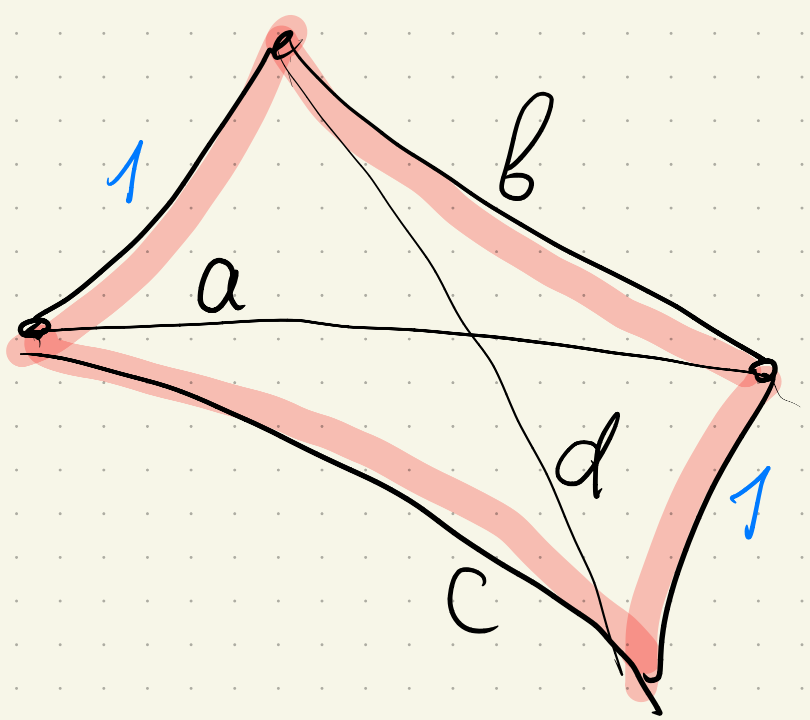
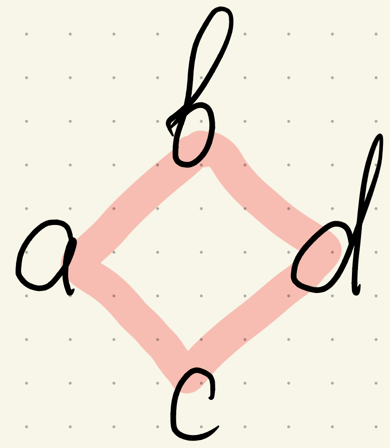


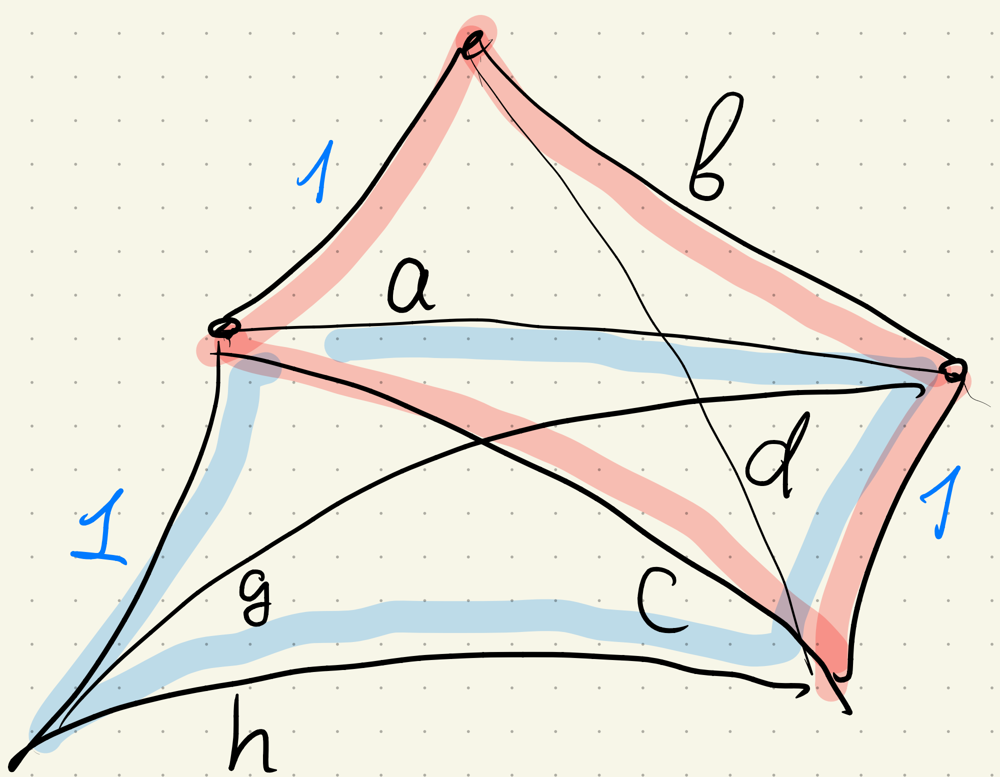
Q1 Does every $A \in SL_2(\mathbb{Z}_+)$ appear?

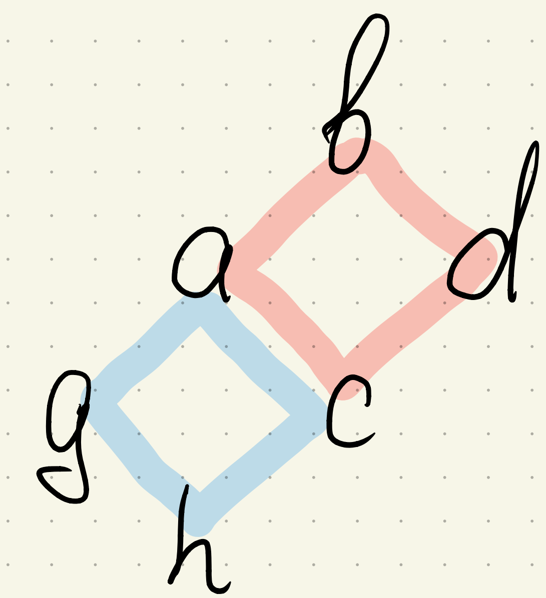
YES!





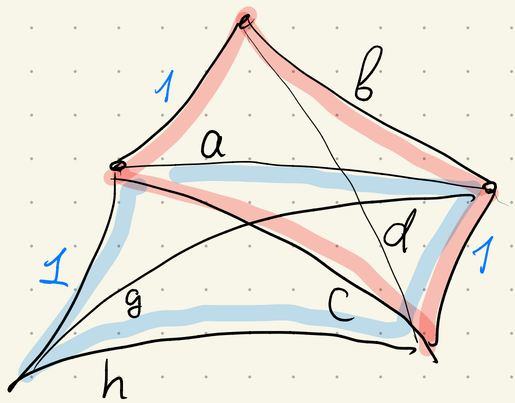
Q1 Does every $A \in SL_2(\mathbb{Z}_+)$ appear? **YES!**

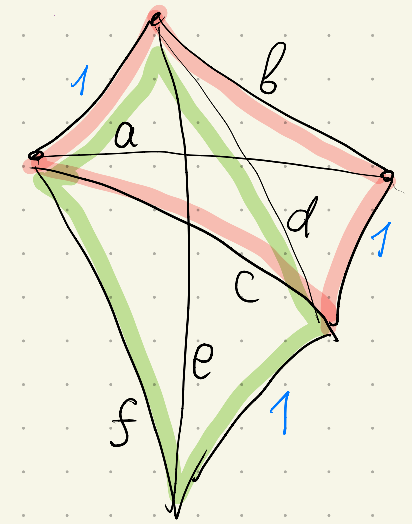


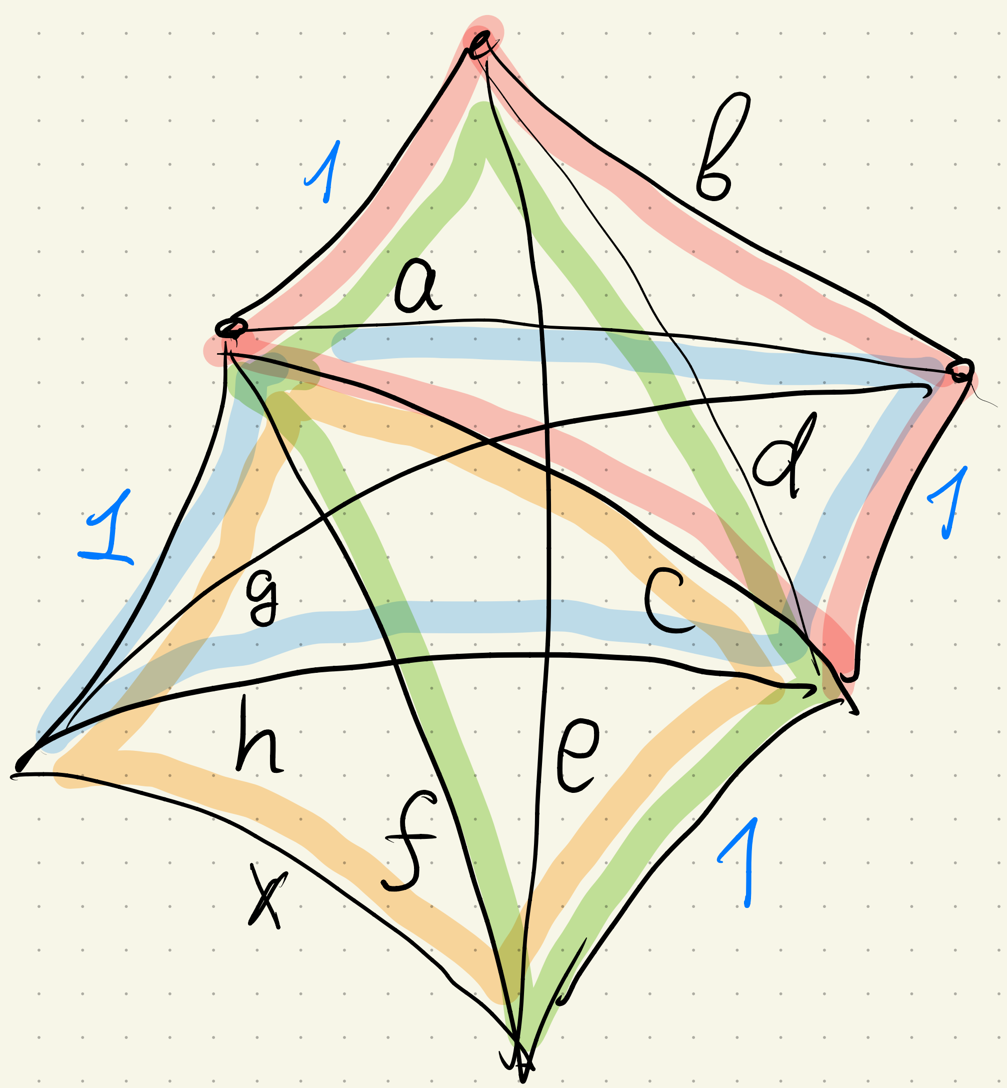


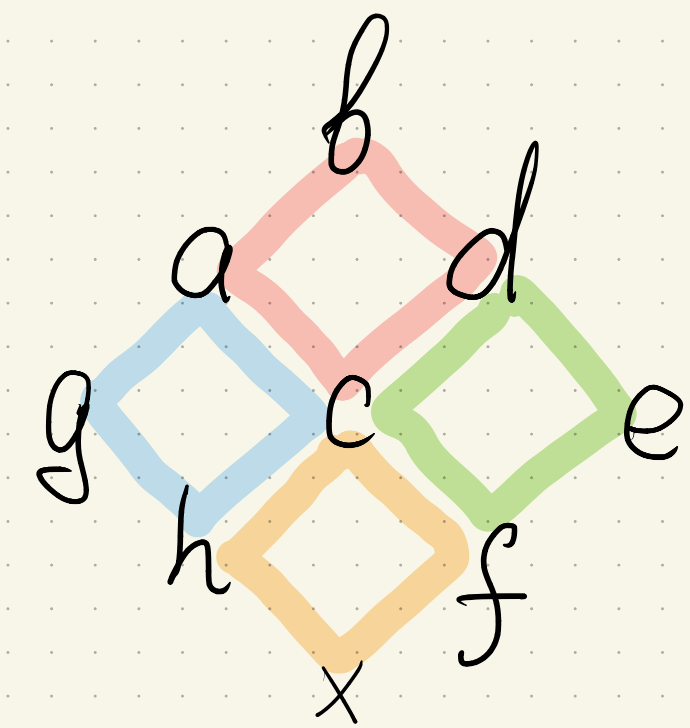
Q1 Does *every* $A \in SL_2(\mathbb{Z}_+)$ appear?

YES!









$hf - xc = 1$

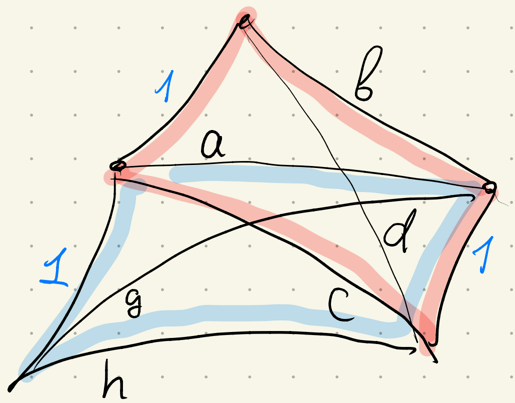
(unimodular rule)

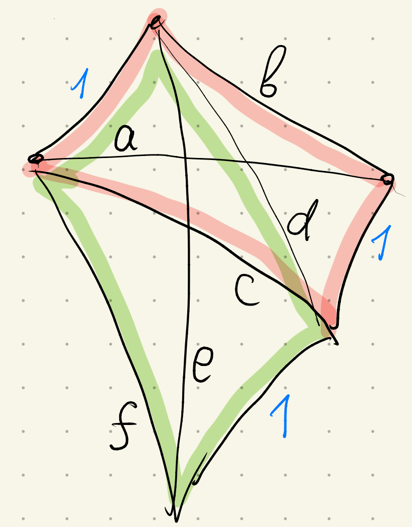
agree!

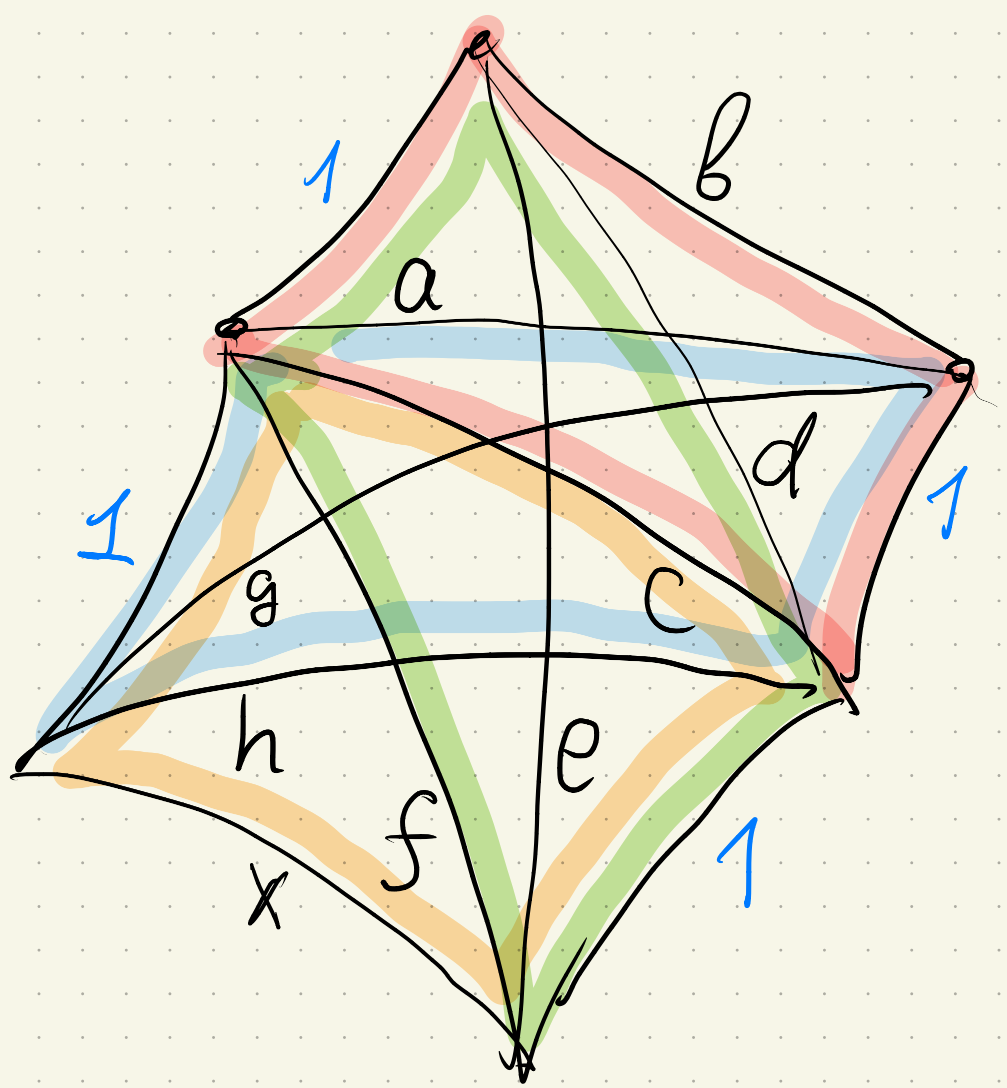
Ptolemy: $xc + 1 = hf$

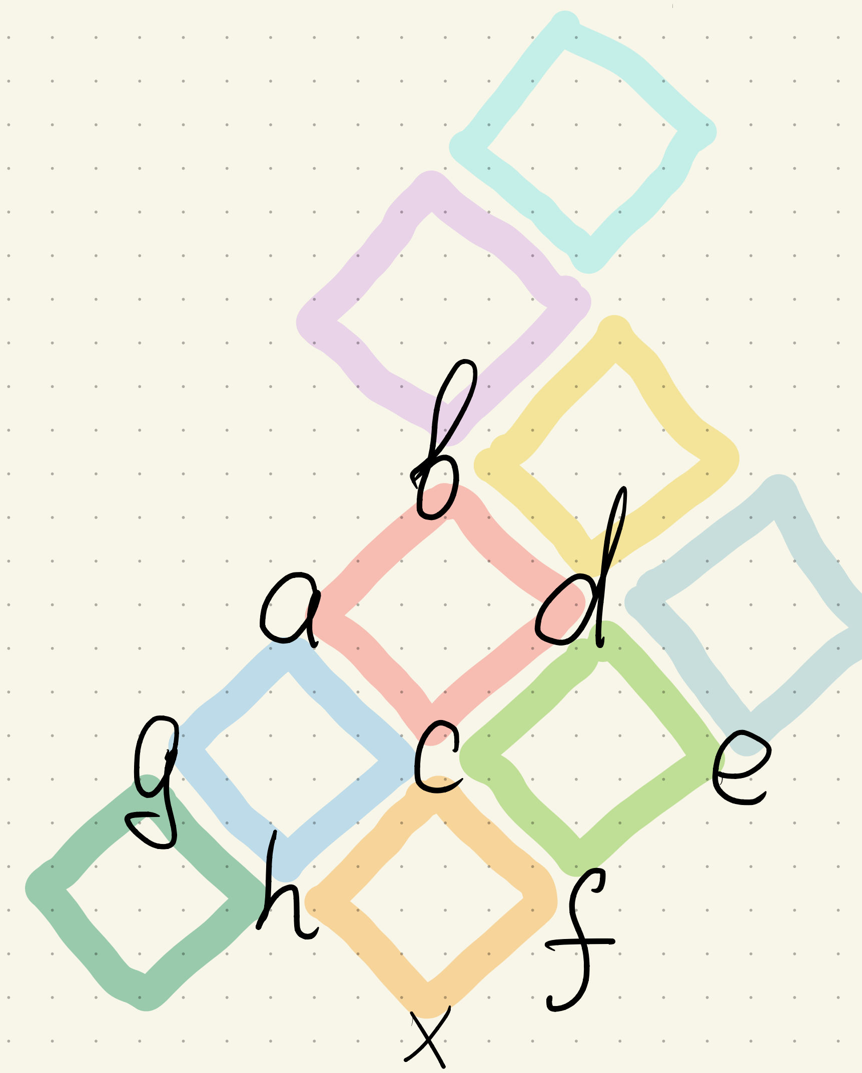
Q1 Does every $A \in SL_2(\mathbb{Z}_+)$ appear?

YES!









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(unimodular rule)

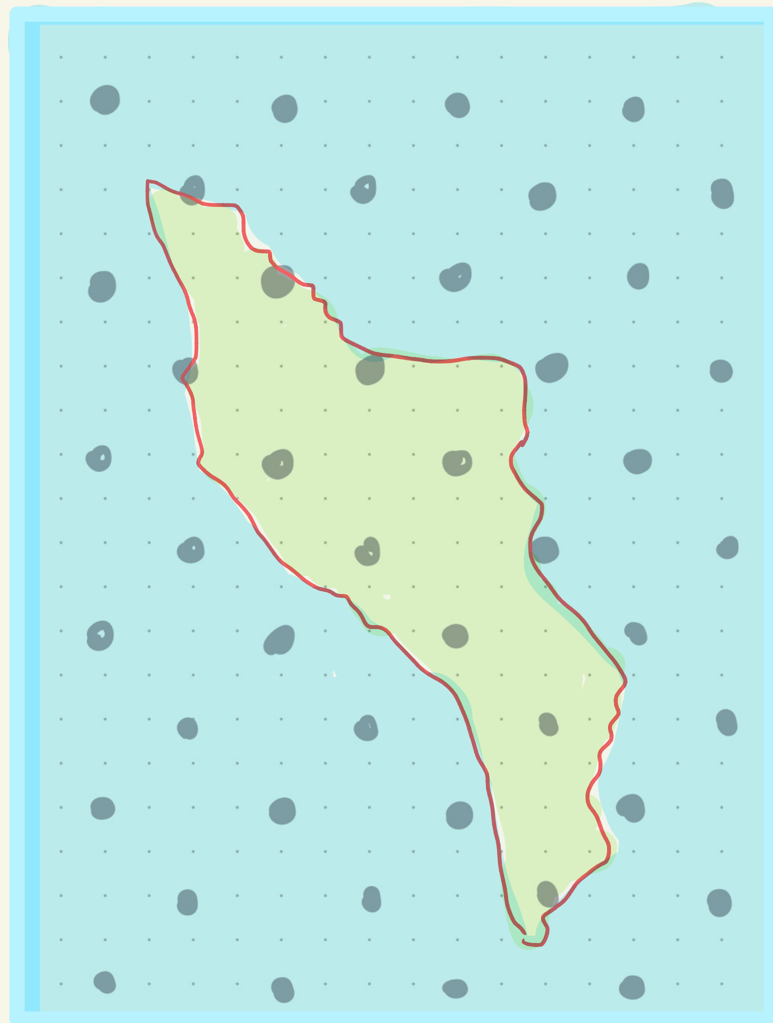
agree!

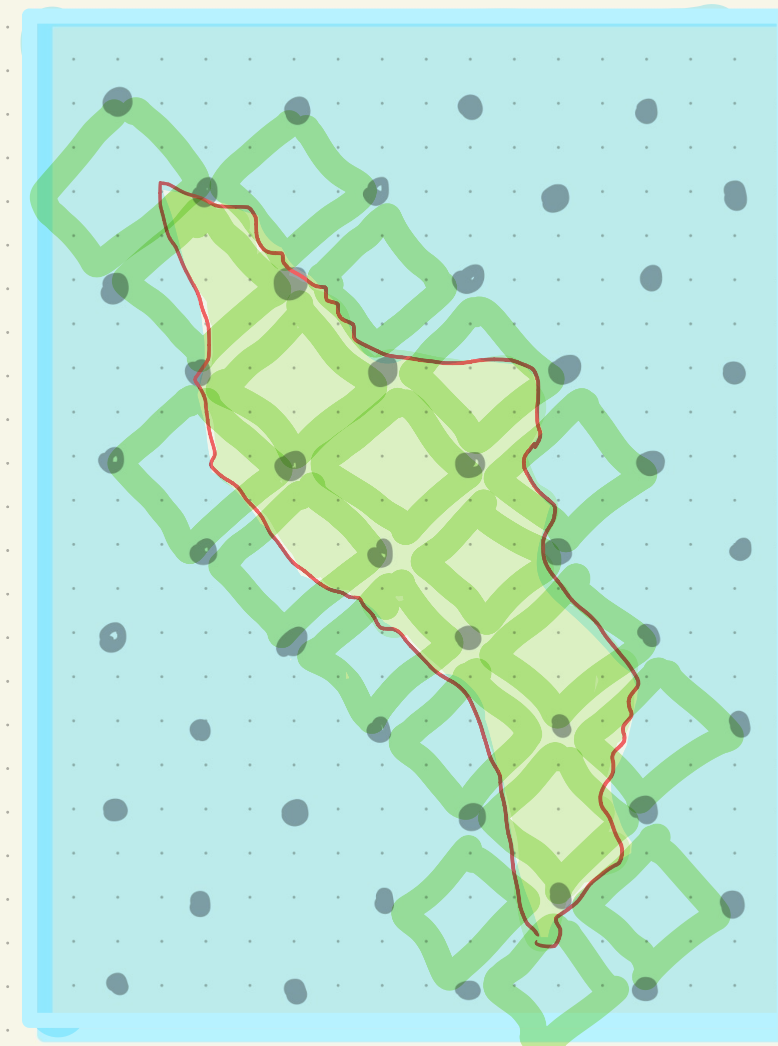
Ptolemy: $xc + 1 = hf$

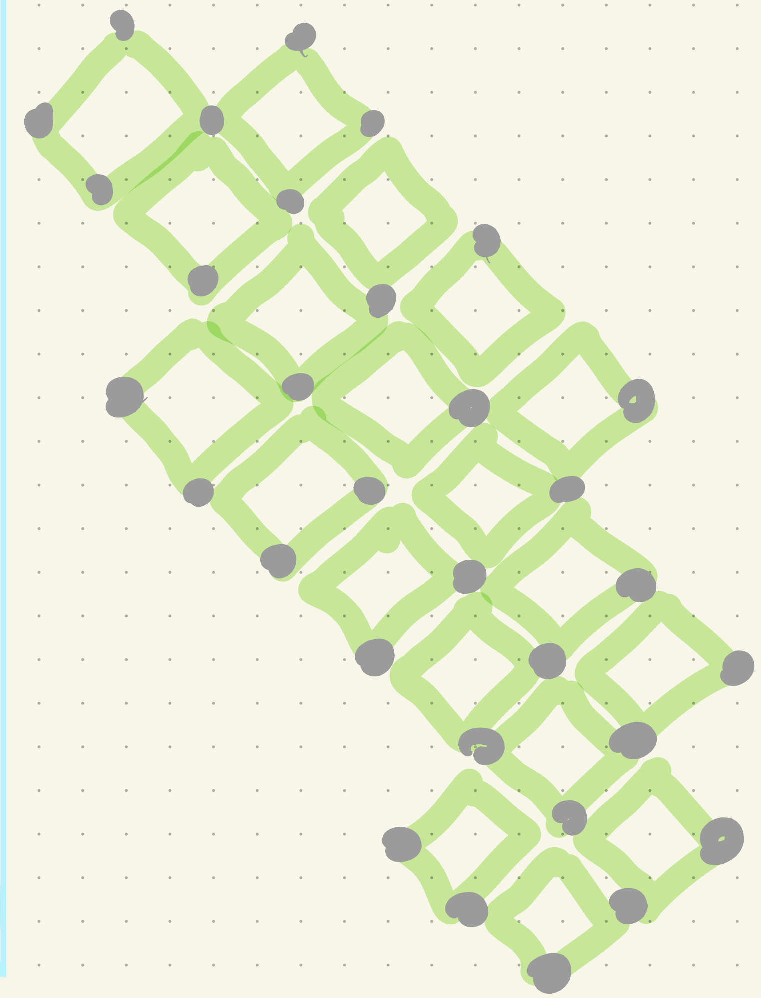
Q1 Does every $A \in SL_2(\mathbb{Z}_+)$ appear?

YES!

Take any island:







connected simply-connected domain $\mathcal{O}$ → fill the diamonds intersecting $\mathcal{O}$ with any positive integers respecting the diamond rule → Get a part of a Conway–Coxeter frieze!

2. More general definition of a frieze:

Let (S,M) be a marked surface
E = set of all arcs on S

Def A frieze on (S,M) is map $F:\gamma\to\mathbb{R}$ for each (tagged) arc $\gamma\in E$ s.t. a Ptolemy relation holds for every quadrilateral

← Ring homomorphism of cluster algebra $A(S)$ to $\mathbb{R}$

- F is positive if $F:\gamma\to\mathbb{R}_+$
  integer if $F:\gamma\to\mathbb{Z}$

Penner

Geometrically:

frieze on S = decorated hyperbolic structure on S

Ex A surface S with a triangulation T defines a frieze by $F(\alpha)=1\ \forall\alpha\in T$
(positive, integer)

$F(\gamma)=\lambda(\gamma)$ for all $\gamma\in E$

2. More general definition of a frieze:

A frieze F on (S, M) is **unitary**
if $\exists$ triangulation T : $F(\alpha) = 1 \quad \forall \alpha \in T$

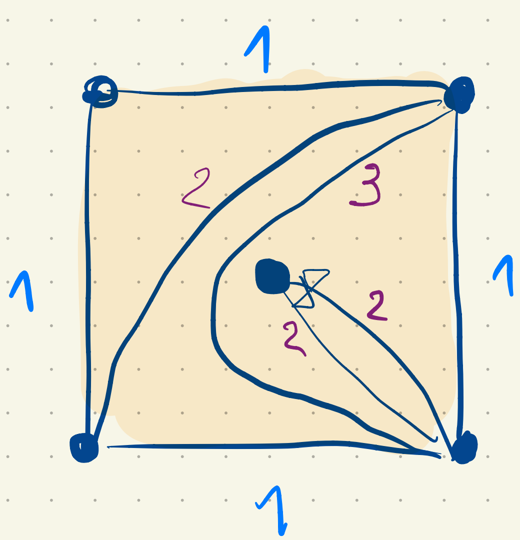
Q2: *Are all friezes on a given surface unitary?*

- Conway, Coxeter '76 : **YES**, if S = polygon (type A)
- Thomas '2009 ← D_4
  Fontaine, Plamondon '2016 **No**, if S = punctured polygon (type D)
- Gunawan, Schiffler '2018 **YES**, if S = annulus (type $\tilde{A}$)
- Çanakçi, AF, PT '2022 **YES**, if S = pair of pants
  García Elsener

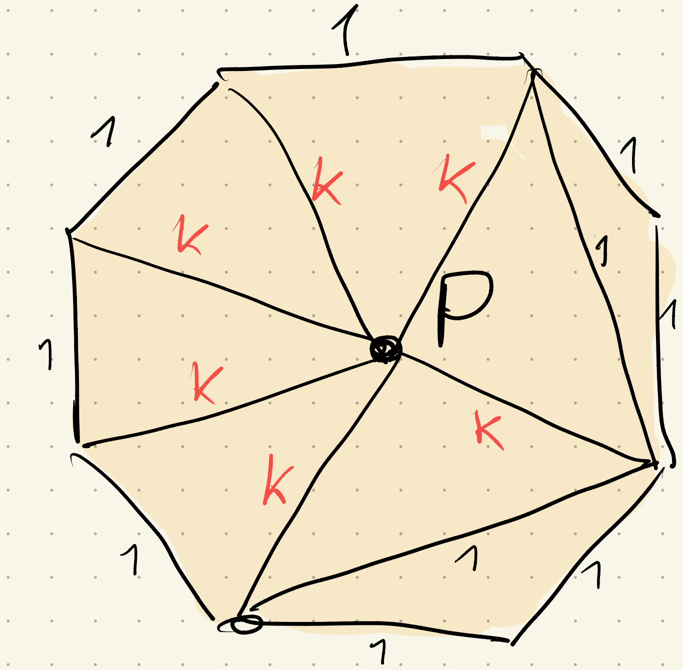
Frieze = $F: \gamma^{\in S} \to \mathbb{R}$ + Ptolemy

Frieze → positive, integer

1 1 1 1
2 2 2 2
3 3 3 3
2 2 2 2
2 2 2 2



3. Construction of non-unitary friezes on punctured surfaces.



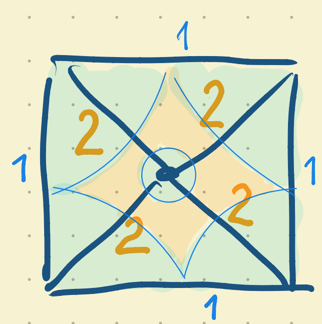
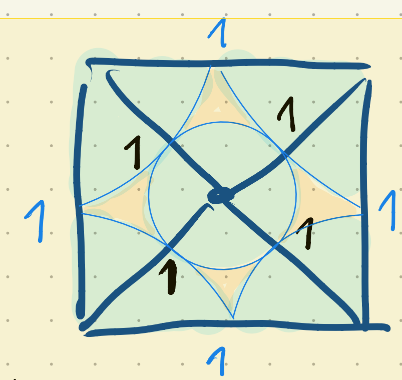
- Fontaine, Plamondon:

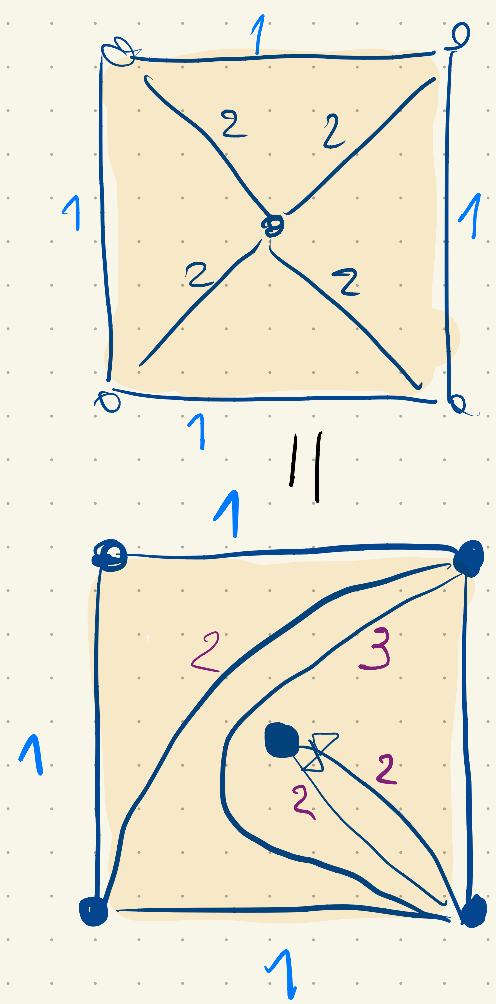
  - Take a triangulated punctured disk,
  - denote m = valency of the puncture P
  - let k be a divisor of m
  - define $F(\gamma) = \begin{cases} k, & \gamma \text{ incident to } P \\ 1, & \text{othewise} \end{cases}$

$m=6$ välency $k = 1, 2, 3, 6$ $k|m$

Get a frieze, non-unitary, if $k \neq 1, m$. ←

F, P: This gives all friezes in $\mathcal{D}_n$!

Geometrically,  and  gives the same hyperbolic structure on S with different choice of horocycle at P



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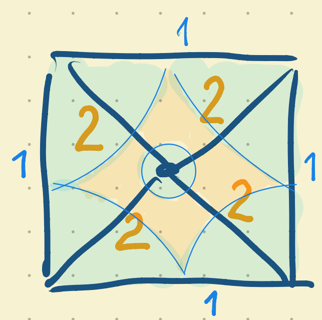
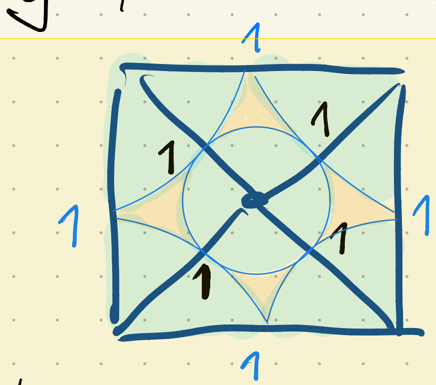
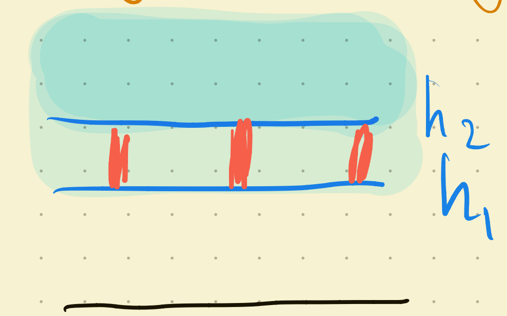
Geometrically,  and  gives the same hyperbolic structure on S with different choice of horocycle at P

Changing a horocycle:

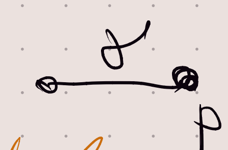


$\ell_1 = d + \ell_2$

$\lambda_1 = e^{\frac{\ell_1}{2}} = e^{\frac{d}{2}} \cdot e^{\frac{\ell}{2}}$

= scaling λ-lengths

$\Rightarrow$ Complete controle on $\lambda(\gamma)$:

- $\lambda(\gamma)$, γ not incident to P is not affected
- $\lambda(\gamma)$ for  multiplied by k
- $\lambda(\gamma)$ for  divided by k

3. Construction of non-unitary friezes on punctured surfaces.

Def: Given unitary F, a triangulation T is unitary $\Leftrightarrow$ $F(\gamma) = 1 \quad \forall \gamma \in T$.

Thm 1

Let $\hat{F}$ be a unitary frieze on a surface S

T the unitary triangulation for F

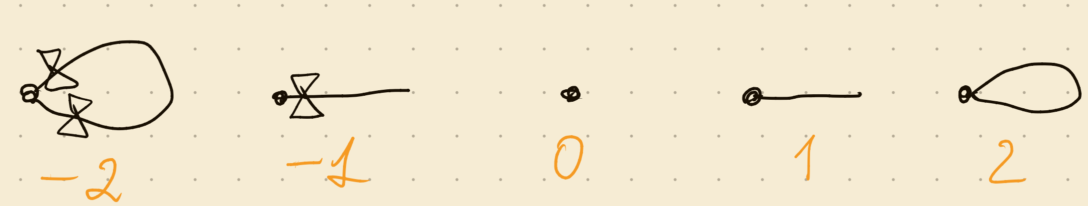
$\{P_i\}$ = punctures on S,

m_i = valency of P_i in T,

k_i = divisor of m_i

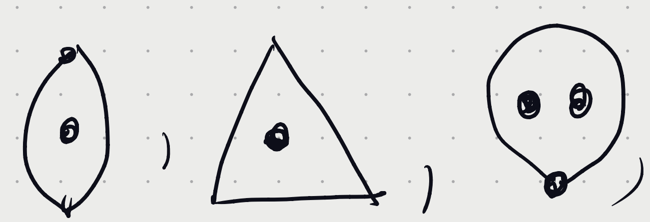
Then there exists a frieze $F(\gamma) = k_i^{\varepsilon_i(\gamma)} k_j^{\varepsilon_j(\gamma)} \hat{F}(\gamma)$

where ε_i = # ends of γ at P_i =



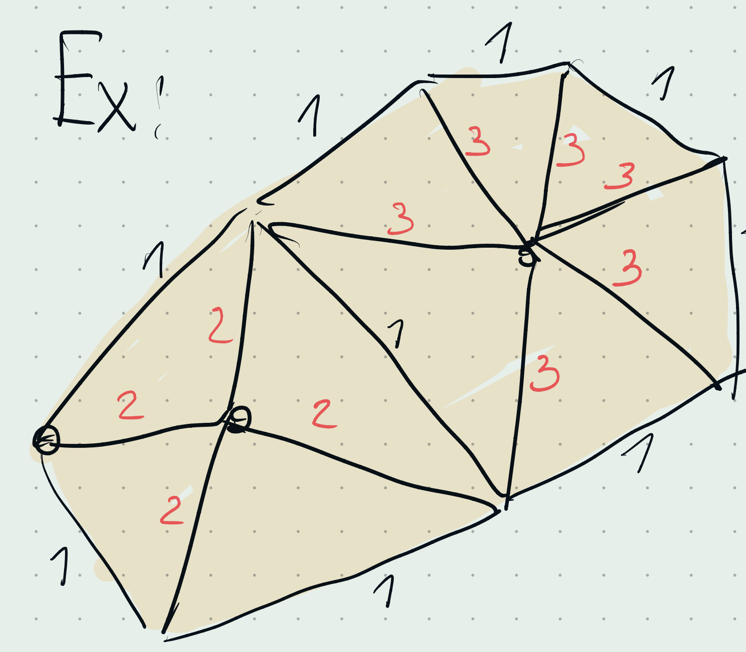
Change horocycles on the same hyperbolic surface. Then

- λ-lengths are changed as prescribed
- remain integer (check based on $k_i | m_i$)

Corollary If S is punctured, $S =$ ,

then there exists a non-unitary frieze on S

Ex:



4. Friezes on bordered surfaces

Idea:

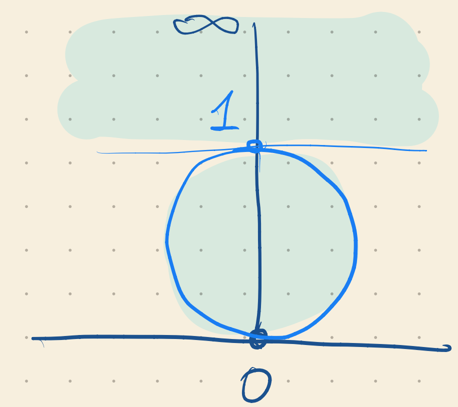
Lift to $\mathbb{H}^2$ → Compare with Farey triangulation

Uniquely (up to isometry of $\mathbb{H}^2$) by Uniformization Thm

$S = \mathbb{H}^2/\Gamma$

some group on $\mathbb{H}^2$

- Lift a boundary segment to 0∞, with Ford horocycles

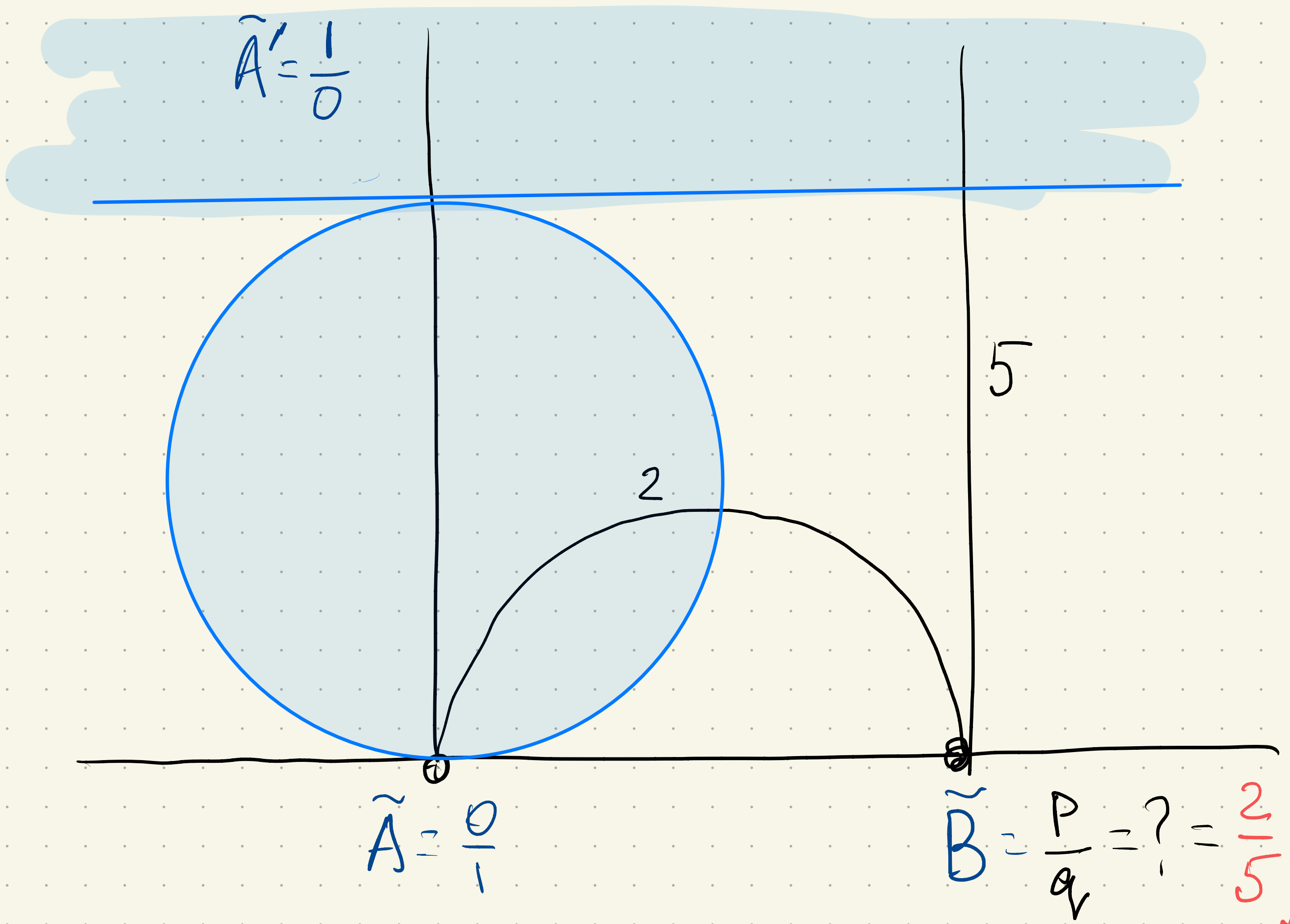


- Show that all lifts of all marked points lie in $\mathbb{Q}$
- and that different lifts of horocycles scale Ford circles by the same coefficient
- Get triangulation on S and (scaled) Ford circles.

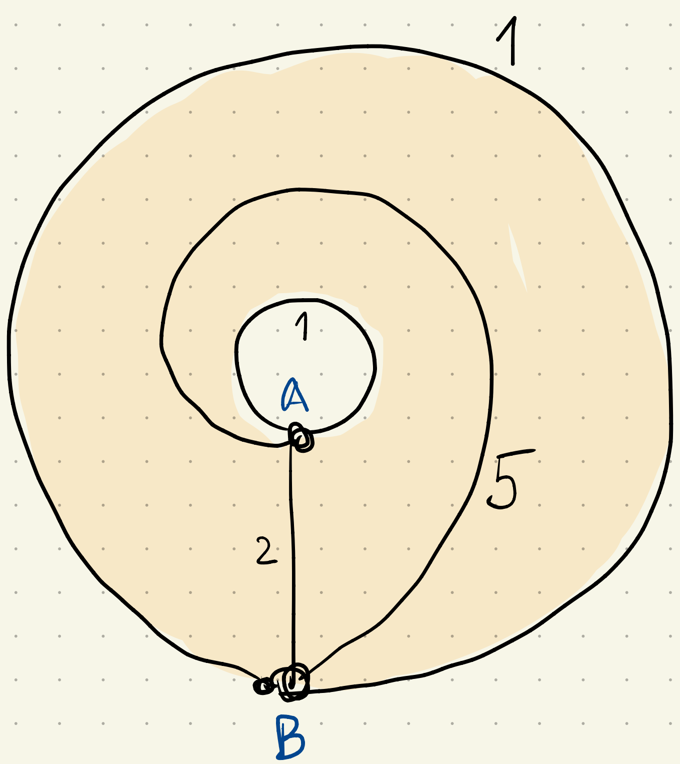
surface S with a frieze = S with a decorated hyperbolic structure

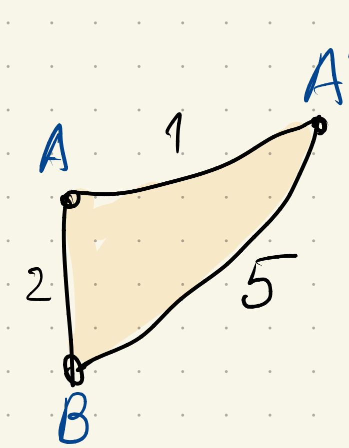
Induce the triangulation on S

4. Friezes on bordered surfaces

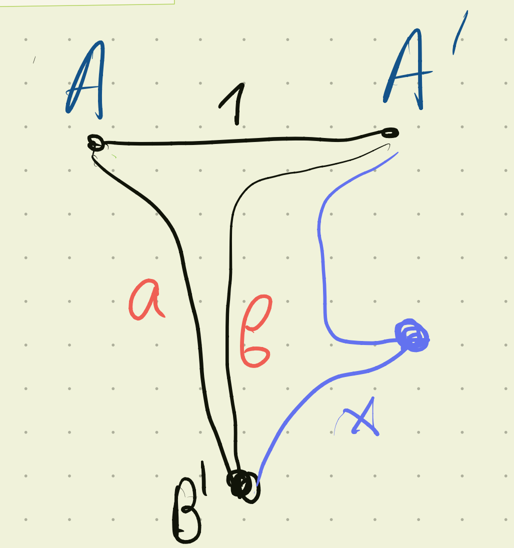


Ex





Any other lift of AB?



$x \cdot 1 + a \cdot y = b \cdot z$

- if $\gcd(a,b) = 1$ same reasoning
- if $\gcd(a,b) = k$ then every curve incident to B' divide by k which is wrong

If $\tilde{B} \in \mathbb{Q}$ and horocycle at $\tilde{B}$ is Ford's:

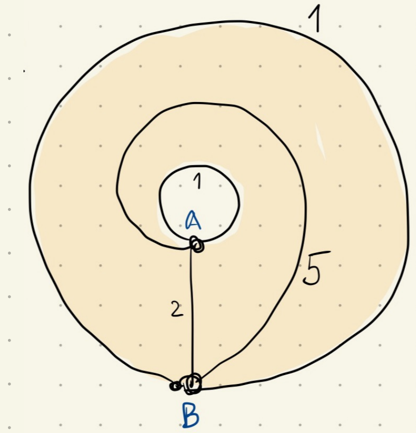
$\lambda_{\tilde{A}\tilde{B}}$: $p \cdot 1 - q \cdot 0 = 2 \Rightarrow p = 2$

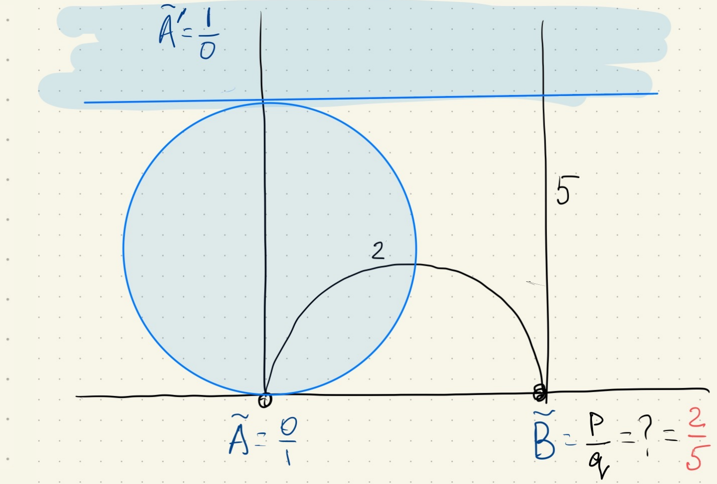
$\lambda_{\tilde{A}'\tilde{B}}$: $q \cdot 1 - p \cdot 0 = 5 \Rightarrow q = 5$

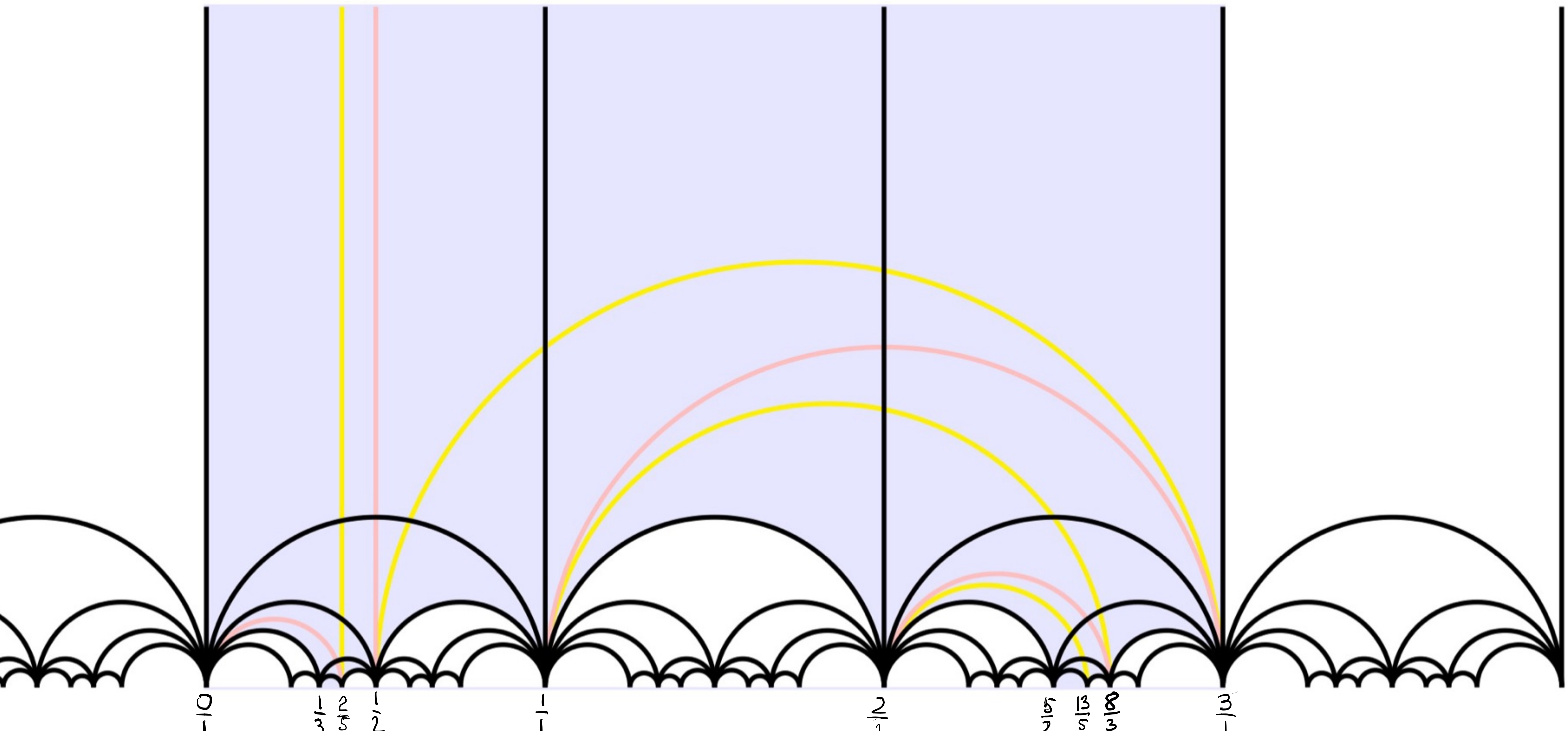
reduced fraction $\Rightarrow$ $\lambda_{(\frac{p}{q}, \frac{r}{s})} = |ps - rq|$ works

$\Rightarrow \tilde{B} = \frac{2}{5}$ – correct lift with Ford circle (unique!)

4. Friezes on bordered surfaces







4. Friezes on bordered surfaces

Thm 2. Let S be a surface with boundary, F frieze on S,
$\{P_i\}$ – marked points on S.
Then there exists a unitary frieze $\hat{F}$ on S
(with unitary triangulation T)
and positive integers $\{k_i\}$ ($k_i = 1$ for $P_i \in \partial S$)
such that k_i divides valency of P_i in T
and

$$F(\gamma) = k_i^{\varepsilon_i(\gamma)} k_j^{\varepsilon_j(\gamma)} \hat{F}(\gamma) \quad \forall \gamma \in S.$$

Corollaries

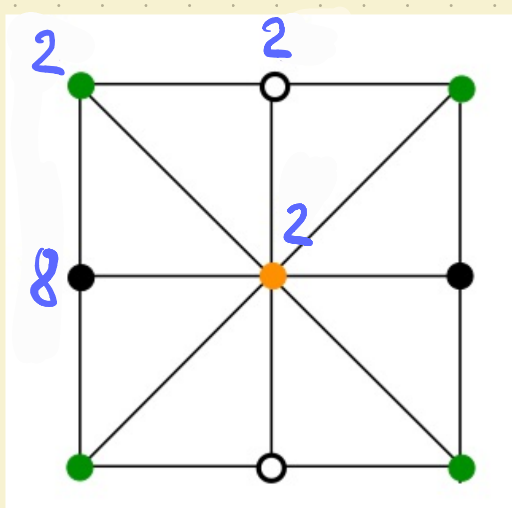
- All friezes on unpunctured surfaces are unitary.
- Friezes on unpunctured S (up to $MCG(S)$) $\overset{1-1}{\longleftrightarrow}$ combinatorial types of ideal triangulations on S
- On punctured surfaces with boundary $\#$ Friezes (up to $MCG(S)$) $< \infty$

5. Friezes on closed surfaces?

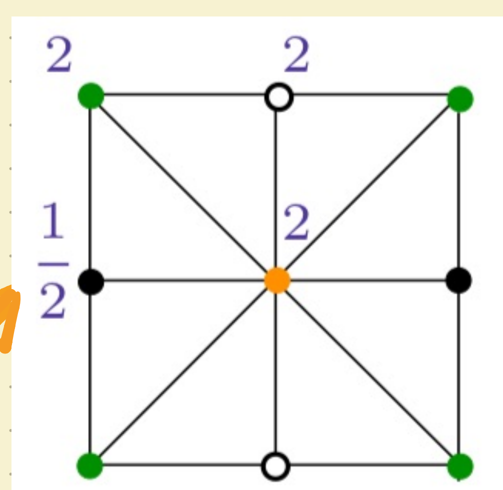
Examples: friezes on torus with 4 punctures with k_i as below:

What goes wrong?

- Can prove P_i are lifted to $\mathbb{Q}$.
- Can NOT prove the horocycles agree

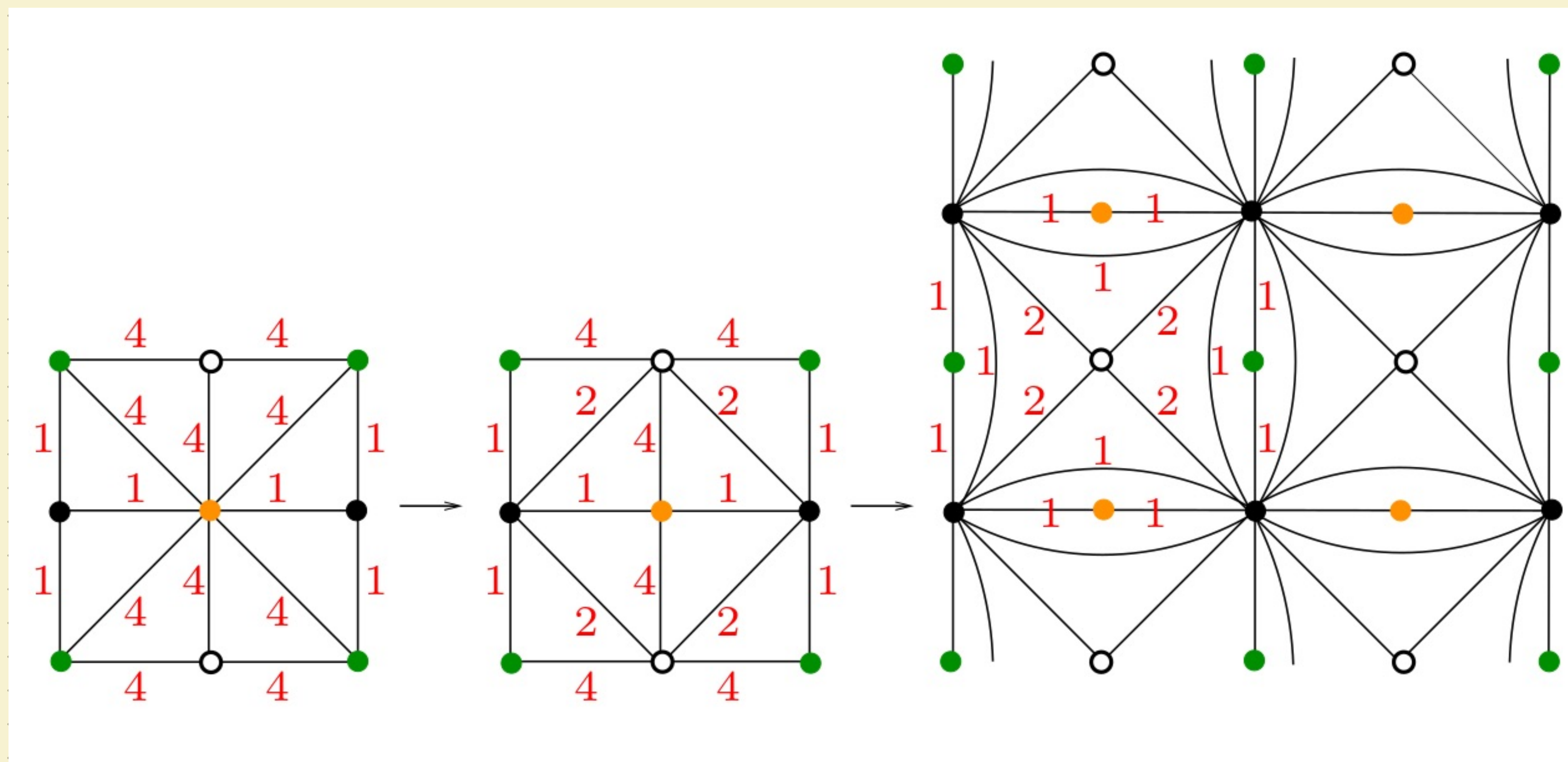


$k_i = 8$ does NOT divide $m_i = 4$



$k_i \notin \mathbb{Z}$

The same frieze is a scaling of two different triangulations:

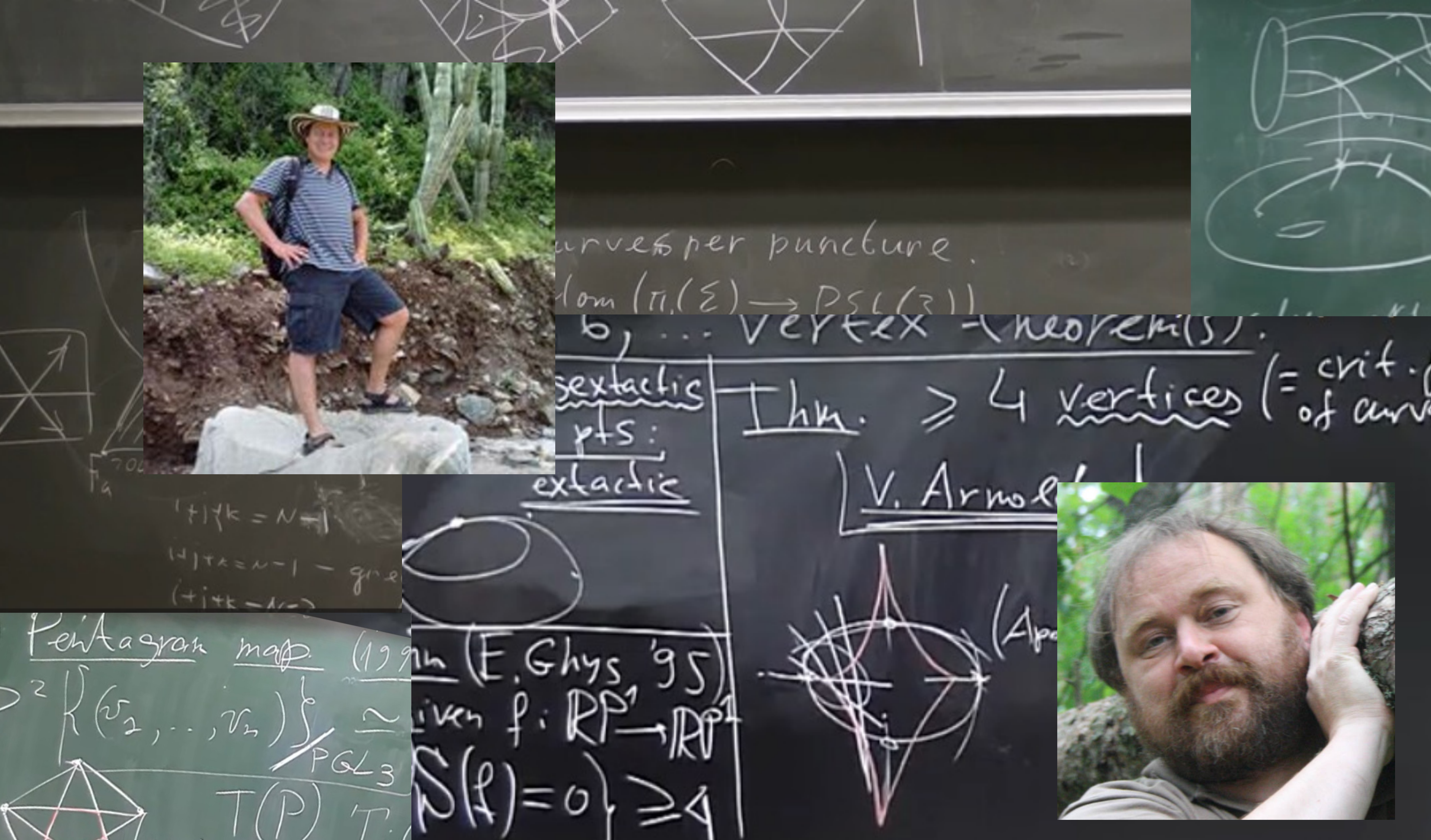


5. Friezes on closed surfaces ?

Thm 3

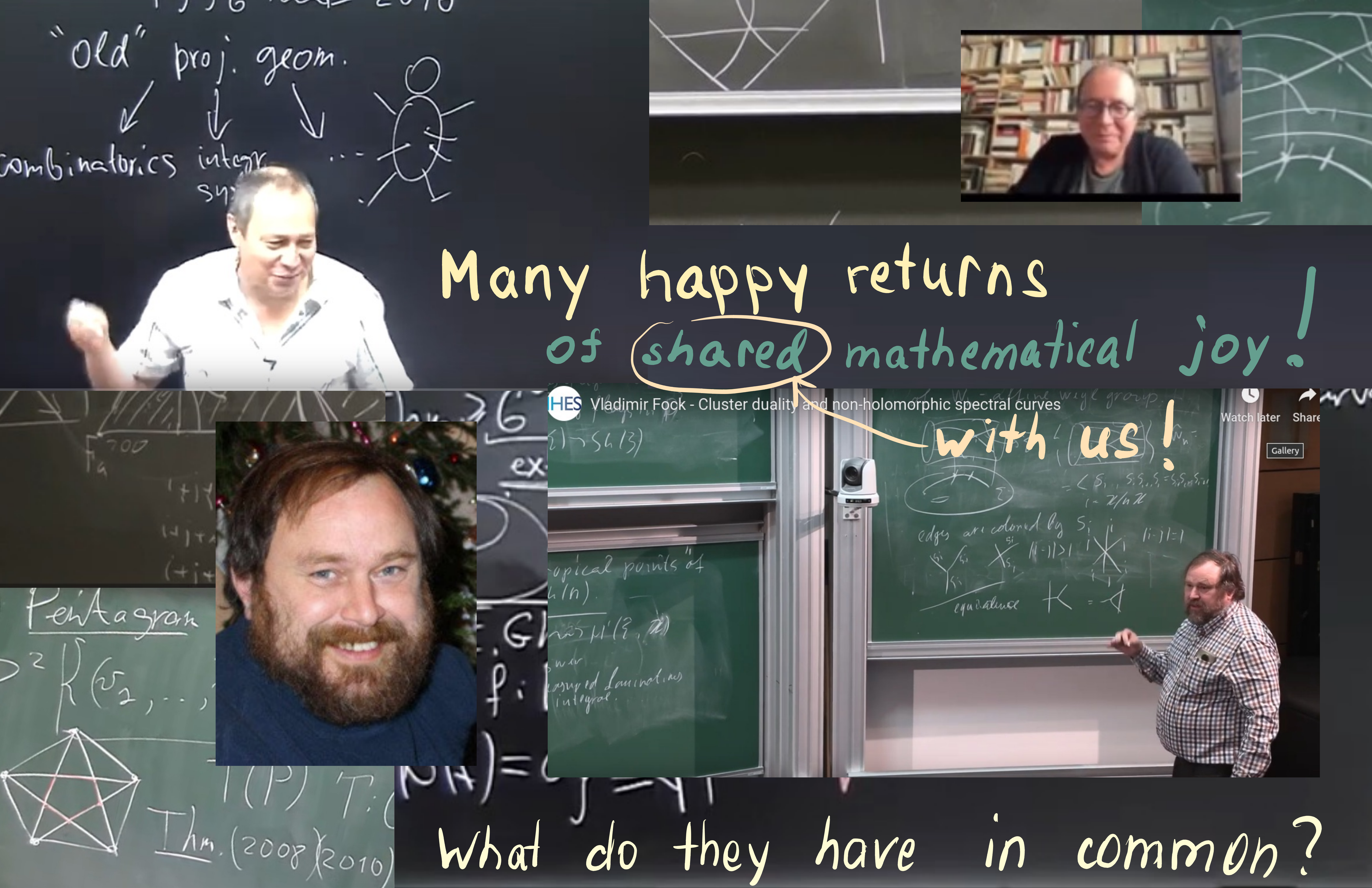
Let F be a frieze on S,
T be a triangulation of S such that
for any triangle $\Delta \in T$
the values on sides of Δ are mutually coprime.

Then F is unitary.

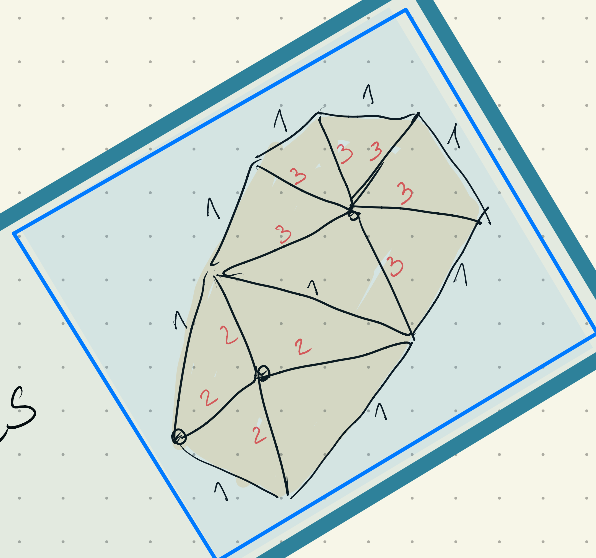


What do they have in common?


Fake! Two different pictures!
What do they have in common?


"old" proj. geom.
combinatorics
Many happy returns
of shared mathematical joy!
with us!
IHES
Vladimir Fock - Cluster duality and non-holomorphic spectral curves
Watch later
Share
Gallery
Pentagram
Thm. (2008)(2010)
What do they have in common?

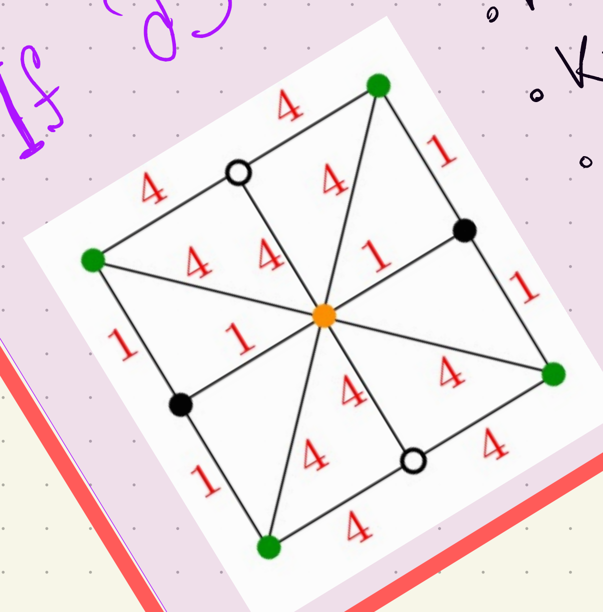
Unitary frieze scaled at punctures with $k_i \mid m_i$ is a frieze



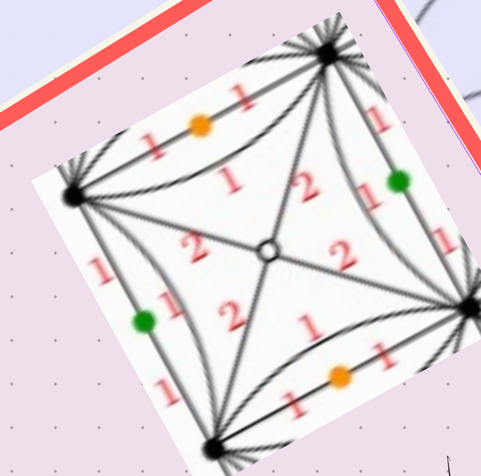
THANKS

If $\partial S \neq \emptyset$: every frieze is like this

If $\partial S = \emptyset$:



- $k_i \nmid m_i$
- $k_i \notin \mathbb{Z}$
- same F may be rescaling of different unitary friezes



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