Algebraic modular functor

(joint w/ Gus Schrader)

"A proof for a theorem is like a machine oil for a car. But if you pour oil over a pile of scrap metal, you still can't drive it."

— V. Fock

Higher quantum Teichmüller theory
[Fock-Goncharov]

S – oriented surface

G – Lie group (e.g. SL_n or PGL_n)

Γ_S – mapping class group of S

$\mathcal{M}_{G,S}$ – moduli space of "stratified" G-local systems on S

(*) $(G,S) \rightsquigarrow \Gamma_S \ltimes \mathbb{L}^q_{G,S} \circlearrowright \mathcal{S}_{G,S} \subset \mathcal{H}_{G,S}$ ← Hilbert space

cluster quantization of $\mathcal{M}_{G,S}$ → $\mathbb{L}^q_{G,S} = \mathcal{O}^{cl}_q(\mathcal{M}_{G,S})$

Schwartz subspace → $\mathcal{S}_{G,S}$

Modular functor conjecture:

Assignment (*) "respects" cutting & gluing of surfaces

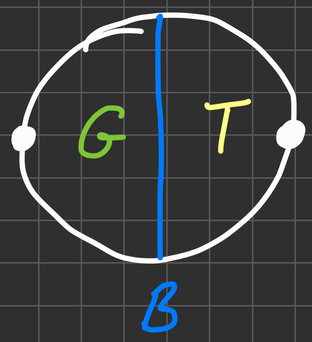
Today: relation between $\mathbb{L}^q_{G,S}$ & $\mathbb{L}^q_{G,S'}$, where $S' = S$ cut along closed simple curve c

<u>Def-n:</u> Given a "decorated" surface, a stratified local system is a

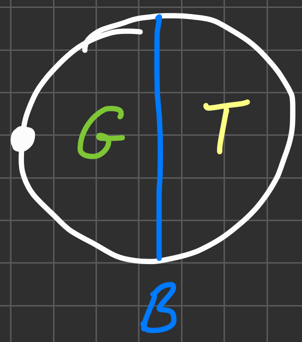
- G-local system on the union of G-regions & same for T
- B-reduction along each wall (i.e. B-subbundle in principal $G\times T$-bundle)
- G-trivialization at every marked pt in a G-region & same for T

<u>Examples</u>

<u>of $\mathcal{M}_{G,S}$</u>:



$G\backslash G\times G\times_B T\times T/T \simeq G/N$



$G\backslash G\times G\times_B T/T \simeq G/B$

More examples:

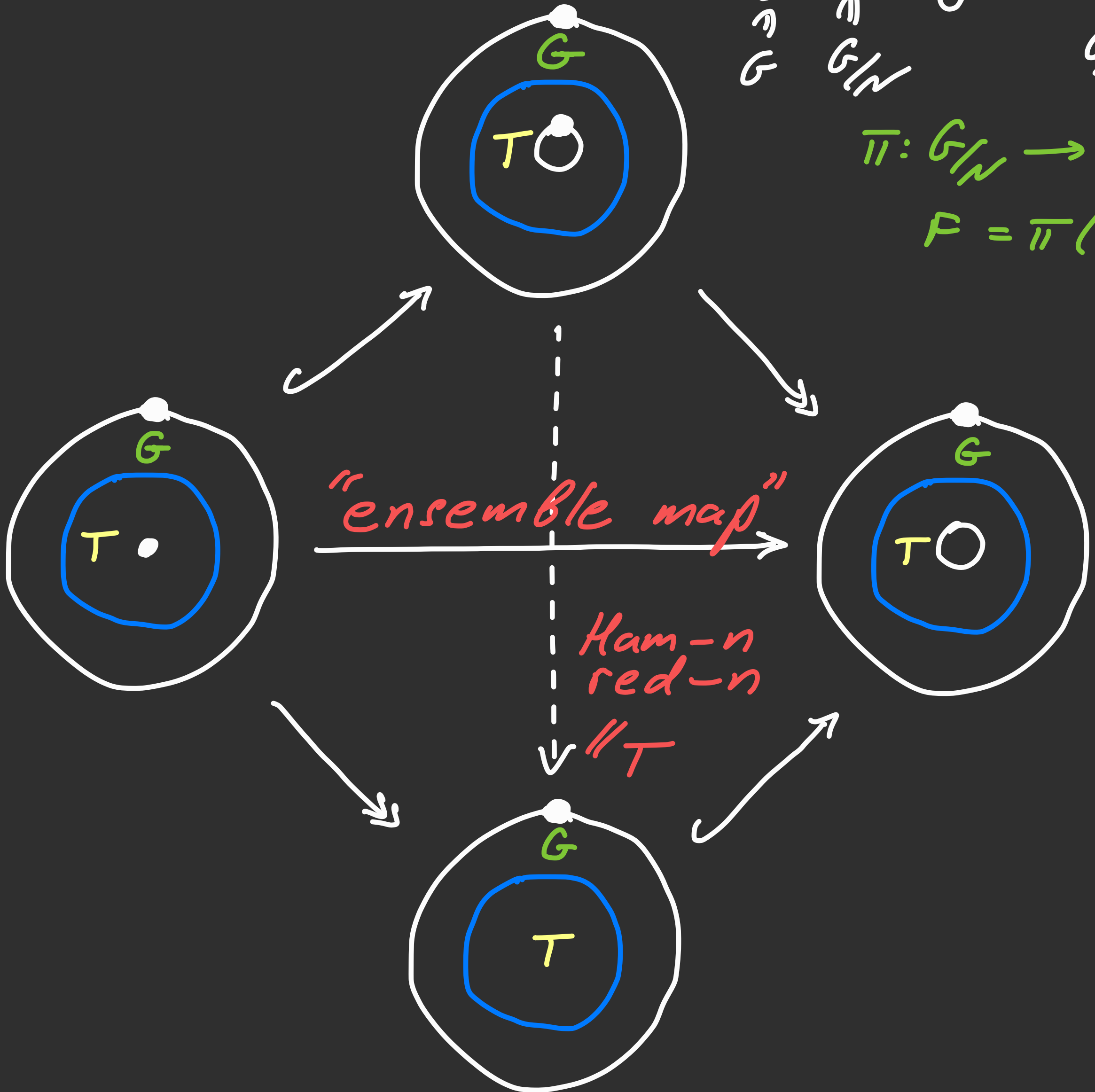
$$\{(\underset{\substack{\cap\\ G}}{g}, \underset{\substack{\cap\\ G/N}}{\bar F}) \mid g F = \underset{\substack{\cap\\ G/B}}{F}\}$$

$\pi : G/N \to G/B$

$F = \pi(\bar F)$

$$\{(\underset{\substack{\cap\\ G}}{g}, \underset{\substack{\cap\\ G/N}}{\bar F}) \mid g\bar F = \bar F\}$$

$\Rightarrow$ g-unipotent



$$\{(\underset{\substack{\cap\\ G}}{g}, \underset{\substack{\cap\\ G/B}}{F}) \mid gF = F\}$$

$\uparrow$ total space of G-S res-n

$$\{(\underset{\substack{\cap\\ G}}{g}, \underset{\substack{\cap\\ G/B}}{F}) \mid gF = F \ \&\ g\text{-unipotent}\}$$

$$\{(g, \bar F_c, \bar F_e, \bar F_r) \mid g F_c = F_c\} / G$$

<u>Thm:</u> [Schrader–S.] $S' = S$ cut along closed simple curve c & $G = SL_n / PGL_n$, e.g.

"residue subalgebra"

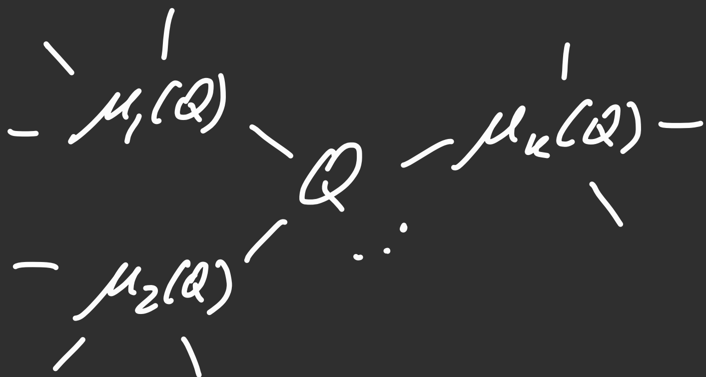
$$\Longrightarrow \quad \mathbb{L}^{q}_{G,S} \simeq \left(\mathbb{L}^{q,\,\mathrm{res}}_{G,S'} /\!\!/ T\right)^{W \simeq S_n}$$

Poisson reduction by max. torus

Weyl group invariants

Cluster varieties

$(\mathbb{C}^\times)^d$ - charts, labelled by quivers, organised into a k-regular tree ($k \le d$).



$\mu_j(Q)$ – mutation of Q

Gluing data is given by "cluster mutations".

Global algebra of functions $\mathbb{L}_Q$ is the "universally Laurent algebra"

quiver mutation class

<u>Thm:</u> [Berenstein–Fomin–Zelevinsky]

Universal Laurentness $\Leftrightarrow$ 1-step Laurentness

Quantum cluster varieties

$(\mathbb{C}^\times)^d$ - chart $\mathcal{X}_Q \rightsquigarrow$ quantum torus $\mathcal{T}_Q^q$

$$\mathcal{T}_Q^q \simeq \mathbb{C}(q)\langle Y_1^{\pm 1}, \dots, Y_d^{\pm 1}\rangle \Big/ q^{\varepsilon_{jk}} Y_j Y_k = q^{\varepsilon_{kj}} Y_k Y_j$$

ε — sign-adjacency matrix of Q

mutations $\mu_j^q : \mathrm{Frac}(\mathcal{T}_Q^q) \xrightarrow{\sim} \mathrm{Frac}(\mathcal{T}^q_{\mu_j(Q)})$

$\sim$ conjugation w/ $\varphi(y_j)$ $\quad y_j = \log Y_j$

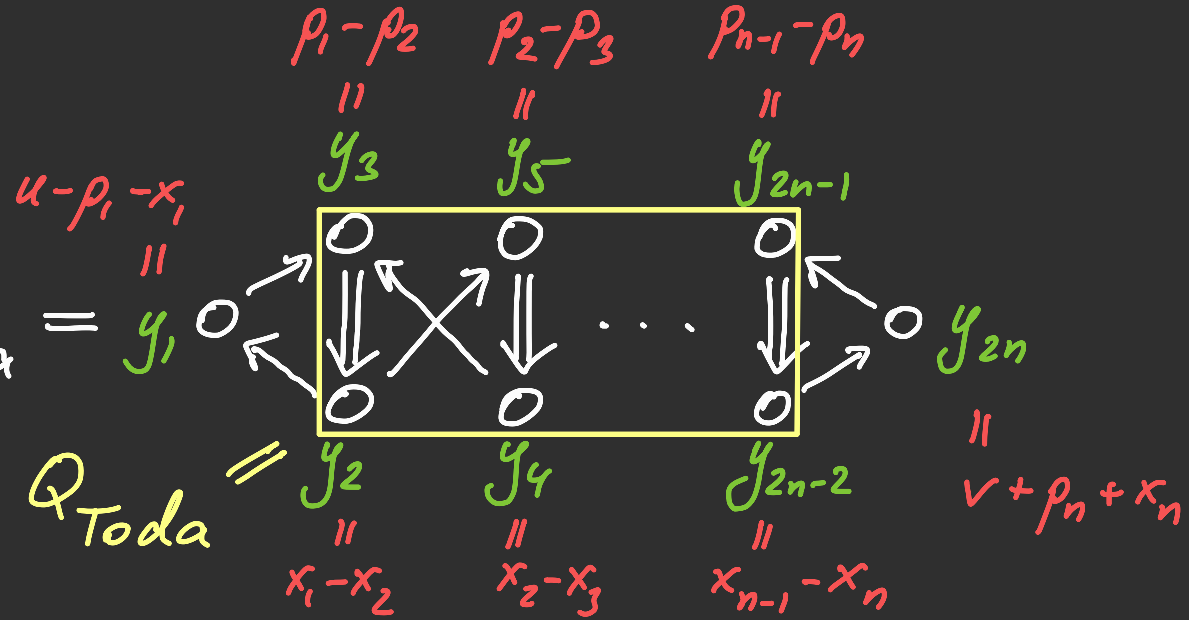
quantum dilogarithm

$$\varphi(y - \underbrace{i\hbar}_{\text{const}}) = (1 + qY)\,\varphi(y) \quad \text{— } q\text{-}\Gamma\text{amma rel-n}$$

$\mathbb{L}_Q^q \simeq \mathcal{O}_q(\mathbb{L}_Q)$ - quantum universally Laurent algebra

Cluster structure on Toda chain.

$\hat{Q}_{Toda} =$



$$p_j = \frac{1}{2\pi i}\frac{\partial}{\partial x_j}$$

$$Y_j = e^{2\pi b y_j}$$

$$q = e^{\pi i b^2}$$

$$B_n(u) := \mu^q_{2n-1} \cdots \mu^q_1 = \prod_{j=1}^{2n-1} \varphi(\ldots)$$

Baxter operator

Prop: [Schrader - S.]

$$B_n(u - {}^{ib}\!/_2)\, B_n(u + {}^{ib}\!/_2)^{-1} = \sum_{j=0}^{n} H_j^{(n)} (e^{2\pi b u})^j$$

$H_j^{(n)}$: j-th Hamilt-n of open q-Toda chain

$\Psi^{(n)}_{\bar{\lambda}}(\bar{x})$ – q-Whittaker functions, i.e.

$$H_j^{(n)}\, \Psi^{(n)}_{\bar{\lambda}}(\bar{x}) = e_j\left(e^{2\pi b \lambda_1}, \dots, e^{2\pi b \lambda_n}\right) \Psi^{(n)}_{\bar{\lambda}}(\bar{x})$$

j-th elementary symm. f-n

$$\iff B_n(u)\, \Psi^{(n)}_{\bar{\lambda}}(\bar{x}) = \prod_{j=1}^{n} \varphi(u-\lambda_j)\, \Psi^{(n)}_{\bar{\lambda}}(\bar{x}) \qquad (**)$$

Toda Bispectrality:

$$e^{2\pi b(x_1+\dots+x_j)}\, \Psi^{(n)}_{\bar{\lambda}}(\bar{x}) = \check{H}_j^{(n)}\left(\bar{\lambda}, \overline{\frac{\partial}{\partial \lambda}}\right) \Psi^{(n)}_{\bar{\lambda}}(\bar{x})$$

$t=0$ Macdonald op-s

Toda spectral transform

Thm: $\mathbb{L}^{q}_{\underline{Q}_{Toda}} \xrightarrow{\sim} \mathbb{SH}_{q,t=0} = \mathbb{C}(q,t)\langle e_j, \check{H}_j \rangle_{j=1}^{n}$

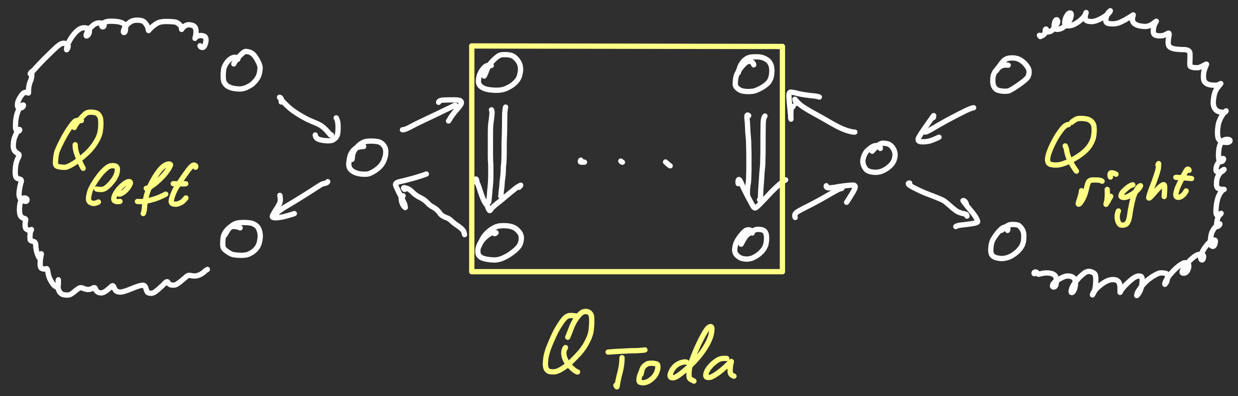
spherical DAHA at $t=0$

Back to surfaces: $S' = S$ cut along c

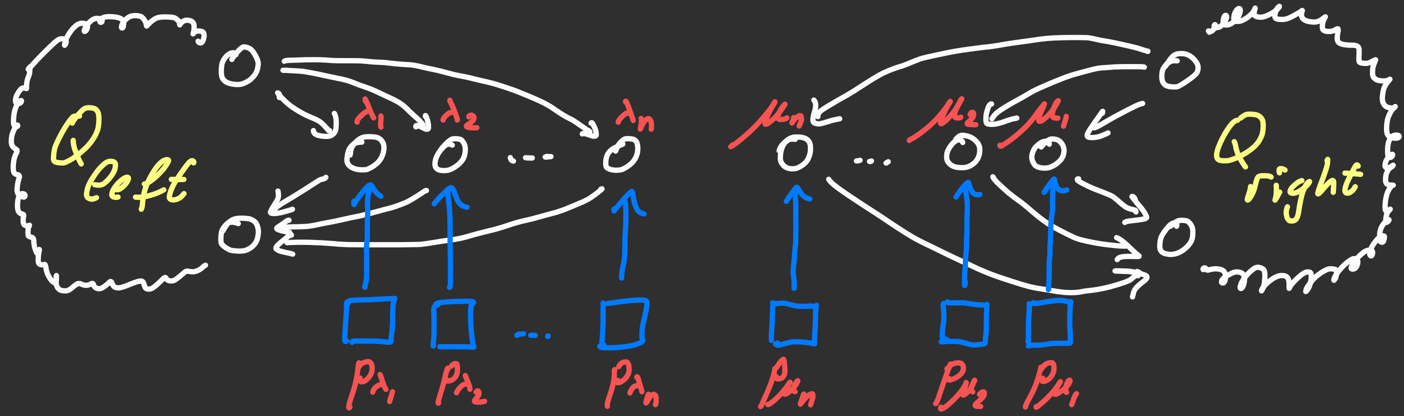
Assume: each connected component of S' contains a marked pt

$\Rightarrow \exists$ charts on $\mathcal{M}_{G,S}$ & $\mathcal{M}_{G,S'}$ s.t.

Q_{glued} =



Q_{cut} =



$$\mathbb{L}^{q,\mathrm{res}}_{G,S'} := \mathbb{L}^{q}_{G,S'}\left[\frac{1}{\lambda_j-\lambda_k}\right]_{j\neq k} \cap \underbrace{\mathbb{SH}_{q,t=0}}_{\text{imposing residue conditions}}$$

T simultaneosly scales p_λ & p_μ

quantum moment map $\iff \lambda_j + \mu_j = 0$

W permutes $\lambda_1, \dots, \lambda_n$

$$\mathcal{W}: \mathbb{L}_{G,S} \hookrightarrow \mathbb{L}_{\underline{Q}_{\mathrm{left}}} \otimes \left(\mathcal{T}^{q,\mathrm{res}}_{\underline{\hat{Q}}_{\mathrm{Toda}}} /\!/ T\right)^{W} \otimes \mathbb{L}_{\underline{Q}_{\mathrm{right}}}$$

(**) $\Longrightarrow$ 1-step Laurentness carries over

$$\Longrightarrow \mathbb{L}^{q}_{G,S} \simeq \left(\mathbb{L}^{q,\mathrm{res}}_{G,S'} /\!/ T\right)^{W}$$

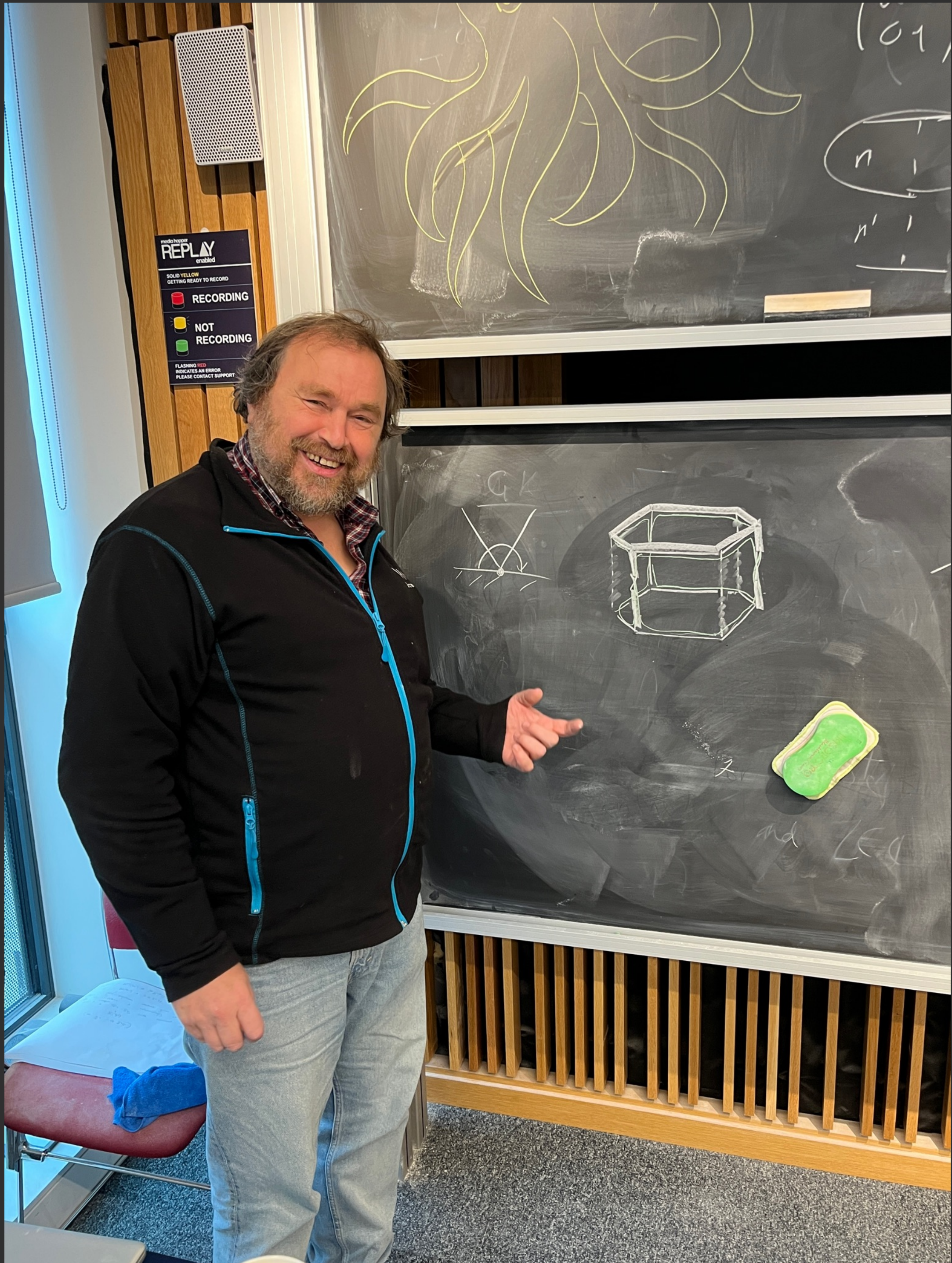
Summary

$$\mathbb{L}_{\underline{Q}_{glued}} \simeq \left(\mathbb{L}^{res}_{\underline{Q}_{cut}/\!/T}\right)^{W}$$

Need: 1) quantization

2) Toda spectral transform

Happy Birthday!


media hopper
REPLAY
enabled
SOLID YELLOW
GETTING READY TO RECORD
RECORDING
NOT
RECORDING
FLASHING RED
INDICATES AN ERROR
PLEASE CONTACT SUPPORT