Quantum Kronecker fractions

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Quantum rationals

Morier-Genoud, Ovsienko 2020: Given a regular continued fraction $x = [a_1, a_2, \ldots, a_{2m}]$, define its *q-deformation* by

$$[x]_q = [a_1, a_2, \ldots, a_{2m}]_q := [a_1]_q + \cfrac{q^{a_1}}{[a_2]_{q^{-1}} + \cfrac{q^{-a_2}}{[a_3]_q + \cfrac{q^{a_3}}{[a_4]_{q^{-1}} + \cfrac{q^{-a_4}}{\ddots + \cfrac{q^{a_{2m-1}}}{[a_{2m}]_{q^{-1}}}}}}}.$$

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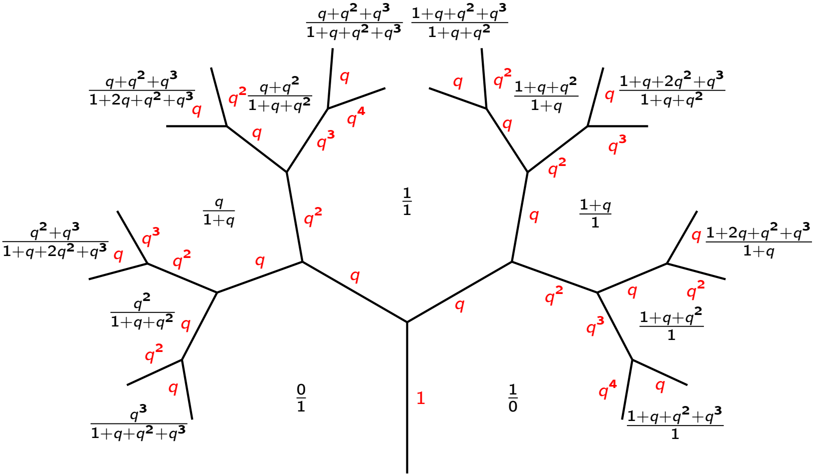
Example. $\frac{2}{5} = [0, 2, 2] = [0, 2, 1, 1]$, so

$$\left[\frac{2}{5}\right]_q = \cfrac{1}{[2]_{q^{-1}} + \cfrac{q^{-2}}{[1]_q + \cfrac{q}{[1]_{q^{-1}}}}} = \frac{q^3 + q^2}{q^3 + 2q^2 + q + 1}.$$

Quantized Conway-Farey Topograph



Quantum rationals

A quantum rational is a rational function of q:

$$\left[\frac{r}{s}\right]_q = \frac{\mathcal{R}(q)}{\mathcal{S}(q)},$$

- $\mathcal{R}(1) = r$, $\mathcal{S}(1) = s$, $\mathcal{S}(0) = 1$,
- $\mathcal{R}$, $\mathcal{S}$ are coprime, monic polynomials with non-negative integer coefficients
- $\deg(\mathcal{R}) = a_1 + a_2 + \cdots + a_{2m} - 1$ and $\deg(\mathcal{S}) = \deg(\mathcal{R}) - a_1$.

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We have the following general formulas

- Shift formula $[x + n]_q = q^n [x]_q + [n]_q, \; n \in \mathbb{N}$,
- Negation formula $[-x]_q = -q^{-1} [x]_{q^{-1}}$,
- Inversion formula $\left[\frac{1}{x}\right]_q = \frac{1}{[x]_{q^{-1}}}$.

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Note that $\mathcal{R}$ and $\mathcal{S}$ depend on both r and s. For example, the denominator "5" is quantised differently in $\frac{1}{5}$ and $\frac{2}{5}$:

$$\left[\frac{1}{5}\right]_q = \frac{q^4}{q^4+q^3+q^2+q+1}, \quad \left[\frac{2}{5}\right]_q = \frac{q^3+q^2}{q^3+2q^2+q+1}.$$

Quantum irrationals

Let $x \in \mathbb{R}$ be irrational, and $(x_n) \subseteq \mathbb{Q}$ a sequence of rationals with $x_n \to x$ as $n \to \infty$.

Morier-Genoud and Ovsienko proved that the sequence of quantised $([x_n]_q = \sum_{k\geq 0} \varkappa_{n,k} q^k)$ *stabilises* in the sense that more and more terms of the Taylor expansion in q become fixed, and defined the quantisation of x by

$$[x]_q := \sum_{k\geq 0} \varkappa_k q^k, \qquad \text{where} \qquad \varkappa_k = \lim_{n\to\infty} \varkappa_{n,k}.$$

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For example, for the golden ratio $\varphi = \frac{1+\sqrt{5}}{2} = [1, 1, 1, \ldots]$ we have

$$[\varphi]_q = [1]_q + \cfrac{q}{[1]_{q^{-1}} + \cfrac{q^{-1}}{[1]_q + \cfrac{q}{[1]_{q^{-1}} + \ddots}}} = 1 + \cfrac{q}{1 + \cfrac{q^{-1}}{[\varphi]_q}},$$

$$[\varphi]_q = \frac{q^2 + q - 1 + \sqrt{(q^2+3q+1)(q^2-q+1)}}{2q},$$

or, as the series

$$[\varphi]_q = 1 + q^2 - q^3 + 2q^4 - 4q^5 + 8q^6 - 17q^7 + 37q^8 - 82q^9 + 185q^{10} - 423q^{11} + 978q^{12} - 2283q^{13} + 5373q^{14} - 12735q^{15} + 30372q^{16} \ldots$$

with the sequence of coefficients in $[\varphi]_q$ coinciding (up to the alternating sign) with

Radius of convergence

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Leclere, Morier-Genoud, Ovsienko, V. 2021 conjectured that for any real $x > 0$ the radius of convergence of $[x]_q$ is at least R_*. This was proved for *metallic numbers* of the form $[0, n, n, n, \ldots]$, $n \in \mathbb{N}$ by **Ren 2022** and in general case recently by **Elzenaar, Gong, Martin and Schillewaert 2024**.

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We will be interested in opposite case: when the radius of convergence of $[x]_q$ is maximal. If we exclude the case of positive integers

$$[n]_q = 1 + q + q^2 + \cdots + q^{n-1},$$

the **maximal radius of convergence is 1**. Indeed, in rational case the denominator $\mathcal{S}(q)$ is a monic polynomial with $\mathcal{S}(0) = 1$, so it cannot have all roots with modulus > 1. In the irrational case the series is clearly diverges at $q = 1$ since all the coefficients are integer.

Kronecker fractions

We call fractions r/s **Kronecker** if the corresponding quantum version $[r/s]_q$ has maximal radius of convergence 1, which means that the denominator $\mathcal{S}(q)$ in $\left[\frac{r}{s}\right]_q = \frac{\mathcal{R}(q)}{\mathcal{S}(q)}$ has all zeros on the unit circle.

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Recall that the nth *cyclotomic polynomial* is defined by

$$\Phi_n(x) = \prod_{\substack{k=1 \\ \gcd(k,n)=1}}^{n} (x - e^{2\pi k/n}) :$$

$\Phi_1(x) = x-1,\ \Phi_2(x) = x+1,\ \Phi_3(x) = x^2+x+1, \dots, \Phi_{10}(x) = x^4-x^3+x^2-x+1, \dots$

Note that apart from Φ_1, all cyclotomic polynomial have *palindromic coefficients*.

Palindromicity theorem

From the known properties of quantum rationals

$$[x+n]_q = q^n [x]_q + [n]_q, \ n \in \mathbb{N}, \quad [-x]_q = -q^{-1}[x]_{q^{-1}}$$

it follows that if $x \in \mathbb{Q} \cap (0,1)$ is Kronecker then the same is true for is $x+n$ and $1-x$. So we can reduce our search of Kronecker fractions to $x \in (0, 1/2)$.

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We say that continued fraction of $x = \frac{r}{s} = [0, a_2, \ldots, a_{2n}]$ is *palindromic* if the tuple $(a_2, \ldots, a_{2n})$ is palindromic.

Evans, Winn and V. 2024: The continued fraction expansion of every Kronecker fraction is palindromic.

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First observe that if $x = r/s$ is Kronecker, then the denominator $\mathcal{S}(q)$ of $[r/s]_q$ is palindromic (by Kronecker's theorem). Now the proof follows from

Lemma (EVW 2024) The denominator $\mathcal{S}(q)$ of $[r/s]_q$ is palindromic if and only if the continued fraction of $r/s = [0, a_2, \ldots, a_{2n}]$ is palindromic.

The proof is based on the result of **Leclere and Morier-Genoud 2021**, who proved the palindromicity of the trace of the corresponding matrix M_x.

Serret criterion

Theorem (Serret, 1848) A rational number r/s with $r < s$ has palindromic continued fraction if and only if s divides $r^2 - 1$.

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In combination with our Theorem we have

Corollary. For any Kronecker fraction $r/s \in (0,1)$ s divides $r^2 - 1$.

This simplifies the search for Kronecker fractions. In particular, there are 9 fractions in $(0,1)$ with palindromic continued fraction and numerator 17:

$$\frac{17}{18}, \quad \frac{17}{24}, \quad \frac{17}{32}, \quad \frac{17}{36}, \quad \frac{17}{48}, \quad \frac{17}{72}, \quad \frac{17}{96}, \quad \frac{17}{144}, \quad \frac{17}{288}.$$

All of them are Kronecker, except $\frac{17}{72} = [0,4,4,4]$.

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One can use this also to identify potential series of the Kronecker functions. For example, if $s = n(n+1)(n+2)$ then for r we have the following possibilities:

$$n^2 + 3n + 1, \; n^2 + n - 1, \; 2n^2 + 4n + 1,$$

leading to 3 families of Kronecker fractions $K6, K7, K8$.

Known Kronecker families

Case	Continued Fraction	Rational	q-Denominator
$K1$	$[0, n+1]$	$\frac{1}{n+1}$	$[n+1]_q$
$K2$	$[0, 2, n-1, 2]$	$\frac{2n-1}{4n}$	$[2]_q[n]_q(1+q^2)$
$K3$	$[0, n, 1, n]$	$\frac{n+1}{n(n+2)}$	$[n]_q[n+2]_q$
$K4$	$[0, n, 2, n]$	$\frac{2n+1}{2n(n+1)}$	$[n]_q[n+1]_q(1+q^2)$
$K5$	$[0, n, 1, n, 1, n]$	$\frac{n^2+3n+1}{n(n+1)(n+3)}$	$[n]_q[n+1]_q[n+3]_q$
$K6$	$[0, n, n+3, n]$	$\frac{n^2+3n+1}{n(n+1)(n+2)}$	$[n]_q[n+1]_q[n+2]_q(1-q+q^2)$
$K7$	$[0, n+2, n-1, n+2]$	$\frac{n^2+n-1}{n(n+1)(n+2)}$	$[n]_q[n+1]_q[n+2]_q(1-q+q^2)$
$K8$	$[0, n, 1, 2n, 1, n]$	$\frac{2n^2+4n+1}{2n(n+1)(n+2)}$	$[n]_q[n+1]_q[n+2]_q(1+q^{n+1})$
$K9$	$[0, n+1, 1, n-1, 1, n+1]$	$\frac{n^2+3n+1}{n(n+2)(n+3)}$	$[n]_q[n+2]_q[n+3]_q$
$K10$	$[0, 2n+1, 1, n-1, 1, 2n+1]$	$\frac{2n^2+4n+1}{4n(n+1)(n+2)}$	$[n]_q[n+1]_q[n+2]_q(1+q^{n+1})^2$
$K11$	$[0, n, 2, 2n, 2, n]$	$\frac{8n^2+8n+1}{4n(n+1)(2n+1)}$	$[n]_q[n+1]_q[2n+1]_q(1+q^2)^2$
$K12$	$[0, n, 1, 2n, 1, 2n, 1, n]$	$\frac{4n^3+12n^2+9n+1}{n(n+2)(2n+1)(2n+3)}$	$[n]_q[n+2]_q[2n+1]_q[2n+3]_q$
$K13$	$[0, n, 1, n, 2n+2, n, 1, n]$	$\frac{2n^4+8n^3+12n^2+8n+1}{2n(n+1)^3(n+2)}$	$[n]_q[n+1]_q^3[n+2]_q(1+q^{n+1})(1-q+q^2)$

Sporadic cases with denominator less than 2000

Case	Continued Fraction	Rational	q-Denominator
$E1$	$[0, 2, 1, 1, 2, 1, 1, 2]$	$\frac{31}{80}$	$[2]_q^3[5]_q(1+q^2)$
$E2$	$[0, 3, 3, 1, 3, 3]$	$\frac{49}{160}$	$[2]_q^2[5]_q(1+q^2)^3$
$E3$	$[0, 3, 2, 1, 1, 1, 2, 3]$	$\frac{71}{240}$	$[2]_q^2[3]_q[5]_q(1+q^2)^2$
$E4$	$[0, 2, 1, 2, 3, 2, 1, 2]$	$\frac{89}{240}$	$[2]_q^2[3]_q[5]_q(1+q^2)^2$
$E5$	$[0, 2, 3, 1, 2, 1, 3, 2]$	$\frac{127}{288}$	$[2]_q^3[3]_q^2(1+q^2)^2(1-q+q^2)$
$E6$	$[0, 2, 2, 1, 5, 1, 2, 2]$	$\frac{134}{315}$	$[3]_q^2[5]_q[7]_q$
$E7$	$[0, 3, 1, 1, 6, 1, 1, 3]$	$\frac{99}{350}$	$[2]_q[5]_q^2[7]_q$
$E8$	$[0, 2, 3, 2, 1, 2, 3, 2]$	$\frac{209}{480}$	$[2]_q^2[3]_q[5]_q(1+q^2)^3$
$E9$	$[0, 3, 2, 1, 4, 1, 2, 3]$	$\frac{161}{540}$	$[2]_q[3]_q^3[5]_q(1+q^2)(1-q+q^2)$
$E10$	$[0, 4, 1, 1, 6, 1, 1, 4]$	$\frac{127}{576}$	$[2]_q^5[3]_q^2(1+q^2)(1-q+q^2)^3$
$E11$	$[0, 2, 2, 1, 1, 3, 1, 1, 2, 2]$	$\frac{251}{600}$	$[2]_q^2[3]_q[5]_q^2(1+q^2)$
$E12$	$[0, 2, 3, 1, 1, 2, 1, 1, 3, 2]$	$\frac{351}{800}$	$[2]_q^3[5]_q^2(1+q^2)^2$
$E13$	$[0, 2, 6, 1, 2, 1, 6, 2]$	$\frac{391}{840}$	$[2]_q[3]_q[5]_q[7]_q(1+q^2)(1+q^4)$
$E14$	$[0, 3, 1, 1, 2, 2, 2, 1, 1, 3]$	$\frac{251}{900}$	$[2]_q[3]_q^2[5]_q^2(1+q^2)$
$E15$	$[0, 2, 7, 4, 7, 2]$	$\frac{449}{960}$	$[2]_q^3[3]_q[5]_q(1+q^2)^2(1-q+q^2)^2(1+q^4)$
$E16$	$[0, 2, 1, 1, 2, 1, 3, 1, 2, 1, 1, 2]$	$\frac{559}{1440}$	$[2]_q^4[3]_q^2[5]_q(1+q^2)(1-q+q^2)$
$E17$	$[0, 4, 1, 2, 7, 2, 1, 4]$	$\frac{323}{1512}$	$[2]_q^2[3]_q^3[7]_q(1+q^2)(1-q+q^2)^2$
$E18$	$[0, 2, 6, 2, 1, 2, 6, 2]$	$\frac{701}{1512}$	$[2]_q^2[3]_q^3[7]_q(1+q^2)(1-q+q^2)^2$
$E19$	$[0, 3, 1, 1, 3, 2, 3, 1, 1, 3]$	$\frac{449}{1600}$	$[2]_q^3[5]_q^2(1+q^2)^3$

Braid groups and Burau representation

Artin 1925: *n-strand braid group* B_n is generated by $\sigma_1, \dots, \sigma_{n-1}$ with relations

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad i = 1, \dots, n-1,$$

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Burau 1936: *Burau representation* $\rho_n : B_n \to GL(n-1, \mathbb{Z}[t, t^{-1}])$. In the simplest case $n = 3$ it is defined by

$$\rho_3 \ : \quad \sigma_1 \quad \mapsto \quad \begin{pmatrix} -t & 1 \\ 0 & 1 \end{pmatrix}, \quad \sigma_2 \quad \mapsto \quad \begin{pmatrix} 1 & 0 \\ t & -t \end{pmatrix}, \qquad (1)$$

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where t is a formal parameter.

It is known after **Arnold 1968, Magnus and Peluso 1969** that ρ_3 is faithful.

(Note that for $n \geq 5$ the Burau representation is known to be non-faithful and that for $n = 4$ the question is still open.)

Quantum rationals and Burau specialization problem

Bharathram and Birman 2022:

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Examples: when $t = 1$ ρ_3 is specialised to the canonical homomorphism $B_3 \to S_3$, when $t = -1$ - to the homomorphism $\varphi : B_3 \to SL(2, \mathbb{Z})$ with kernel $< (\sigma_1\sigma_2)^6 >$:

$$\varphi(\sigma_1) = R := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \varphi(\sigma_2) = L^{-1} := \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

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Define $\Sigma \subset \mathbb{C}^*$ as the union of complex poles of all q-rationals and $\Sigma_* := \Sigma \cup \{1\}$.

Morier-Genoud, Ovsienko, V. 2023 The Burau representation ρ_3 specialized at $t_0 \in \mathbb{C}^*$ is faithful if and only if $-t_0 \notin \Sigma_*$.

Bharathram and Birman 2022:

At which complex specializations of $t \in \mathbb{C}^*$ is the Burau representation ρ_3 faithful?

Examples: when $t = 1$ ρ_3 is specialised to the canonical homomorphism $B_3 \to S_3$, when $t = -1$ - to the homomorphism $\varphi : B_3 \to SL(2, \mathbb{Z})$ with kernel $< (\sigma_1\sigma_2)^6 >$:

$$\varphi(\sigma_1) = R := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \varphi(\sigma_2) = L^{-1} := \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

Define $\Sigma \subset \mathbb{C}^*$ as the union of complex poles of all q-rationals and $\Sigma_* := \Sigma \cup \{1\}$.

Morier-Genoud, Ovsienko, V. 2023 The Burau representation ρ_3 specialized at $t_0 \in \mathbb{C}^*$ is faithful if and only if $-t_0 \notin \Sigma_*$.

Key observation: let

$$\varphi(\beta) = \begin{pmatrix} r & v \\ s & u \end{pmatrix}, \quad \rho_3(\beta) = \begin{pmatrix} \mathcal{R}(t) & \mathcal{V}(t) \\ \mathcal{S}(t) & \mathcal{U}(t) \end{pmatrix},$$

then $[r/s]_q = \frac{\mathcal{R}(t)}{\mathcal{S}(t)}$ and $[v/u]_q = \frac{\mathcal{V}(t)}{\mathcal{U}(t)}$ with $t = -q$.

The braids, corresponding to the Kronecker fractions, are special since they belong to the kernel of the Burau representation specialized **only at some roots on unity**.

Kronecker knots?

There is a class of *rational (or two-bridge) knots and links* labelled by the continued fractions (**Simony 1882, Schubert 1954, Conway 1967**).

Lee, Schiffler 2019, Morier-Genoud, Ovsienko 2020: For rational knot $K_{r/s}$ the (normalised) Jones polynomial can be expressed as

$$J_{\frac{r}{s}}(q) = q\mathcal{R}(q) + (1-q)\mathcal{S}(q).$$

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Here are the examples of knots and links (taken from **Rolfsen 1976**) corresponding to some fractions from the Kronecker families $K1 - K5$:

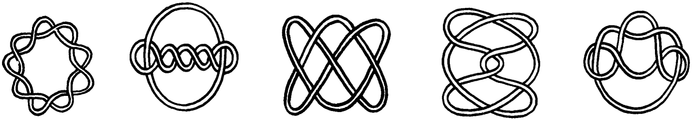

Figure: Knots and links corresponding to the Kronecker fractions $[0, 9]$, $[0, 2, 4, 2]$, $[0, 3, 1, 3]$, $[0, 3, 2, 3]$, $[0, 2, 1, 2, 1, 2]$.

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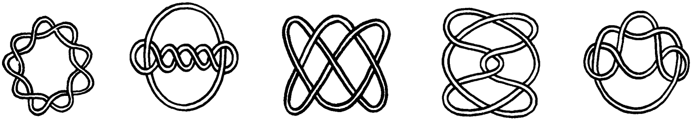

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What is special (apart from palindromic symmetry) about knots/links, corresponding to Kronecker fractions?

Lyon-2022

Happy-60!

Many Happy Returns, Valya and Volodya!