GROUPS G_n. (TALK AT THE CONFERENCE "NEW TRENDS IN GEOMETRY, COMBINATORICS, AND MATHEMATICAL PHYSICS", IN OLERON ISLAND, FRANCE, OCTOBER 21-25, 2024.)

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Contents

1. Introduction

Recall that the orbit method in representation theory, is based on the notion of **coadjoint orbit**, i.e. an orbit of a Lie group G in the space $\mathfrak{g}^*$, dual to the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$. The method works the better, the closer the Lie group is to its Lie algebra. Therefore the case of connected and simply connected nilpotent Lie groups is an ideal situation, because in this case the exponential map $\exp : \mathfrak{g} \to G$ is a diffeomorphism (a smooth bijection). The most important example of such groups is the subgroup $N_n(\mathbb{R})$ of upper unitriangular matrices in $\mathrm{GL}(n, \mathbb{R})$.

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The notion of coadjoint orbit makes sense not only for ordinary Lie groups, but for all groups for which one can define the analog of a Lie algebra and its dual space. A challenging example is the class of algebraic matrix groups over arbitrary field K. The main obstacle for applying the orbit method in this case is the absence of exponential map. But for unitriangular matrix groups, i.e., subgroups of $N_n(K)$, it can be partially compensated by introducing the so-called "fake" exponential map expf, defined as

$$\operatorname{expf}(A) := 1 + A. \tag{1}$$

This map shares the two main properties of the exponential map: it is a bijection of $\mathfrak{g}$ to G and is compatible with adjoint action. Thus, it establishes a bijection between adjoint orbits in $\mathfrak{g}$ and conjugacy classes in G.

There is also a question about the definition of the dual space $\mathfrak{g}^*$ for $\mathfrak{g}$. The good solution is to assume that K is a topological ring, whose additive group K_+ is Pontryagin self dual. [1] It includes the cases of finite fields $\mathbb{F}_q$, p-adic fields $\mathbb{Q}_p$, and the ring of adeles.

Unfortunately, the attempt to apply the orbit method to the class of unitriangular matrix groups over finite fields was until now only partially successful (see [K4]).

In this talk I want to speak about a smaller family G_n of finite groups, introduced in our forthcoming joint paper with D. B. Fuchs. For these groups, despite the above mentioned difficulties, the orbit method gives the simple answers to all questions of representation theory.

2. Definition and properties of groups $G_n(\mathbb{F}_q)$

Let $N_n(\mathbb{F}_q)$ (or simply N_n) be the group of upper unitriangular $(n \times n)$ matrices over a finite field $\mathbb{F}_q$. It is an affine algebraic group over K with the Lie algebra $\mathfrak{n}_n(\mathbb{F}_q)$ consisting of upper triangular nilpotent matrices over K. Denote by G_n the quotient group of N_{n+1} by its second commutant $[[N_{n+1},\, N_{n+1}],\, N_{n+1}]$. As usual, we denote by $\mathfrak{g}_n$ the Lie algebra of G_n and by $\mathfrak{g}_n^*$ the dual vector space.

2.1. **Coordinates.** Both $\mathfrak{g}_n$ and $\mathfrak{g}_n^*$ are vector spaces of dimension $2n-1$ over $\mathbb{F}_q$. They can be conveniently realized as subquotients of the full matrix space $\mathrm{Mat}_{n+1}(\mathbb{F}_q)$.

We choose the coordinates a_i, b_j for $A \in \mathfrak{g}_n$ and x_i, y_j for $F \in \mathfrak{g}_n^*$, where $1 \le i \le n,\ 1 \le j \le n-1$, so that

$$A = \begin{pmatrix} 0 & a_1 & b_1 & \dots & 0 & 0 \\ 0 & 0 & a_2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & a_{n-1} & b_{n-1} \\ 0 & 0 & 0 & \dots & 0 & a_n \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & 0 \\ x_1 & 0 & \dots & 0 & 0 & 0 \\ y_1 & x_2 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & x_{n-1} & 0 & 0 \\ 0 & 0 & \dots & y_{n-1} & x_n & 0 \end{pmatrix}.$$

[1] The equivalent formulation of this condition: if $\mathbf{e}$ is a non-trivial additive character of K, then every additive character χ has the form $\chi_\lambda(x) = \mathbf{e}(\lambda x)$ for some $\lambda \in K$.

For elements of the group G_n we use also the coordinates $(\vec{a}, \vec{b})$, writing $g_{\vec{a},\vec{b}} := \operatorname{expf} A_{\vec{a},\vec{b}}$.

The duality between $\mathfrak{g}_n^*$ and $\mathfrak{g}_n$ has the form $\langle F_{\vec{x},\vec{y}},\, A_{\vec{a},\vec{b}}\rangle = \sum_i a_i x_i + \sum_j b_j y_j$.

The group law in terms of coordinates has the form

$$g_{\vec{a}',\vec{b}'} \cdot g_{\vec{a}'',\vec{b}''} = g_{\vec{a},\vec{b}}, \text{ where } a_i = a_i' + a_i'', \qquad b_j = b_j' + b_j'' + a_j' a_{j+1}''.$$

All groups G_n admit the outer group automorphism $\tau\colon g \to (\check{g})^{-1}$, where "check" means the transposition with respect to the second diagonal. In terms of coordinates, τ permutes the elements in each of the pairs

$$(a_i,\, a_{n+1-i}),\ (b_j,\, b_{n-j}),\ (x_i,\, x_{n+1-i}),\ (y_j,\, y_{n-j}).$$

The automorphism τ gives rise to the involutive transformations (denoted by the same letter τ) of many other objects, related to the group G_n: the Lie algebra $\mathfrak{g}_n$, its dual space $\mathfrak{g}_n^*$, the set $Cl(G_n)$ of conjugacy classes, the set $\mathfrak{g}_n/G_n$ of adjoint orbits and set $\mathfrak{g}_n^*/G_n$ of coadjoint orbits.

The two different objects, related by τ, are called **twins**. Most of properties of an object are also valid for its twin.

2.2. **The adjoint and coadjoint actions.** They are are given by formulas:

$$(2) \qquad Ad(g_{\vec{\alpha},\vec{\beta}}) : A_{\vec{a},\vec{b}} \mapsto A_{\vec{a},\vec{b}+\alpha S_n(\vec{a})}, \qquad K(g_{\vec{\alpha},\vec{\beta}}) : F_{\vec{x},\vec{y}} \mapsto F_{\vec{x}+\vec{\alpha}T(\vec{y}),\vec{y}}.$$

Here $S_n(\vec{a})$ and $T_n(\vec{y})$ are matrices with elements from $\mathbb{F}_q$; the first matrix is rectangular with n rows and $n-1$ columns, while the second is square of size n. Explicitly they are

$$S = \begin{pmatrix} a_2 & 0 & * & 0 & 0 \\ -a_1 & a_3 & * & 0 & 0 \\ * & * & * & * & * \\ 0 & 0 & * & a_{n-1} & 0 \\ 0 & 0 & * & -a_{n-2} & a_n \\ 0 & 0 & * & 0 & -a_{n-1} \end{pmatrix}, \quad T = \begin{pmatrix} 0 & y_1 & * & 0 & 0 & 0 \\ -y_1 & 0 & * & 0 & 0 & 0 \\ * & * & * & * & * & * \\ 0 & 0 & * & 0 & y_{n-1} & 0 \\ 0 & 0 & * & -y_{n-1} & 0 & y_n \\ 0 & 0 & * & 0 & -y_n & 0 \end{pmatrix}.$$

The orbits for both actions are affine submanifolds on which G_n acts by affine transformations. The adjoint action preserves $\vec{a}$ and shifts $\vec{b}$, while the coadjoint action preserves $\vec{y}$ and shifts $\vec{x}$. The dimension of the conjugacy class, containing $g_{\vec{a},\vec{b}}$, is $\operatorname{rk} S(\vec{a})$ and the dimension of a coadjoint orbits, passing through $F_{\vec{x},\vec{y}}$, is $\operatorname{rk} T(\vec{y})$.

Note also that the center Z_n of G_n in both cases acts trivially, so the stabilizers of a point $g \in G_n$ or a point $F \in \mathfrak{g}_n^*$ contain the center. Hence, they are normal subgroups and are the same for all points $g \in C$ or all points $F \in \Omega$. Therefore, we denote them $\operatorname{Stab}(C)$ or $\operatorname{Stab}(\Omega)$ respectfully.

3. Description of unirreps of $G_n(\mathbb{F}_q)$

We show that the equivalence classes of unirreps of $G_n(\mathbb{F}_q)$ are naturally labeled by coadjoint orbits.

3.1. **Basic orbits, representations and characters.** The coadjoint orbit passing through the point $F_{\vec{x},\vec{y}} \in \mathfrak{g}_n^*$ is called **basic**, if all y-coordinates are non-zero. The unirreps corresponding to basic orbits, their equivalence classes, and their characters are also called **basic**. The structure of basic orbits is a bit different for odd and even n, so we consider separately these two types.

Type 1. $n = 2k$ is even. The matrix $T_n(\vec{y})$ has the full rank $2k$. All basic orbits have dimension $2k$. They cover the open subset $V_n \subset \mathfrak{g}_n^*$, defined by the condition $y_1 y_2 \dots y_{2k} \neq 0$. The stabilizer of any point $F \in V_n$ is the center Z_n of G_n. All elements of a basic orbit Ω have the same y-coordinates, while the x-coordinates take arbitrary values. Therefore, all basic objects - orbits, equivalence classes of unirreps and characters, - can be labeled by vector $\vec{y}$ and we denote them respectively $\Omega_{\vec{y}},\ \lambda_{\vec{y}},\ \chi_{\vec{y}}$.

Type 2. $n = 2k+1$ is odd. Now $\operatorname{rk} T_n = n-1$, while the size of T_n is n.

Proposition 1. *The 1-dimensional kernel of $T_n(\vec{y})$ is spanned by the row n-vector $\vec{A}(\vec{y})$ with coordinates $A_{even} = 0$ and coordinates A_{odd} given by*
(3)

$$A_1 = \prod_{s=1}^{k} y_{2s},\ \ A_{2i+1} = \prod_{s=1}^{i} y_{2s-1} \prod_{s=i}^{k} y_{2s}\ \ \textit{for } 1 \le i < k,\ \ A_{2k+1} = \prod_{s=1}^{k} y_{2s-1}.$$

All basic orbits have dimension $2k = n-1$ and cover the open subset $V_n \subset \mathfrak{g}_n^*$ defined by the condition $y_1 y_2 \dots y_{2k} \neq 0$. But now the y-coordinates no longer separate the orbits.

The stabilizer of the point $F_{\vec{x},\vec{y}}$ is spanned by the center Z_n ant the 1-parametric subgroup of elements $\{g_{\vec{a},\vec{0}}\}$, for with $\vec{a} = \varkappa \cdot \vec{A}(\vec{y})$ for some $\varkappa \in \mathbb{F}_q$. The coefficient $\varkappa$, is an invariant of the coadjoin action. This invariant together with coordinates $y_1, \dots y_{n-1}$ form the full system of invariants, separating the orbits. In terms of coordinates it is a polynomial function $J^{(n)}(\vec{x},\vec{y}$, linear in x-coordinates and homogeneous of degee k in y-coordinates. The explicit formula is

$$J^{(n)} = \sum_{i\,odd} A_i(\vec{y}) x_i. \tag{4}$$

Thus, all basic objects are labeled by pairs $\{\vec{y},\ \varkappa\}$ and we denote them $\Omega_{\vec{y},\varkappa},\ \lambda_{\vec{y},\varkappa},\ \chi_{\vec{y},\varkappa}$ respectively. In total, there are $Q^{2k}q$ basic orbits in V_n.

3.2. **Deviation. The case of Lie groups.** For a simply connected nilpotent Lie group G there is a standard procedure (see [K4]) for the construction of a concrete unirrep π_Ω of the class λ_Ω. Let us recall it briefly here.

We have to choose a point $F_0 \in \Omega$ and a so-called **polarization** of F_0, which is a Lie group H, satisfying two conditions:

1. $StabF_0 \subset H \subset G$ and $\dim H = \frac{\dim StabF_0 + \dim G}{2}$.
2. The restriction of F_0 to the commutant $[\mathfrak{h}, \mathfrak{h}]$ vanishes.

The second condition means that the restriction of $F_0|_{\mathfrak{h}}$ is a Lie algebra homomorphism of $\mathfrak{h}$ to the field $\mathbb{F}_q$, considered as a 1-dimensional Lie algebra with zero bracket. We denote by ρ the 1-dimensional unirrep of H

$$\rho(\mathrm{expf} A) = e^{2\pi i F_0(A)}. \tag{5}$$

The H-orbits in Ω are Lagrangian submanifolds of Ω. Let E be a complex 1-dimensional G-vector bundle over Ω with a connection ∇, whose curvature is the symplectic structure form σ on Ω. The desired unirrep π_Ω acts on the space $\Gamma(E,\, H,\, \Omega)$ of sections of E, which are covariantly constant along the H-orbits.

It is also known as the induced representation $\mathrm{Ind}_H^G \rho$ and can be realized in the space $L(G,\, H,\, \rho)$ of complex-valued functions ϕ on G, satisfying the condition

$$\phi(hg) = \rho(h)\phi(g) \text{ for all } h \in H,\, g \in G. \tag{6}$$

The group G acts on $L(G,\, H,\, \rho)$ by right shifts and $\Big(\pi_\Omega(g')\phi\Big)(g) = \phi(gg')$.

3.3. **Basic unirreps of G_n.** We imitate the construction described above, but now instead of the differential geometry we shall use combinatorics and number theory. Again we consider separately the cases of even and odd n.

Case I. $n = 2k$. We choose as F_0 the point $F_{\vec{0},\vec{y}} \in \Omega_{\vec{y}}$. The polarization H consists of those $g_{\vec{\alpha},\vec{\beta}}$, for which all coordinates α_{even} are zero. It is a normal abelian subgroup of G_n.

Let $\mathcal{T}$ be the abelian subgroup in G_n, which consists of elements

$$s(\vec{\tau}) = g_{\vec{\alpha},\vec{0}} \text{ with } \alpha_{odd} = 0 \text{ and } \alpha_{2i} = \tau_i, 1 \le i \le k. \tag{7}$$

Clearly, $\mathcal{T}$ is isomorphic to the vector group $\mathbb{F}_q^k$ and is complementary to H, so that the whole group G_n is a semi-direct product $\mathcal{T} \ltimes H$.

The unirrep $\pi_{\vec{y}}$ acts in the space $L(G_n,\, H,\, \rho)$ of functions ϕ on G_n, satisfying the condition (6) above. Actually, there is a bijection between $L(G_n,\, H,\, \rho)$ and $\mathrm{Fun}(\mathcal{T})$, because every function $\phi \in L(G_n,\, H,\, \rho)$ is completely determined by its restriction to $\mathcal{T}$, which is $f(\vec{\tau}) := \phi(s(\vec{\tau}))$.

To rewrite the unirrep $\pi_{\vec{y}}$ in terms of functions $f \in \mathrm{Fun}T$, we have to solve the so-called **Master Equation**:

$$s(t)g = hs(t') \text{ with given } t \in \mathcal{T}, g \in G_n \text{ and unknown } h \in H, t' \in \mathcal{T}. \tag{8}$$

The result is

$$\Big(\phi(g)f\Big)(t) = \rho(h)f(t') \text{ where } h \text{ and } t' \text{ are solutions to (8).} \tag{9}$$

This formula allows to prove

Proposition 2. *The basic character has the form*

$$\chi_{\vec{y}}(g_{\vec{a},\vec{b}}) = q^k \prod_{i=1}^{2k} \delta(a_i) \prod_{s=1}^{2k-1} \mathbf{e}(y_s b_s). \tag{10}$$

Case II. $n = 2k + 1$. Now $\operatorname{rk} T_n = n - 1$ while the size of T_n is n. All basic orbits have dimension $n-1$ and cover the open subset $V_n \subset \mathfrak{g}_n^*$ defined by the condition $y_1 y_2 \ldots y_{2k} \neq 0$. The polarization H is the same for every $F_{\vec{x},\vec{y}} \in V_n$. It consists of $g_{\vec{a},\vec{b}}$ with $\alpha_{even} = 0$. The geometry of orbits is described by

Proposition 3. *1. The 1-dimensional kernel of $T_n(\vec{y})$ is spanned by the row n-vector $\vec{A}(\vec{y})$ with zero coordinates A_{even}, and non-zero A_{odd} given by* (11)

$$A_1 = \prod_{s=1}^{k} y_{2s}, \;\; A_{2i+1} = \prod_{s=1}^{i} y_{2s-1} \prod_{s=i}^{k} y_{2s} \text{ for } 1 \le i < k, \;\; A_{2k+1} = \prod_{s=1}^{k} y_{2s-1}.$$

2. The complete system, separating orbits in V_n, consists of coordinates y_j, $1 \le j \le 2k$, and the polynomial [2]

$$J^{(n)}(\vec{x},\vec{y}) = \sum_{i=0}^{k} x_{2i+1} A_{2i+1}(\vec{y}). \tag{12}$$

3. The stabilizer of $F_{\vec{x},\vec{y}} \in V_n$ is spanned by the center Z_n and the 1-parametric abelian subgroup of elements $\{g_{\vec{a},\vec{0}}\}$, satisfying the condition

$$\vec{a} = \varkappa \cdot \vec{A}(\vec{y}) \text{ for some } \varkappa \in \mathbb{F}_q. \tag{13}$$

4. The character $\chi_{\vec{y},J^{(n)}}(g_{\vec{a}\vec{b}})$ vanishes outside the domain where $\vec{a}$ is proportional to $\vec{A}$ and is $q^k \prod_{i=0}^{k} \mathbf{e}(a_{2i+1}x_{2i+1}) \prod_{j=1}^{2k} \mathbf{e}(b_j y_j)$ on this domain.

Thus, the basic objects are labeled by pairs $\{\vec{y},\, J^n\}$ and we denote them $\Omega_{\vec{y},\,J^n},\ \pi_{\vec{y},\,J^n},\ \chi_{\vec{y},\,J^n}$ respectively. In total, there are $Q^{2k}q$ basic orbits in V_n.

It remains the problem, how to express the value $\chi_{\vec{y},\,J_n}(g_{\vec{a},\vec{b}})$ in terms of $J_n, \vec{y}$ and the parameters of the class, containing g.

E.g., for $n = 3$ the class, containing $g_{\vec{a},\vec{b}}$ with $a_2 = 0$ is determined by the adjoint invariant $I_2 = a_1 b_2 + b_1 a_3$ and we have $\chi_{\vec{y},\,\varkappa}(g) = q\mathbf{e}(\varkappa J^{(1)} + \varkappa^{-1} I_2)$.

3.4. **Ordered partitions.** We say, that a finite ordered set $S = \{n_1,\, n_2, \ldots,\, n_m\}$ of positive integers (possibly, with repetitions) is a **ordered partition** of number $n \in \mathbb{N}$, if $\sum_i n_i = n$. Denote by $\widetilde{P}_n$ the set of all such partitions for given n and by $\tilde{p}_n$ the cardinality of $\widetilde{P}_n$.

In the case $n = 0$ our definition needs a special consideration. It is convenient to agree that the set $\widetilde{P}_0$ consists of the only element, an empty set $\emptyset$, and therefore $\tilde{p}_0 = 1$. For positive n we have

Proposition 4.

$$\tilde{p}_n = 2^{n-1} \textit{ for all } n > 0. \tag{14}$$

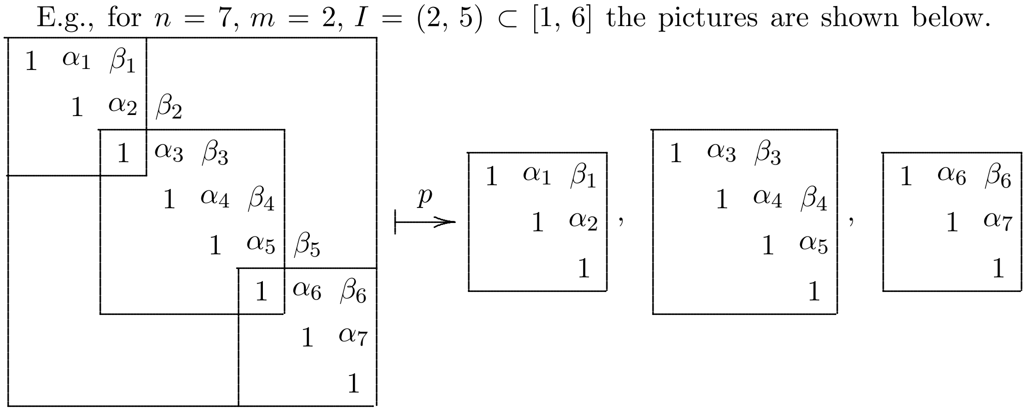
[2]To memorize the form of coordinates A_{2i+1} and monomials entering in $J^{(n)}$, note that all of them have the is $x_1 y_2 y_4 \ldots y_{2n} + y_1 x_3 y_4 \ldots y_{2n} + y_1 y_3 x_5 \ldots$

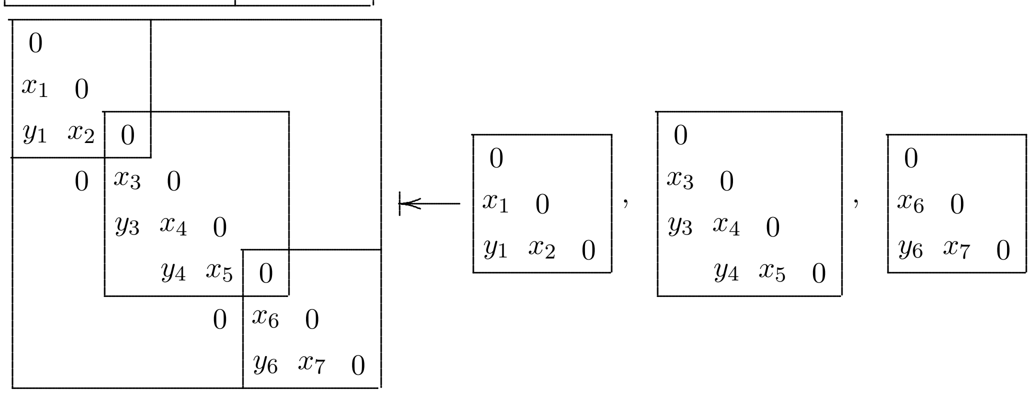
Proof. In this case all elements $S \in \widetilde{P}_n$ are non-empty. Let us sort them by the first term n_1. If $n_1 = k$, then the other terms define the ordered partition of $n-k$, hence, there are $\tilde{p}_{n-k}$ possibiliiies. We come to the recurrent relation: $\tilde{p}_n = \sum_{k\geq 1} \tilde{p}_{n-k}$[3] Comparing the recurrent relations for n and for $n-1$, we get the equality $\tilde{p}_n = \tilde{p}_{n-1} + \tilde{p}_{n-1}$. Hence, $\tilde{p}_n = 2\tilde{p}_{n-1}$ and we are done.

3.5. **The coupling operation.** For any $I \subset [1,\, n-1]$, denote by V_n^I the subset of $\mathfrak{g}_n^*$, consisted of those $F_{\vec{x},\vec{y}}$, for which $y_i = 0$ exactly when $i \in I$. When I is empty, V_n^I is an open domain in $\mathfrak{g}_n^*$, covered by basic orbits.

For a nonempty $I = \{i_1,\, \ldots,\, i_m\}$ we put $i_0 = 0$, $i_{m+1} = n$, and define the numbers n_k as $i_k - i_{k-1}$ for $k = 1, 2, \ldots, m+1$.

There is a remarkable pair of dual maps p, p^*. The first is the surjection $p : G_n \to \prod_{i=1}^{m+1} G_{n_i}$. The second is the injection $p^* : V_n^I \to \mathfrak{g}_n^*$. The definition of p and p^* in terms of coordinates looks rather cumbersome and we postpone it. Much better is to illustrate these maps by pictures. E.g., for $n = 7$, $m = 2$, $I = (2,\, 5) \subset [1,\, 6]$ the pictures are shown below.





The first picture shows that the kernel of p consists of those $g_{\vec{\alpha},\vec{\beta}}$ for wich $\beta_2 = \beta_5 = 0$. (In general case, this condition is $\beta_{i_1} = \cdots = \beta_{i_m} = 0$.) The image $p(g_{\vec{\alpha},\vec{\beta}})$ is the element $(g_1, \ldots g_{m+1})$, where coordinates of g_i are those of coordinates of $(\vec{\alpha}, \vec{\beta})$, which are inside the i-th square.

[3]For $n = 1$ this formula takes the form $\tilde{p}_1 = \tilde{p}_0$. It it justifies our choice $\tilde{p}_0 = 1$.

The second picture shows that the image of p^* is exactly $V_7^{(2,5)}$. (In general, it is V_n^I.) Let $F^{(s)}, 1 \leq s \leq m+1$, be the points in $\mathfrak{g}_{n_s}^*$ with coordinates $(\vec{x}_i^{(s)}, \vec{y}_j^{(s)}), 1 \leq i \leq n_i, 1 \leq j \leq n_j - 1$ and let $\Omega^{(s)}$ be the coadjoint orbit in $\mathfrak{g}_{n_s}^*$, passing through $F^{(s)}$. Then the point $F_{\vec{x},\vec{y}}$ in V_n^I will be the image $(F^{(1)}, \ldots, F^{(m+1)})$ under p^* iff $x_i^{(s)} = x_{i+s-1}$.

Thus, any coadjoint orbit Ω in V_n^I is in bijection with the product $\prod_{s=1}^{m+1} \Omega(s)$ of basic orbits $\Omega(s) \in \mathfrak{g}_{n_s}^*$.

The orbits $\Omega^{(s)}$ determine basic unirreps $\pi^{(s)}$ of groups G_{n_s} with the characters $\chi^{(s)}$. We can define the unirrep π of $\prod_{s=1}^{m+1} G_{n_s}$ as the direct product of $\pi^{(s)}$, and finally define the desired unirrep π_Ω of G_n by the formula $\pi_\Omega(g) = \pi(p(g)$. Clearly, the character of this representation is $\chi_\Omega(g) = \prod_{s=1}^{m=1} \chi_s(g^{(s)})$ where $g(s)$ is the s-th component of $p(g)$ in $\prod_{s=1}^{m+1} G_{n_s}$.

The general orbits in $\mathfrak{g}_n^*$ are labeled by **rigged ordered** partitions of n. Practically, such a label is visualized by a strip of height 1 and length ?, split on rectangular boxes of different size. The box, corresponding to an even summand $n_i = 2m_i$, has the format $1 \times n_i - 1$ and is filled up by $n_i - 1$ y-coordinates. To an odd summand $n_i = 2m_i - 1$ we associate the box of format $1 \times n_i - 1$ and is fill it up by $n_i - 1$letters y; sizes $1 \times n_i$. A box of odd length 2s-1 is rigged by the letter J^{2s-1} and $2(s-1)$ letters y, and a box of even length $2s$ is rigged by $2s-1$ letters y. E.g.:

J^0	y	$y \quad J^1 \quad y$	$y \quad y \quad J^2 \quad y \quad y$	$y \quad y \quad y$	15=1+2+3+5+4

Proposition 5. *Every unirrep of G_n can be obtain by the coupling procedure from exactly one ordered partition S of n and exactly one choice of basic orbits $\Omega(s) \in \mathfrak{g}_{n_s}^*$ for $1 \leq s \leq m$.*

4. The manifold M

4.1. **General facts and notations.** There is a remarkable submanifold $M \subset G_n(\mathbb{F}_q)$, which consist of those $g_{\vec{a}\vec{b}}$, for which $A_{\vec{a}\vec{b}}^2 = 0$. When q is even, the manifold M is just the set $\mathrm{Inv}(G_n)$ of all involutions in $G_n(\mathbb{F}_q)$.[4] Indeed, for even q we have $g_{\vec{a}\vec{b}}^2 = (1 + A_{\vec{a}\vec{b}})^2 = 1 + A_{\vec{a}\vec{b}}^2$. So, $g \in M \iff g^2 = 1$.

The set M clearly is stable under inner automorphisms. Hence, it is the union of conjugacy classes. We call them M**-classes**.

4.2. **Sparse sequences.** Introduce in $\mathbb{Z}$ the relation $a << b$ which means that the integers a, b satisfy the inequality $a < b - 1$. A sequence $I = (i_1, \ldots, i_k) \subset \mathbb{Z}$ is called **sparse**, if it satisfies the condition $i_s << i_{s+1}$ for all s. A set $\{i_1, \ldots, i_k\} \subset \mathbb{Z}$ is call **sparse**, if $|i_s - i_t| > 1$ for any $s \neq t$. If we list its elements in increasing order, we get a sparse sequence.

We denote by $S(n, k)$ the collection of all sparse k-subsets in $[1, n]$ and by $s(n, k)$ its cardinality. Sometimes it is convenient to extend $I \in S(n, k)$

[4] Some authors define involutions as elements of order 2. We prefer to include in $\mathrm{Inv}(G)$ the unit element e of order 1.

to a bigger subset $\tilde{I}$, by adding two more points $i_0 = -1$ and $i_{k+1} = n+1$. Then $i_0, i_1, \ldots, i_k, i_{k+1}$ will be also a sparse sequence.

Proposition 6. *The number of sparse sequences of the type* $(n,\, k)$ *is*

$$s(n,k) = \binom{n-k}{k}. \tag{15}$$

It can be easily proven by induction, but there is more simple way. By definition, the number $\binom{n-k}{k}$ is the cardinality of the set C_n^{n-k} of all $(k-1)$-point subsets in $[1, \ldots n-k]$. So, to prove the proposition, it is enough to find a bijection between $S(n,\, k$ and C_n^{n-k}. Let $I = (i_1, \ldots, i_k)$ corresponds the subset $\Delta = (\delta_1, \ldots, \delta_{k-1})$, where $\delta_1 = i_1, \delta_s = i_s - i_{s-1} - 1$ for $2 \le s \le k-2, \delta_{k-1} = n - i_k$. Conversely, to $\Delta = (\delta_1, \ldots, \delta_{k-1})$ there corresponds $I \in S(n,\, k)$ with $i_1 = d_1, i_s = 2s - 1 + \sum_{t=1^s} \delta_t$ for$2 \le s \le k-2, \delta_{k-1} = n - i_k$.

As a curious remark, observe that the sum $\sum_{k \ge 0} s(n,\, k)$, which is the number of all sparse sequence in the segment $[1,\, n-1]$, is the well-known **Fibonacci number $\mathbf{F}_{n+1}$**.

For the amateurs of combinatorics, I can suggest the following question. Find the explicit formula for the sum of those binomial coefficients $\frac{(k+l)!}{k!\, l!}$ for which the point $(k,\, l) \in \mathbb{N}^2$ is on the line $L_{a,b,c} : ak + bl + c = 0$.

5. Anatomy of $G_n(\mathbb{F}_q)$

Here we discuss some questions about unirreps of G_n which can be answered using the orbit method.

5.1. **The function ζ_G.** Recall, that the ζ-function for a compact group G is defined as $\zeta_G(s) = \sum_{i=1}^{k} d_i^{-s}$, where $d_1, \ldots, d_k$ are dimensions of unirreps of G.[5]

It is well-known that the values of ζ_G at the points 0 and -2 have nice interpretations. Namely, $\zeta_G(0)$ is equal to the numbef $|CL(G)|$ of conjugacy classes in G and the value $\zeta_G(-2)$ is equal to the numbef $|G|$ of points in G.

Less known is the interpretation of $\zeta_G(-1)$. It is related with the manifold M, introduced in the Section 4.

Consider the linear operation J in the space Fun$G$ defined by $(Jf)(g) = f(g^{-1})$. Let us compute the trace of J, using two different bases in Fun$G$.

The first basis consists of functions $\delta_x(g) = \begin{cases} 1 \text{ if } g = x \\ 0 \text{ if } g \ne x \end{cases}$. Clearly, the matrix of J in this basis is equal to the number of fixed points for the map $g \mapsto g^{-1}$, i.e. the number of involutions in G.

The second basis consists of matrix coefficients π_{ij} of unirreps of G. Recall that the unirreps of a compact group G are of three types: **real, complex** and **quaternionic**. A unirrep π belongs to the real type, if in an appropriate

[5]For the group $SU(2, \mathbb{C})$ it is the classical Riemann ζ-function.

basis all operators $\pi(g)$ have real matrices; it belongs to the quaternionic type, if the operators $\pi(g)$ can be realized by matrices with quaternionic entries. In both cases the character χ_λ is a real-valued function. Finally, π belongs to the complex type, if its character is not real-valued. The Frobenius-Schur index of π, takes by definition the value 1, -1, 0 for the real, quaternionc or complex unirrep.

There is a nice formula for the index:

$$\operatorname{ind}\pi = \sum_{g\in G} \chi_\lambda(g^2). \tag{16}$$

Proposition 7. *For a group G, for which all unirreps belong to the real type, we have*

$$\zeta_{G_n}(-1) = |M_n|. \tag{17}$$

The proof consists of two parts. First, consider the case of even q. Here the set M_n coincides with $Inv(G_n)$ and $|M_n|$ is the number of involutions in G_n. On the other hand, the character $\mathbf{e}$ for even q takes the values ± 1, hence all unirreps are of real type. Thus, we can assume that matrices π_{ij} have real entries. Therefore, they are real orthogonal, the operator J is just the transposition and the contribution of π to the trace of J is equal to $\dim \pi$ and we are done.

Consider now the case of general q. We use the fact, that both $\mathrm{tr}\pi(g)$ and $\zeta_{G_n}(-1)$ are polynomial functions of q We already know, that they are equal for all q of the form 2^k, hence, they coincide for all q.

5.2. **Numbers $m_k(G_n)$.** Denote by m_k the number of the coadjoint orbits of dimension $2k$ in $\mathfrak{g}_n^*$, equal to the number of q^k-dimensional unirreps of $G_n(\mathbb{F}_q)$. The collection of these numbers is the important characteristic of the group. The ζ-function for the group $G_n(\mathbb{F}_q)$ is by definition

$$\zeta_{G_n}(s) = \sum_k m_k(G_n) q^{-sk}.$$

Proposition 8.

$$m_k(G_n) = Q^k q^{n-k-2}\Big[\binom{n-k-2}{k} q + \binom{n-k-1}{k-1}\Big]. \tag{18}$$

The formula (18) shows that $m_k(G_n)$ are polynomial functions of q. Therefore, many other quantities, which are defined in terms of numbers m_k, are also polynomial functions of q.

Table. Numbers m_k for the groups $G_n(\mathbb{F}_q)$ (To be corrected)

group	m_0	m_1	m_2	m_3	$\zeta(0)$	$\zeta(-1)$	$\zeta(-2)$
G_3	q^2	Q	0	0	$q(2q-1)$	q^2+q-1	q^3
G_4	q^3	$qQ(q+1)$	0	0	?	$5q^2$	q^5
G_5	q^4	$q^2Q(2q+1)$	$q^{n-4}Q^2((n-5)q+1)$	0	?	?	q^7
G_6	q^5	$q^3Q(3q+1)$	qQ	?	?	?	q^9
G_7	q^6	$q^4Q(4q+1)$	?	?	?	?	q^{11}
G_8	q^7	$q^5Q(5q+1)$	?	?	?	?	q^{13}

5.3. **The model representation.** By definition, it is a (reducible) unitary representation of a group G, which contains every equivalence class of unirreps with multiplicity 1. Some of them admit a simple geometric description.

E.g., for $G = \mathrm{SO}(3, \mathbb{R})$ the natural representation π in the space of functions on S^2, which are restrictions of polynomial funcions in coordinates (x, y, z), is a model. The $n+1$-dimensional unirrep π_n is realised here in the space of restrictions of homogeneous polynomials of degree n.

The natural generalizations of this example are the so-called **geometrical** model representations. They are acting in the space of sections of some G-vector bundle E over a G-set B.

In our paper we showed that all groups G_n have geometric model representations for which the base space is the manifold M. Unfortunately, the approprtate vector bundle is not unique.

The construction of this bundle is a sort of the packing problem, because the problem is to put "things", labeled by $\lambda \in \widehat{G}_n$, into several "containers", labeled by symbols $C(I)$, where C is a M-class and $I = \{i_1, \ldots, i_m\}$ is the sparce set of indices, for which coordinates $a_i \neq 0$ for elements of C. Every container $C(I)$ is the union of $q^\ell(q-1)^m$ "boxes" of capacity $q^{n-1-\ell}$, labeled by M-classes.

The goal of the packing procedure is: every thing wilmust be packed and every box must be completely filled up. The necessary condition for the existing such a packing is that sum of sizes of all things is equal to the sum of capacities of all containers. And it is true according to (17).

We have seen in the section 3.3 and 3.5 that the properties of a basic representation of G_n depend on parity of n and the properties of general representation, related to the given ordered partition $\tilde{p}$, depend on the number $\nu(\tilde{p})$ of odd parts. It turns out that $\nu(\tilde{p})$ also plays a role in the solution of the packing problem.

Namely, let Stab C be the common stabilizer for all elements of the M-class C. Every box in a container, related to C, is labeled by some character of $StabC$. The number of things, going to the same box is $2^{\nu(\tilde{p})}$.

6. References

[IK] I. Isaacs and D. Karagueuzian, Conjugacy in groups of uppertriangular matrices, J. Algebra, 202(1998), 704?711. See also : Erratum, J. Algebra, v. 208 (1998), no. 2, p.722.

[K1] A. Kirillov, Unitary representations of nilpotent Lie groups, (in Russian), Dokl. Akad. Nauk, vol 138, N2, 1961, pp 283-284 MR [22:740].

[K2] A. Kirillov, Variations on the triangular theme, in: Lie groups and Lie algebras, AMS, Providence RI, 1995, 43-73.

[K3] A. Kirillov, Two more variations on the triangular theme, in: The Orbit Method in Geometry and Physics, Birkhäuser, Boston, MA, 2003, 243?258.

[K4] A. Kirillov, Lectures on the orbit method, AMS, Providence, RI, 2004.

[L] Lehrer G.I., Discrete Series and the unipotent subgroup, Compositio Math. v. 28 (1974), 9–19.

[VA1] A. Vera-López and J. M. Arregi, Conjugacy classes in unitriangular matrices, J. Algebra 152(1992), 1?19.

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