Higher q-Continued Fractions

Nick Ovenhouse
(Yale University)

Higher q-Continued Fractions

Nick Ovenhouse
(Yale University)

Based on joint work with Amanda Burcroff, Gregg Musiker, Ralf Schiffler, and Sylvester Zhang

Outline

Outline

* q-rationals

Outline

* q-rationals
* Properties and expressions

Outline

* q-rationals
* Properties and expressions
* Higher Version

Outline

* q-rationals
* Properties and expressions
* Higher Version
* Results and Further Questions

q - Rationals

q- Rationals

q-integers: $[n]_q = 1+q+q^2+\cdots+q^{n-1} = \frac{1-q^n}{1-q}$

q-Rationals

q-integers: $[n]_q = 1+q+q^2+\cdots+q^{n-1} = \dfrac{1-q^n}{1-q}$

What about $[\alpha]_q$ for $\alpha \notin \mathbb{Z}$?

q-Rationals

For $\alpha = \frac{r}{s} \in \mathbb{Q}$ $(\alpha \geq 1)$ with continued fraction $\alpha = [a_1, a_2, \cdots, a_{2n}] = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots + \cfrac{1}{a_{2n}}}}}$

q-Rationals

For $\alpha = \frac{r}{s} \in \mathbb{Q}$ ($\alpha \geq 1$) with continued fraction $\alpha = [a_1, a_2, \cdots, a_{2n}] = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots + \cfrac{1}{a_{2n}}}}}$

define

$$[\alpha]_q := [a_1]_q + \cfrac{q^{a_1}}{[a_2]_{q^{-1}} + \cfrac{q^{-a_2}}{[a_3]_q + \cfrac{q^{a_3}}{\ddots + \cfrac{q^{a_{2n-1}}}{[a_{2n}]_{q^{-1}}}}}}$$

Example: $\frac{5}{2} = [2,2] = 2 + \frac{1}{2}$

so $\left[\frac{5}{2}\right]_q = [2]_q + \frac{q^2}{[2]_{q^{-1}}}$

$= (1+q) + \frac{q^2}{1+q^{-1}}$

$= \frac{1+2q+q^2+q^3}{1+q}$

Positivity

Thm (Morier-Genoud, Ovsienko)

If $\frac{r}{s} > \frac{a}{b}$, and $\left[\frac{r}{s}\right]_q = \frac{R(q)}{S(q)}$, $\left[\frac{a}{b}\right]_q = \frac{A(q)}{B(q)}$

then $R(q)B(q) - A(q)S(q)$

has non-negative integer coefficients.

Stabilization

Thm (Morier-Genoud, Ovsenko)

irrational

If $\alpha_n \in \mathbb{Q}$ have $\lim_{n\to\infty} \alpha_n = \alpha$,

then $\lim_{n\to\infty} [\alpha_n]_q$ is a well-defined

formal series.

Stabilization

Thm (Morier-Genoud, Ovsienko)

If $\alpha_n \in \mathbb{Q}$ have $\lim_{n\to\infty} \alpha_n = \alpha$, irrational

then $\lim_{n\to\infty} [\alpha_n]_q$ is a well-defined formal series.

Define $[\alpha]_q$ as this limiting series.

"q-real number"

Example $\varphi = \frac{1+\sqrt{5}}{2} = \lim_{n\to\infty} \frac{f_n}{f_{n-1}}$ Fibonacci #'s

Example

$\varphi = \dfrac{1+\sqrt{5}}{2} = \lim\limits_{n\to\infty} \dfrac{f_n}{f_{n-1}}$ Fibonacci #'s

$$\left[\tfrac{3}{2}\right]_q = \frac{1+q+q^2}{1+q} = 1 + q^2 - q^3 + q^4 - q^5 + q^6 - q^7 + \cdots$$

Example

$$\varphi = \frac{1+\sqrt{5}}{2} = \lim_{n\to\infty} \frac{f_n}{f_{n-1}}$$ Fibonacci #'s

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix}_q = \frac{1+q+q^2}{1+q} = 1 + q^2 - q^3 + q^4 - q^5 + q^6 - q^7 + \cdots$$

$$\begin{bmatrix} 5 \\ 3 \end{bmatrix}_q = \frac{1+q+2q^2+q^3}{1+q+q^2} = 1 + q^2 - q^4 + q^5 - q^7 + q^8 - q^{10} + \cdots$$

Example

$$\varphi = \frac{1+\sqrt{5}}{2} = \lim_{n\to\infty} \frac{f_n}{f_{n-1}}$$

Fibonacci #'s

$$\left[\frac{3}{2}\right]_q = \frac{1+q+q^2}{1+q} = 1+q^2-q^3+q^4-q^5+q^6-q^7+\cdots$$

$$\left[\frac{5}{3}\right]_q = \frac{1+q+2q^2+q^3}{1+q+q^2} = 1+q^2-q^4+q^5-q^7+q^8-q^{10}+\cdots$$

$$\left[\frac{8}{5}\right]_q = \frac{1+2q+2q^2+2q^3+q^4}{1+2q+q^2+q^3} = 1+q^2-q^3+2q^4-4q^5+7q^6-12q^7+\cdots$$

Example

$$\varphi = \frac{1+\sqrt{5}}{2} = \lim_{n\to\infty} \frac{f_n}{f_{n-1}}$$

Fibonacci #'s

$$\left[\frac{3}{2}\right]_q = \frac{1+q+q^2}{1+q} = 1 + q^2 - q^3 + q^4 - q^5 + q^6 - q^7 + \cdots$$

$$\left[\frac{5}{3}\right]_q = \frac{1+q+2q^2+q^3}{1+q+q^2} = 1 + q^2 - q^4 + q^5 - q^7 + q^8 - q^{10} + \cdots$$

$$\left[\frac{8}{5}\right]_q = \frac{1+2q+2q^2+2q^3+q^4}{1+2q+q^2+q^3} = 1 + q^2 - q^3 + 2q^4 - 4q^5 + 7q^6 - 12q^7 + \cdots$$

$$\left[\frac{13}{8}\right]_q = 1 + q^2 - q^3 + 2q^4 - 3q^5 + 3q^6 - 3q^7 + \cdots$$

Example

$$\varphi = \frac{1+\sqrt{5}}{2} = \lim_{n\to\infty} \frac{f_n}{f_{n-1}}$$ ← Fibonacci #'s

$$\begin{bmatrix} \frac{3}{2} \end{bmatrix}_q = \frac{1+q+q^2}{1+q} = 1 + q^2 - q^3 + q^4 - q^5 + q^6 - q^7 + \cdots$$

$$\begin{bmatrix} \frac{5}{3} \end{bmatrix}_q = \frac{1+q+2q^2+q^3}{1+q+q^2} = 1 + q^2 - q^4 + q^5 - q^7 + q^8 - q^{10} + \cdots$$

$$\begin{bmatrix} \frac{8}{5} \end{bmatrix}_q = \frac{1+2q+2q^2+2q^3+q^4}{1+2q+q^2+q^3} = 1 + q^2 - q^3 + 2q^4 - 4q^5 + 7q^6 - 12q^7 + \cdots$$

$$\begin{bmatrix} \frac{13}{8} \end{bmatrix}_q = 1 + q^2 - q^3 + 2q^4 - 3q^5 + 3q^6 - 3q^7 + \cdots$$

$$\begin{bmatrix} \frac{21}{13} \end{bmatrix}_q = 1 + q^2 - q^3 + 2q^4 - 4q^5 + 8q^6 - 17q^7 + \cdots$$

Generating Function

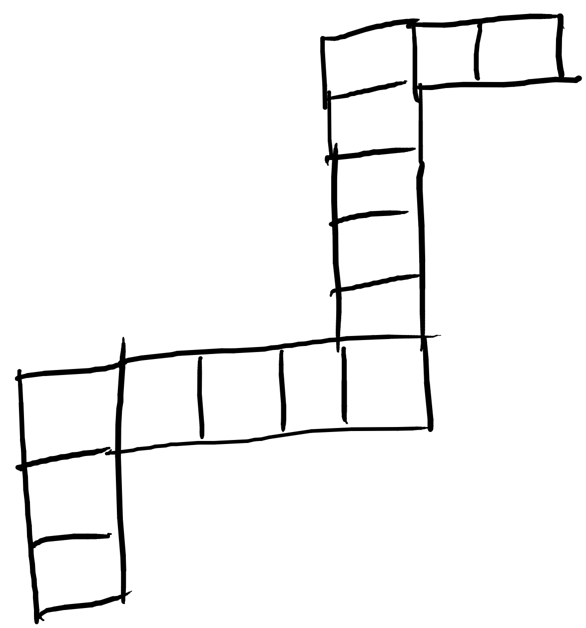
Generating Function

For $\alpha = [a_1, a_2, \cdots, a_n]$, draw skew shape λ/μ whose row/col lengths are $a_1, a_2, \cdots, a_n$

Generating Function

For $\alpha = [a_1, a_2, \cdots, a_n]$, draw skew shape λ/μ whose row/col lengths are $a_1, a_2, \cdots, a_n$

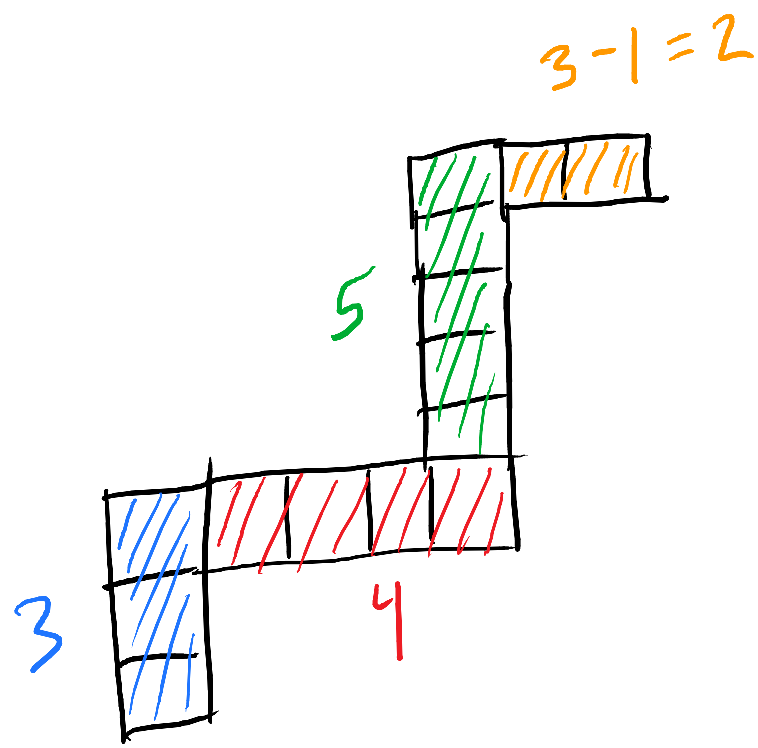
$\alpha = [3,4,5,3]$



Generating Function

For $\alpha = [a_1, a_2, \cdots, a_n]$, draw skew shape λ/μ whose row/col lengths are $a_1, a_2, \cdots, a_n$

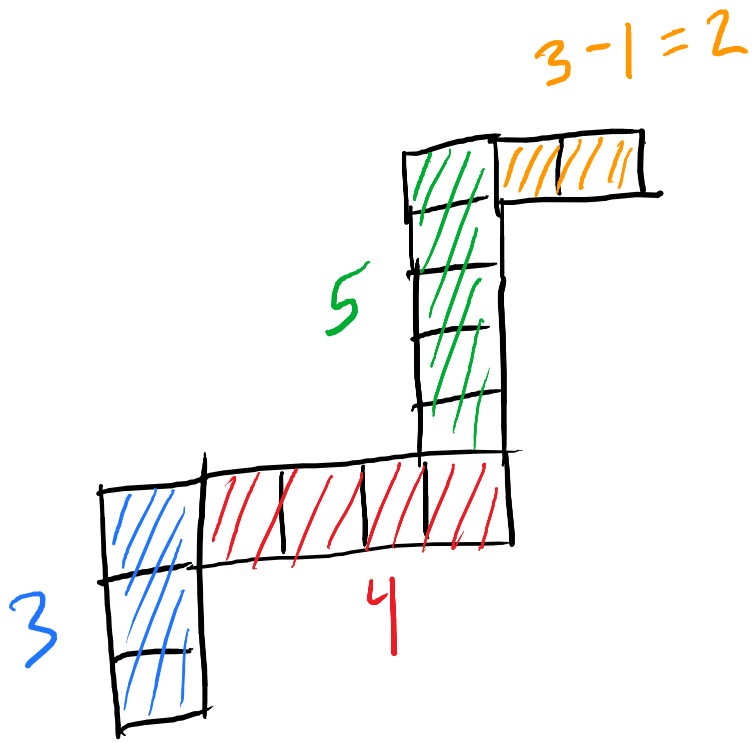
$\alpha = [3,4,5,3]$



Generating Function

For $\alpha = [a_1, a_2, \cdots, a_n]$, draw skew shape λ/μ whose row/col lengths are $a_1, a_2, \cdots, a_n$

$\alpha = [3,4,5,3]$



Call this graph G_α

Generating Function

Thm (Morier-Genoud, Ovsienko)

Let $\alpha = [a_1, a_2, \cdots, a_n]$, $[\alpha]_q = \dfrac{R(q)}{S(q)}$

and $\alpha' = [a_2, a_3, \cdots, a_n]$.

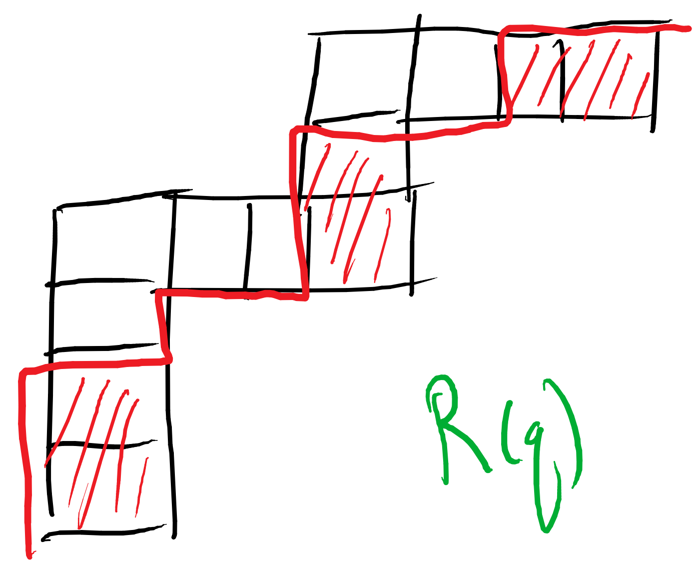
Generating Function

Thm (Morier-Genoud, Ovsienko)

Let $\alpha = [a_1, a_2, \cdots, a_n]$, $[\alpha]_q = \dfrac{R(q)}{S(q)}$

and $\alpha' = [a_2, a_3, \cdots, a_n]$. Then

$$R(q) = \sum_{\substack{NE \text{ paths} \\ \text{in } G_\alpha}} q^{\text{area}}$$



Generating Function

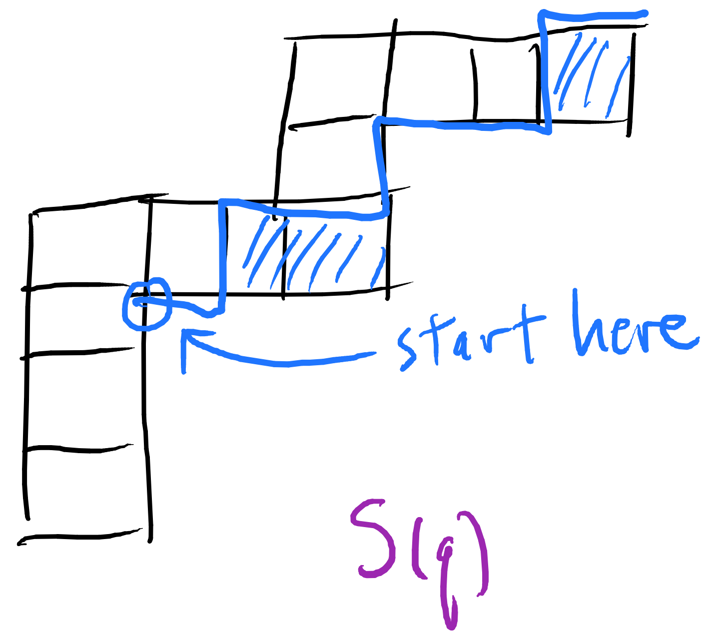
Thm (Morier-Genoud, Ovsienko)

Let $\alpha = [a_1, a_2, \cdots, a_n]$, $[\alpha]_q = \dfrac{R(q)}{S(q)}$

and $\alpha' = [a_2, a_3, \cdots, a_n]$. Then

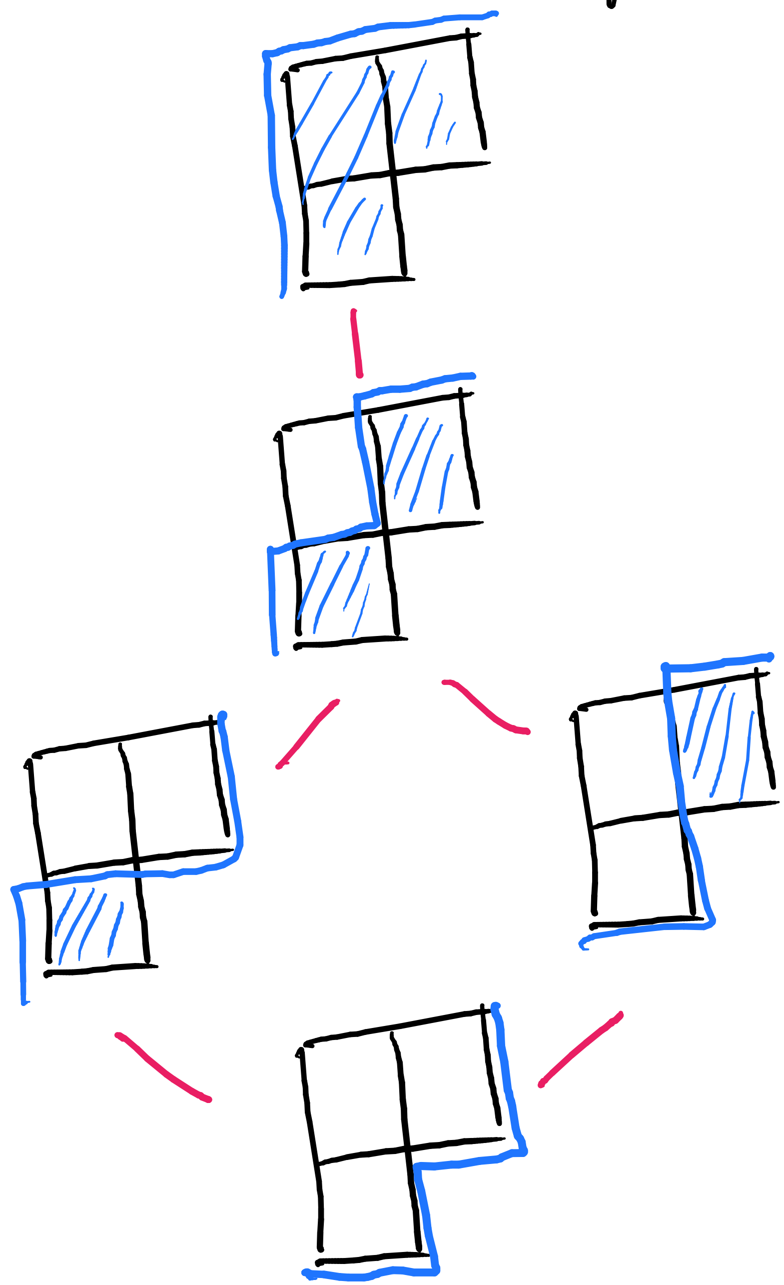
$$R(q) = \sum_{\substack{NE \text{ paths} \\ \text{in } G_\alpha}} q^{\text{area}}$$

$$S(q) = \sum_{\substack{NE \text{ paths} \\ \text{in } G_{\alpha'}}} q^{\text{area}}$$

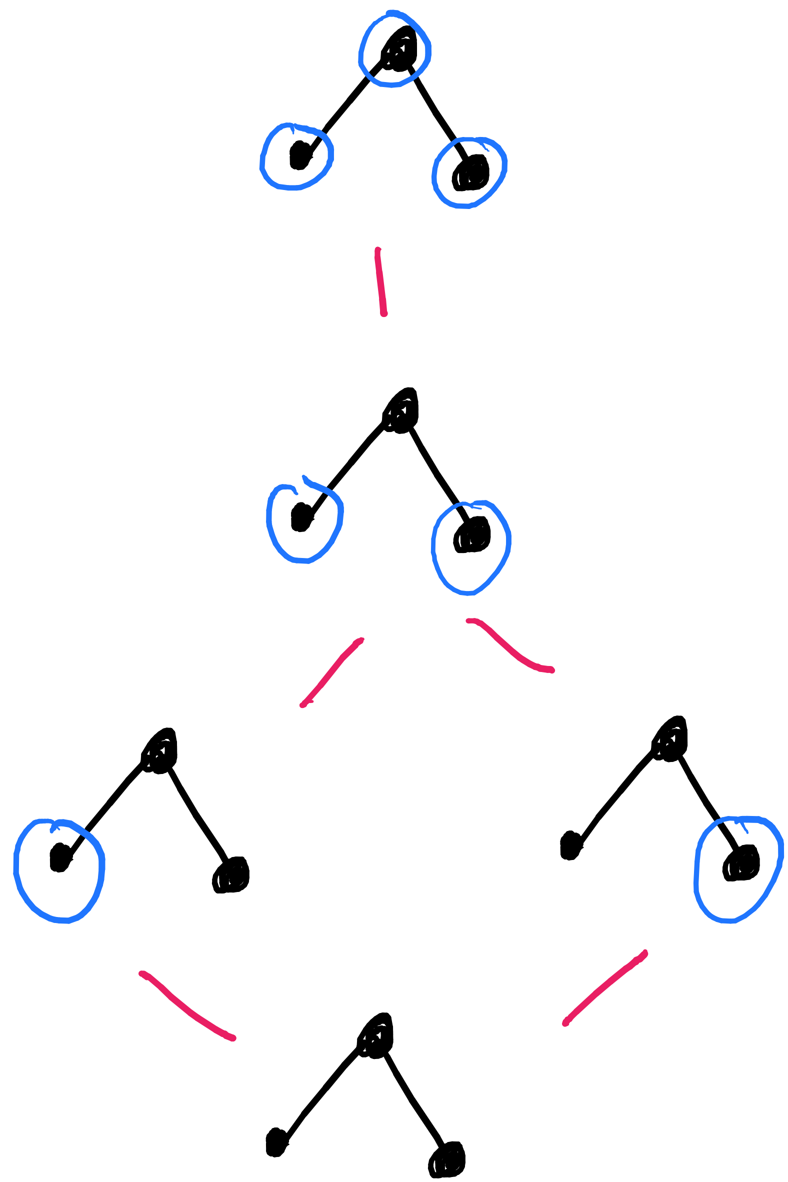


Example

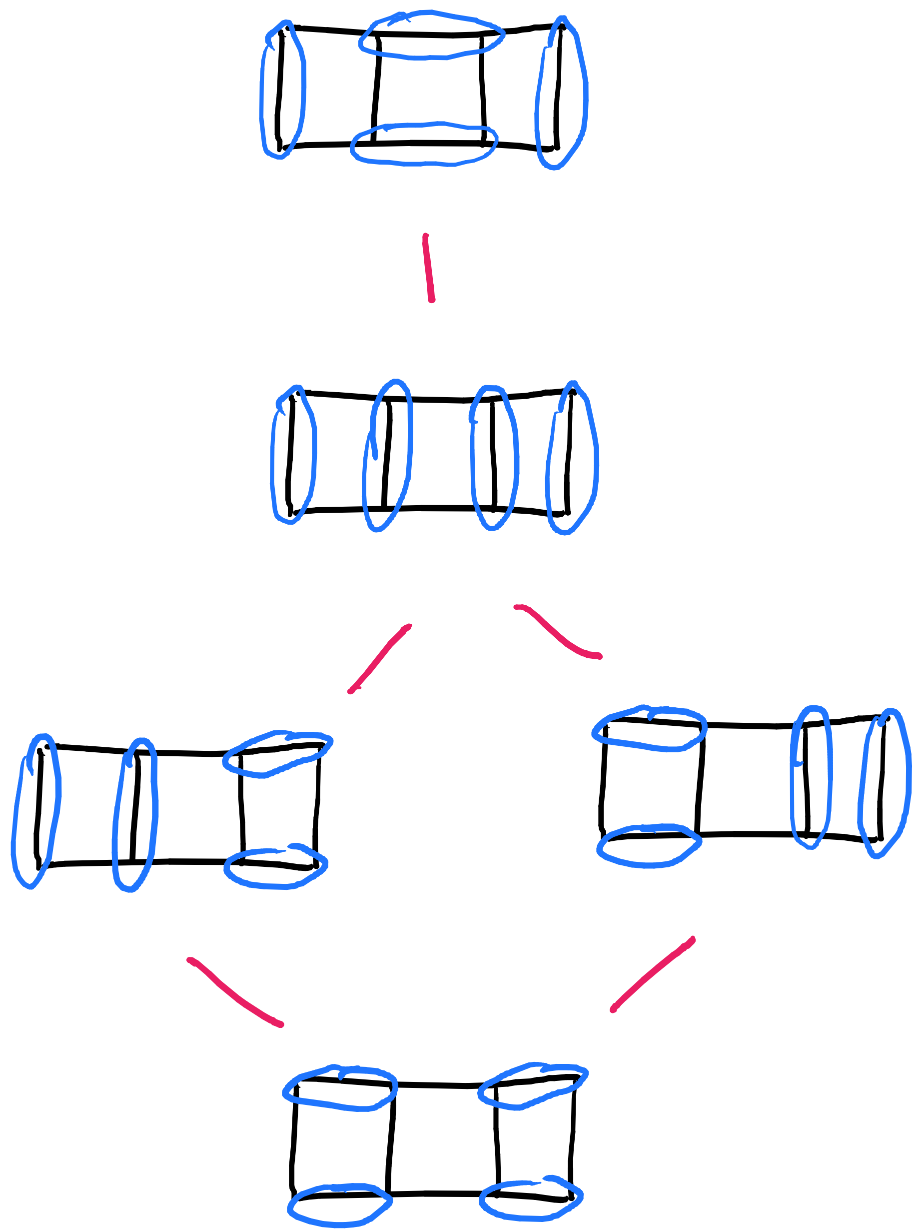
$\alpha = 5/2, \quad \left[\tfrac{5}{2}\right]_q = \dfrac{1+2q+q^2+q^3}{1+q}$



Other Combinatorial Interpretations: Order Ideals



Other Combinatorial Interpretations: Perfect Matchings



Other Combinatorial Interpretations: Permutations

$$
\begin{tikzcd}
& 2413 \arrow[d, dash] & \\
& 2143 \arrow[dl, dash] \arrow[dr, dash] & \\
2134 \arrow[dr, dash] & & 1243 \arrow[dl, dash] \\
& 1234 &
\end{tikzcd}
$$

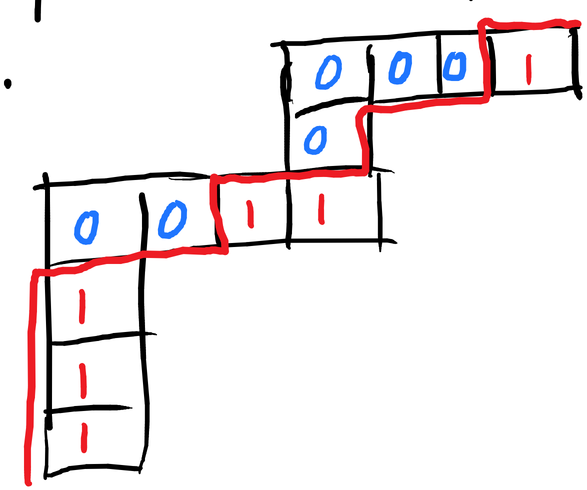
P - Partitions

Def: On G_α, a "reverse plane partition" (or "P-partition") is a labeling of the faces with non-negative integers which are weakly increasing in rows and columns.

P - Partitions

Def: On G_α, a "reverse plane partition" (or "P-partition") is a labeling of the faces with non-negative integers which are weakly increasing in rows and columns.

Note: NE lattice paths correspond to P-partitions with parts 0 or 1.



Natural Generalization is to allow parts $\leq m$ for any m.

Natural Generalization is to allow parts $\leq m$ for any m.

Motivation:

Multi-variable version of $R(q)$ is a cluster variable in surface cluster algebra.

Natural Generalization is to allow parts $\leq m$ for any m.

Motivation:

Multi-variable version of $R(q)$ is a cluster variable in surface cluster algebra.

Thm (Musiker, O, Zhang)

In (super) Teichmüller space for $OSp(2|1)$, super cluster variables are weighted sums over P-partitions of G_α with parts ≤ 2.

Natural Generalization is to allow parts $\leq m$ for any m.

Def: For $\alpha \in Q$, let $\Omega_m(G_\alpha)$ be the set of P-partitions with parts $\leq m$, and let $\Omega_m(G_\alpha, q) = \sum_{\sigma \in \Omega_m(G_\alpha)} q^{|\sigma|}$

Natural Generalization is to allow parts $\leq m$ for any m.

Def: For $\alpha \in \mathbb{Q}$, let $\Omega_m(G_\alpha)$ be the set of P-partitions with parts $\leq m$, and let $\Omega_m(G_\alpha, q) = \sum_{\sigma \in \Omega_m(G_\alpha)} q^{|\sigma|}$

Define "higher q-rational"

$$[\alpha]_q^{(m)} = \frac{\Omega_m(G_\alpha, q)}{\Omega_m(G_{\alpha'}, q)}$$

Natural Generalization is to allow parts $\leq m$ for any m.

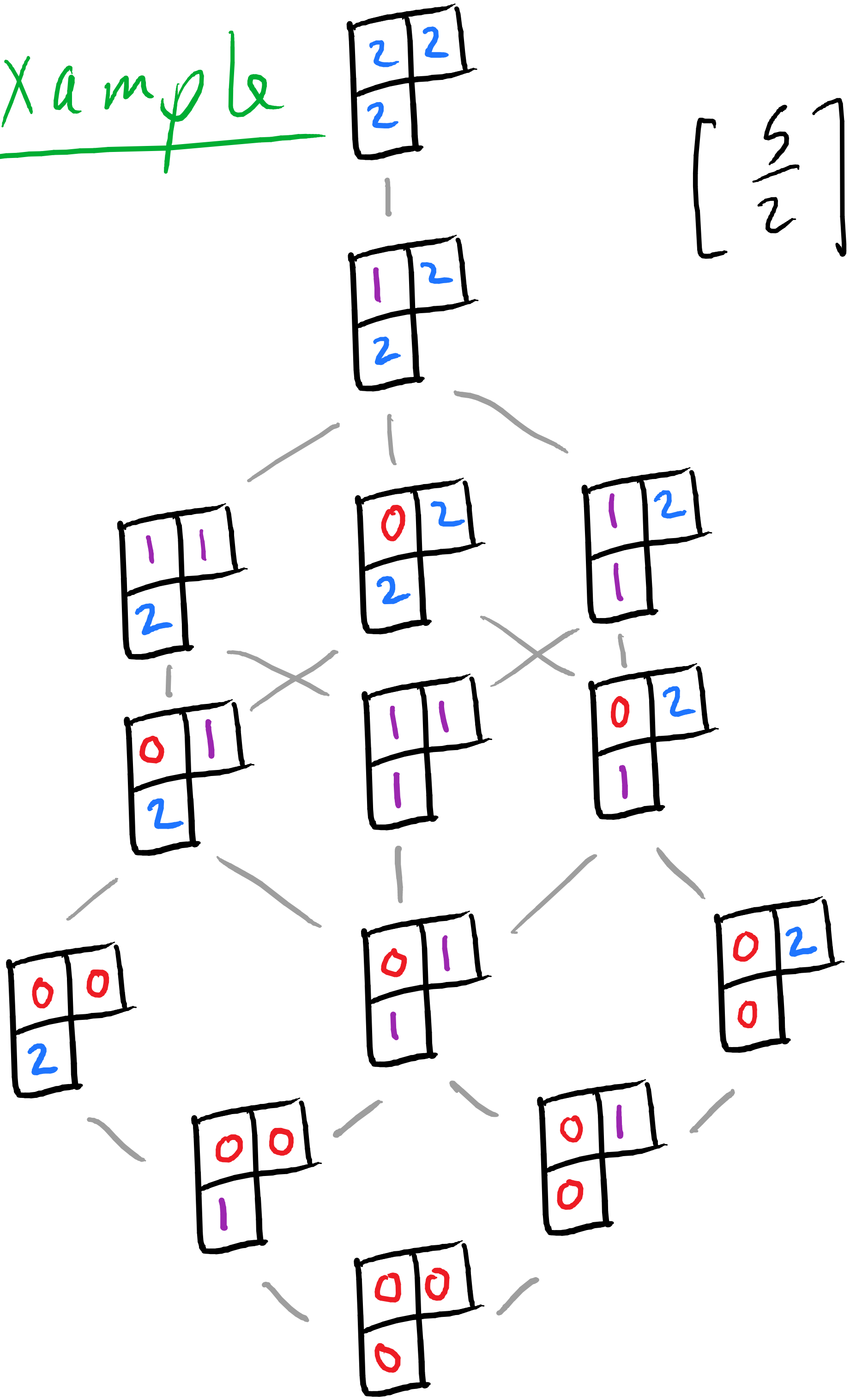
Def: For $\alpha \in \mathbb{Q}$, let $\Omega_m(G_\alpha)$ be the set of P-partitions with parts $\leq m$, and let $\Omega_m(G_\alpha, q) = \sum_{\sigma \in \Omega_m(G_\alpha)} q^{|\sigma|}$

Define "higher q-rational"

$$[\alpha]_q^{(m)} = \frac{\Omega_m(G_\alpha, q)}{\Omega_m(G_{\alpha'}, q)}$$

$[\alpha]_q^{(1)} = [\alpha]_q$

Example



$$\begin{bmatrix} 5 \\ 2 \end{bmatrix}_q^{(2)} = \frac{1+2q+3q^2+3q^3+3q^4+q^5+q^6}{1+q+q^2}$$

Some of the nice properties of q-rationals are special cases of similar things for higher m.

Some of the nice properties of q-rationals are special cases of similar things for higher m.

Thm (Burcroft, O, Schiffler, Zhong) ("Positivity")

If $\left[\frac{r}{s}\right]_q^{(m)} = \frac{R(q)}{S(q)}$, $\left[\frac{a}{b}\right]_q^{(m)} = \frac{A(q)}{B(q)}$,

and if $\frac{r}{s} > \frac{a}{b}$, then

$R(q)\,B(q) - A(q)\,S(q)$ has non-negative coefficients.

Thm (Burcroft, O, Schiffler, Zhang)
("Stabilization")

If $\alpha_n \in \mathbb{Q}$ with $\lim_{n\to\infty} \alpha_n = \alpha \in \mathbb{R}\setminus\mathbb{Q}$,

then $\lim_{n\to\infty} [\alpha_n]_q^{(m)}$ is a well-defined series with integer coefficients.

(use this to define $[\alpha]_q^{(m)}$)

Thm (Burcroff, O, Schiffler, Zhang)

If $\alpha_n \in \mathbb{Q}$ with $\lim_{n\to\infty} \alpha_n = \alpha \in \mathbb{R}\setminus\mathbb{Q}$, then $\lim_{n\to\infty} [\alpha_n]_q^{(m)}$ is a well-defined series with integer coefficients.

More specifically, $[\alpha_n]_q^{(m)}$ and $[\alpha_{n-1}]_q^{(m)}$ agree up to $q^{a_1+a_2+\cdots+a_n-1}$ term.

Thm (Burcroff, O, Schiffler, Zhang)

If $\alpha_n \in \mathbb{Q}$ with $\lim_{n\to\infty} \alpha_n = \alpha \in \mathbb{R}\setminus\mathbb{Q}$,

then $\lim_{n\to\infty} [\alpha_n]_q^{(m)}$ is a well-defined series with integer coefficients.

More specifically, $[\alpha_n]_q^{(m)}$ and $[\alpha_{n-1}]_q^{(m)}$ agree up to $\underbrace{q^{a_1+a_2+\cdots+a_n-1} \text{ term}}_{\text{independent of } m!}$.

Matrix Recurrence

Define $(m+1)\times(m+1)$ matrices

$$R = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ & 1 & 1 & \cdots & 1 \\ & & 1 & \ddots & \vdots \\ & 0 & & \ddots & 1 \\ & & & & 1 \end{pmatrix}, \quad L = \begin{pmatrix} 1 & & & & \\ 1 & 1 & & 0 & \\ 1 & 1 & \ddots & & \\ \vdots & \vdots & & \ddots & \\ 1 & 1 & \cdots & \cdots & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} q^m & & & & 0 \\ & q^{m-1} & & & \\ & & \ddots & & \\ & & & q & \\ 0 & & & & 1 \end{pmatrix}$$

Matrix Recurrence

Define $(m+1)\times(m+1)$ matrices

$$R = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ & 1 & 1 & \cdots & 1 \\ & & 1 & \cdots & \vdots \\ & 0 & & \ddots & 1 \\ & & & & 1 \end{pmatrix}, \quad L = \begin{pmatrix} 1 & & & & 0 \\ 1 & 1 & & & \\ \vdots & & \ddots & & \\ 1 & & & \ddots & \\ 1 & 1 & \cdots & \cdots & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} q^m & & & & 0 \\ & q^{m-1} & & & \\ & & \ddots & & \\ & & & q & \\ 0 & & & & 1 \end{pmatrix}$$

and $\quad R(q) := R \cdot Q, \quad L(q) := L \cdot Q$

Matrix Recurrence

Define $(m+1)\times(m+1)$ matrices

$$R = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ & 1 & 1 & \cdots & 1 \\ & & 1 & \cdots & \vdots \\ & 0 & & \ddots & \vdots \\ & & & & 1 \end{pmatrix}, \quad L = \begin{pmatrix} 1 & & & 0 \\ 1 & 1 & & \\ \vdots & & \ddots & \\ 1 & 1 & \cdots & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} q^m & & & 0 \\ & q^{m-1} & & \\ & & \ddots & \\ & & & q \\ 0 & & & & 1 \end{pmatrix}$$

and $\quad R(q) := R\cdot Q, \quad L(q) := L\cdot Q$

For $\alpha = [a_1, a_2, \cdots, a_n]$, define

$$M_\alpha := R(q)^{a_1} L(q)^{a_2} R(q)^{a_3} L(q)^{a_4} \cdots$$

Thm (B, O, S, Z)

$$[\alpha]_q^{(m)} = \frac{(M_\alpha)_{11}}{(M_\alpha)_{m+1,1}}$$

Thm (B, O, S, Z)

$$[\alpha]_q^{(m)} = \frac{(M_\alpha)_{11}}{(M_\alpha)_{m+1,\,1}}$$

$$\begin{pmatrix} * & & \\ & & \\ * & & \end{pmatrix}$$

Thm (B, O, S, Z)

$$[\alpha]_q^{(m)} = \frac{(M_\alpha)_{11}}{(M_\alpha)_{m+1,1}}$$

$$\begin{pmatrix} * & \\ & \\ * & \end{pmatrix}$$

When $m=1$

$$\begin{pmatrix} q & 1 \\ 0 & 1 \end{pmatrix}^{a_1} \begin{pmatrix} q & 0 \\ q & 1 \end{pmatrix}^{a_2} \cdots = \begin{pmatrix} q\,R(q) & \star \\ q\,S(q) & \star \end{pmatrix}$$

<u>Open Question</u>: Quantizing Algebraic numbers

Open Question: Quantizing Algebraic numbers

Thm (Leclere, Morier-Genoud)

If $\alpha = \dfrac{a+\sqrt{b}}{c}$, then $\exists\, A(q), B(q), C(q) \in \mathbb{Z}[q]$

such that $[\alpha]_q = \dfrac{A(q) + \sqrt{B(q)}}{C(q)}$

Open Question: Quantizing Algebraic numbers

Thm (Leclere, Morier-Genoud)

If $\alpha = \frac{a+\sqrt{b}}{c}$, then $\exists\, A(q), B(q), C(q) \in \mathbb{Z}[q]$

such that $[\alpha]_q = \frac{A(q)+\sqrt{B(q)}}{C(q)}$

What about algebraic numbers of degree ≥ 3?

The $q=1$ Version

Def: Let $[\alpha]^{(m)} := [\alpha]_1^{(m)}$

The $q=1$ Version

Def: Let $[\alpha]^{(m)} := [\alpha]_1^{(m)}$

We get $[-]^{(m)} : \mathbb{Q} \to \mathbb{Q}$

The $q=1$ Version

Def: Let $[\alpha]^{(m)} := [\alpha]_1^{(m)}$

We get $[-]^{(m)} : \mathbb{Q} \to \mathbb{Q}$

Thm (Musiker, O, Schiffler, Zhang)

$[-]^{(m)}$ extends to a map $\mathbb{R} \to \mathbb{R}$.

i.e. if $\underset{\substack{\Uparrow \\ \mathbb{Q}}}{\alpha_n} \longrightarrow \alpha$, then $\lim_{n\to\infty} [\alpha]^{(m)}$ exists.

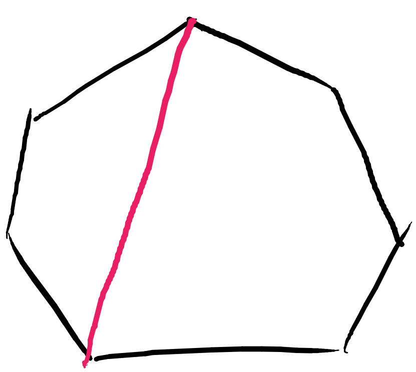
The $q=1$ Version

Thm (MOSZ) If α is a quadratic irrational, then $[\alpha]^{(m)}$ is a degree $(m+1)$ algebraic number.

The $q=1$ Version

Thm (MOSZ) If α is a quadratic irrational, then $[\alpha]^{(m)}$ is a degree $(m+1)$ algebraic number.

Example: $[\varphi]^{(2)} = 4\cos^2\left(\frac{\pi}{7}\right) - 1 \approx 2.247\cdots$

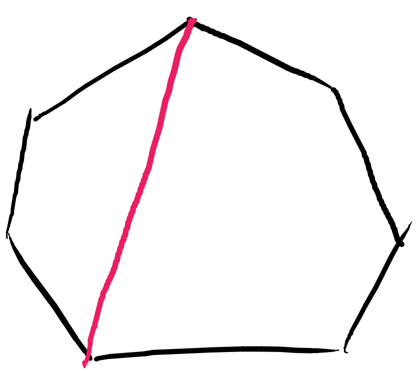


The $q=1$ Version

Thm (MOSZ) If α is a quadratic irrational, then $[\alpha]^{(m)}$ is a degree $(m+1)$ algebraic number.

Example: $[\varphi]^{(2)} = 4\cos^2\left(\frac{\pi}{7}\right) - 1 \approx 2.247\ldots$

which satisfies



$$x^3 - 2x^2 - x + 1 = 0$$

$\alpha \in \mathbb{R}$ quadratic

$\alpha \in \mathbb{R}$ quadratic

$[\alpha]_q$ is alg over $\mathbb{Z}[q]$

$[\alpha]^{(m)} \in \mathbb{R}$ is alg. of degree $(m+1)$

$\alpha \in \mathbb{R}$ quadratic

$[\alpha]_q$ is alg over $\mathbb{Z}[q]$

$[\alpha]^{(m)} \in \mathbb{R}$ is alg. of degree $(m+1)$

Is $[\alpha]_q^{(m)}$ alg (over $\mathbb{Z}[q]$) of degree $(m+1)$?

Example: For $\varphi = \frac{1+\sqrt{5}}{2}$

$[\varphi]_q^{(2)}$ satisfies

$$0 = q^7 x^3 - \left(1 + (q-1)q^2[3]_q\right) \cdot q^3[2]_q X^2 - \left(1 + q(q-1)(1+3q+3q^2+3q^3+q^4)\right) \cdot q^2 X + q^2$$

<u>Example</u>: For $\varphi = \frac{1+\sqrt{5}}{2}$

$[\varphi]_q^{(2)}$ satisfies

$$0 = q^7 x^3 - \left(1 + (q-1)q^2[3]_q\right) \cdot q^3[2]_q x^2$$
$$- \left(1 + q(q-1)(1+3q+3q^2+3q^3+q^4)\right) \cdot q^2 x$$
$$+ q^2$$

(q-analog of $x^3 - 2q^2 - x + 1 = 0$)

Thank You!