Multidimensional Farey summation algorithm and frieze patterns

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21 October 2024

Contents

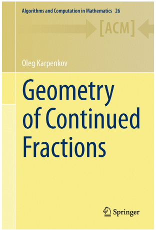
Contents

I. Geometry of CF.

II. Boats, sails, prismatic diagrams

III. Farey polyhedra, λ-lengths and Ptolemy relation.

IV. Frieze patterns.



Geometry of continued fractions.

Continued fractions for 7/5

$$\frac{7}{5} =$$

Continued fractions for 7/5

$$\frac{7}{5} = 1 + \frac{2}{5}$$

Continued fractions for 7/5

$$\frac{7}{5} = 1 + \frac{1}{5/2}$$

Continued fractions for 7/5

$$\frac{7}{5} = 1 + \frac{1}{2 + \frac{1}{2}}$$

Continued fractions for 7/5

$$\frac{7}{5} = 1 + \frac{1}{2 + \frac{1}{2}} = 1 + \frac{1}{2 + \frac{1}{1+\frac{1}{1}}}$$

Ordinary continued fractions

The expression

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\dots + \cfrac{1}{a_n}}}}$$

is an *ordinary continued fraction* if $a_0 \in \mathbb{Z}$, $a_k \in \mathbb{Z}_+$ for $k > 0$.
Denote it $[a_0 : a_1; \dots; a_n]$.

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Ordinary continued fraction is *odd* (*even*) if it has odd (even) number of elements.

$$\frac{7}{5} = 1 + \frac{1}{2 + \frac{1}{2}} = 1 + \frac{1}{2 + \frac{1}{1+1/1}}$$

$$\frac{7}{5} = [1 : 2; 2] = [1 : 2; 1; 1]$$

Ordinary continued fractions

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$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\dots + \frac{1}{a_n}}}}$$

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Ordinary continued fraction is *odd* (*even*) if it has odd (even) number of elements.

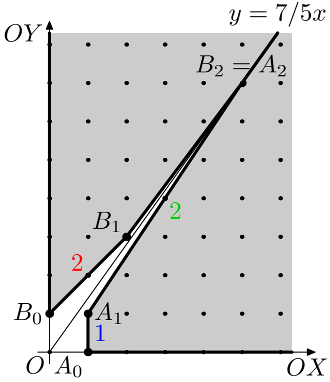
Proposition

Any rational number has a unique odd and even ordinary continued fractions.

Integer geometry

No slide on integer geometry.

Integer trigonometry (O.K. '08)



$a_0 = \mathrm{l}\ell(A_0A_1) = 1;$

$a_1 = \mathrm{l}\ell(B_0B_1) = 2;$

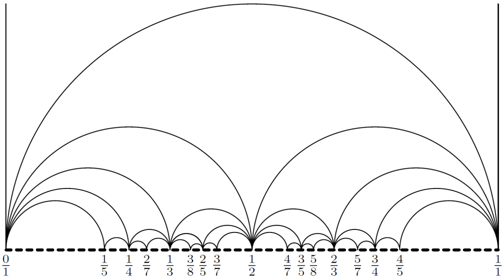
$a_2 = \mathrm{l}\ell(A_1A_2) = 2.$

$$7/5 = [1; 2 : 2].$$

$$\mathrm{ltan}\, \alpha = [a_0 : \ldots : a_{2n}].$$

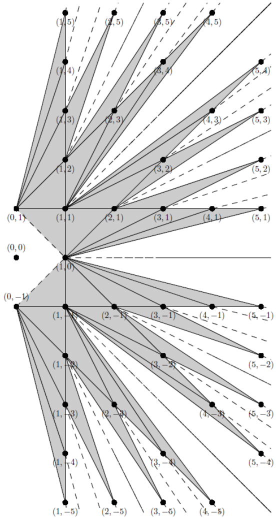
$$\mathrm{ltan}\, AOB = [1 : 2; 2] = \frac{7}{5} \quad \Longrightarrow \quad \begin{cases} \mathrm{lsin}\, AOB = 7 \\ \mathrm{lcos}\, AOB = 5 \end{cases}$$

Boats and prismatic diagrams



Farey tessellation of $\mathbb{H}^2$ with chords

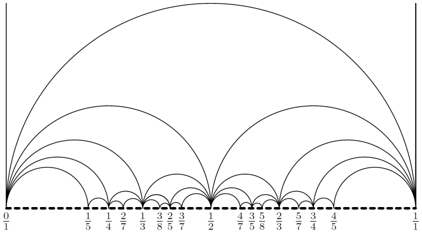
Boats and prismatic diagrams



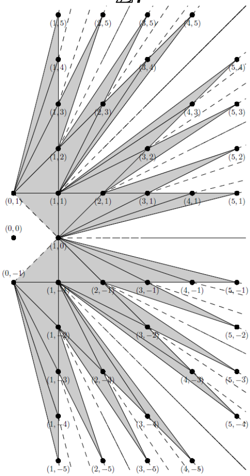
Disc bounded by the integer unit half-circle (or, even by $\mathbb{Z}P^1$) with chords

Boats and prismatic diagrams

$\mathbb{H}^2$

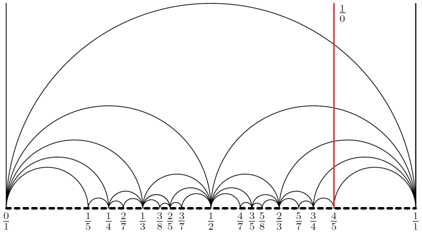


$\mathbb{Z}P^1$



Boats and prismatic diagrams

$\mathbb{H}^2$



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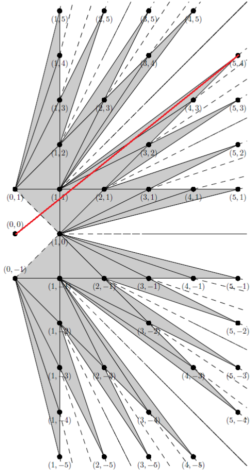

Boats and prismatic diagrams

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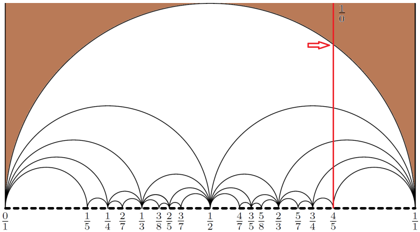

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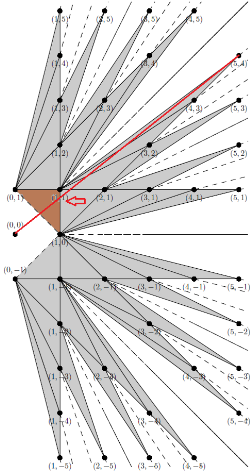

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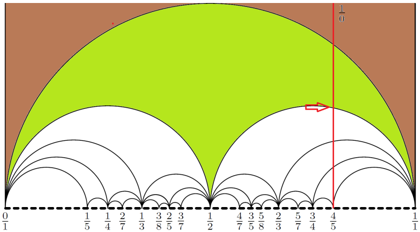

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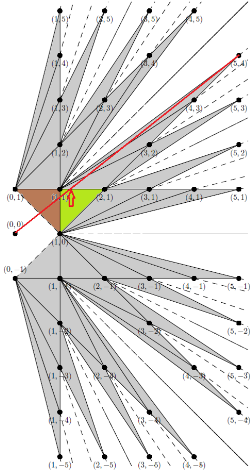

Boats and prismatic diagrams

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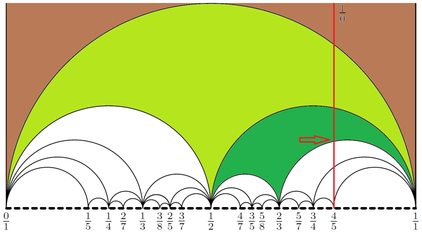

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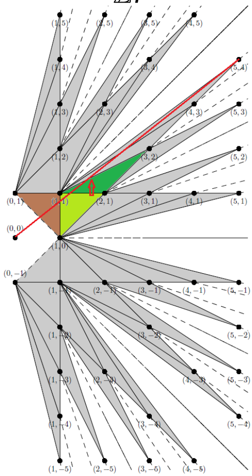

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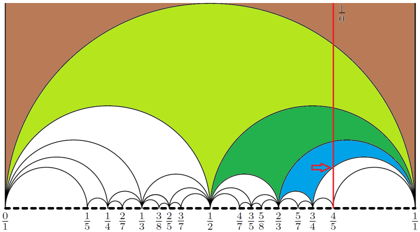
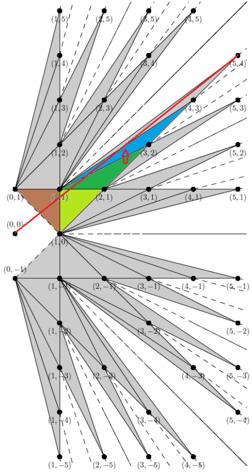


$\mathbb{Z}P^1$



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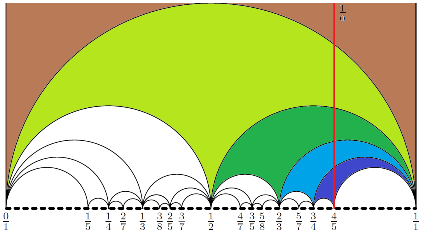

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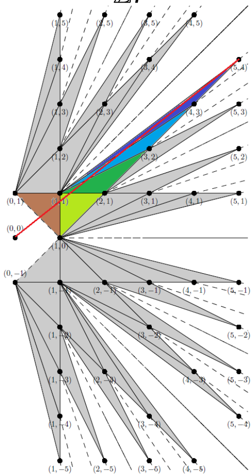

Boats and prismatic diagrams

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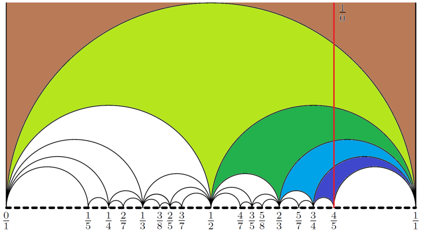
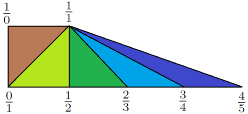


Morier-Genoud—Ovsienko boat

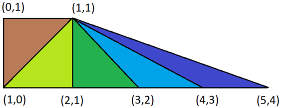
$\mathbb{Z}P^1$



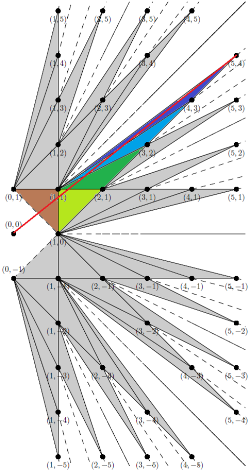
Boats and prismatic diagrams

$\mathbb{H}^2$

$\mathbb{Z}P^1$



Prismatic diagram



Boats and prismatic diagrams

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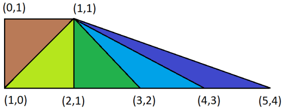

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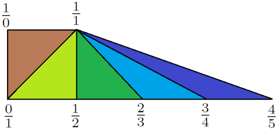
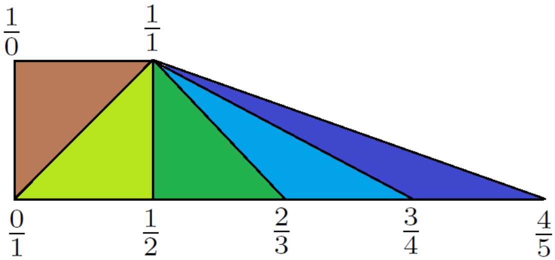


Some properties of boats



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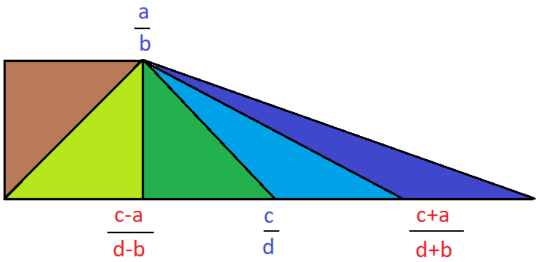
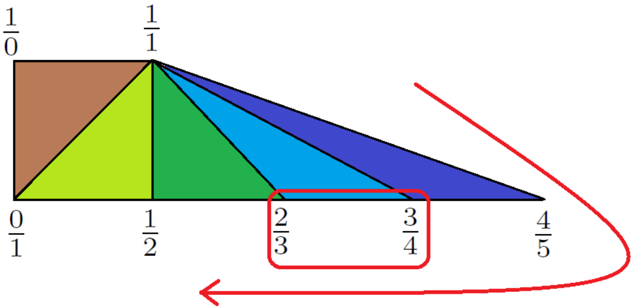


A nice rule to write numbers (Farey summation)

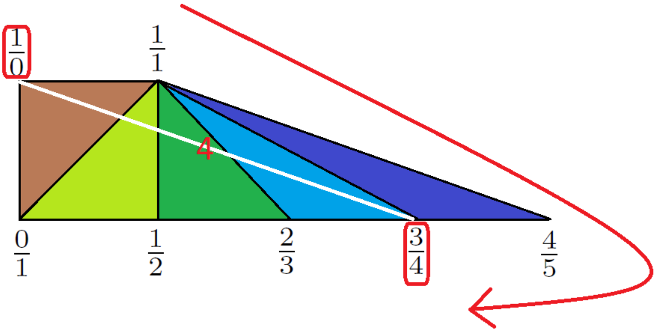
Some properties of boats



Neighbours determinant 1:

$$\det \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix} = 1$$

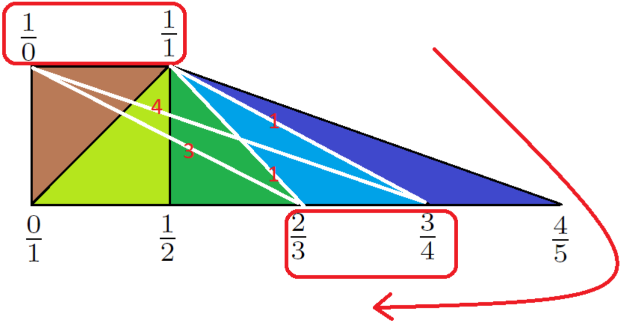
Some properties of boats



λ-length are determinants (order is important)
In integer geometry these are integer sines.

$$\lambda(1/0, 3/4) = \operatorname{lsin}\big((1,0)(0,0)(3,4)\big) = \det\begin{pmatrix}1 & 3\\ 0 & 4\end{pmatrix} = 4$$

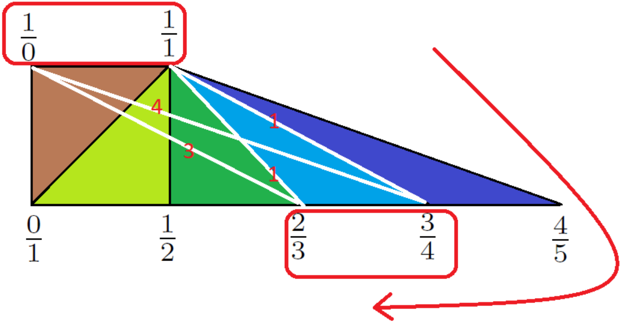
Some properties of boats



Ptolemy relation for λ-lengths:

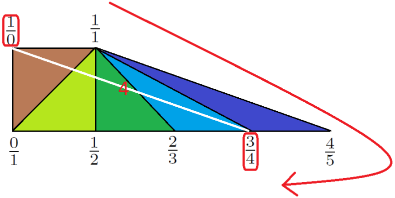
$$\det\begin{pmatrix} 4 & 1 \\ 3 & 1 \end{pmatrix} = 1$$

Some properties of boats



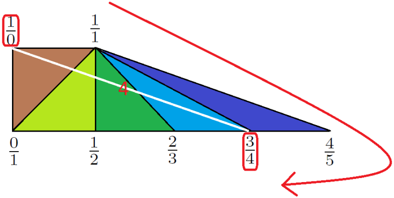
Frieze patterns are tables with λ-length for universal coverings of the boundary (multiplied by $(-1)^{\#\text{layers in between}}$).

Some properties of boats



0	1	5	4	3	2	1	0	−1	−5	−4	−3	−2	−1
−1	0	1	1	1	1	1	1	0	−1	−1	−1	−1	−1
−5	−1	0	1	2	3	4	5	1	0	−1	−2	−3	−4
−4	−1	−1	0	1	2	3	4	1	1	0	−1	−2	−3
−3	−1	−2	−1	0	1	2	3	1	2	1	0	−1	−1
−2	−1	−3	−2	−1	0	1	2	1	3	2	1	0	−1
−1	−1	−1	−1	−1	−1	0	1	1	1	1	1	1	0
0	−1	−5	−4	−3	−2	−1	0	1	5	4	3	2	1

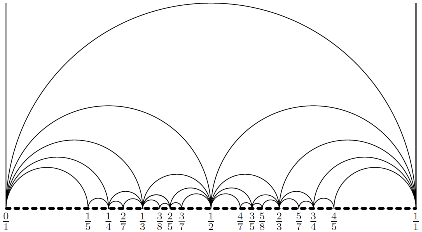
Some properties of boats



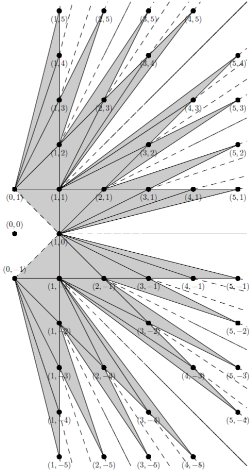
0		0		0		0		0		0		0	
	1		1		1		1		1		1		1
5		1		2		2		2		2		1	
	4		1		3		3		3		1		4
1		3		1		4		4		1		3	
	2		2		1		5		1		2		2
1		1		1		1		1		1		1	
	0		0		0		0		0		0		0

What in three dimensions?

$\mathbb{H}^3$



$\mathbb{Z}P^2$

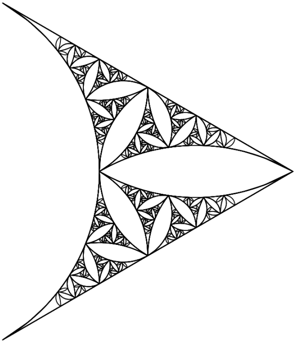


What in three dimensions?

$\mathbb{H}^3$	$\mathbb{Z}P^2$
???	???

What in three dimensions?

$\mathbb{H}^3$



$\mathbb{Z}P^2$

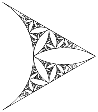
A. Pratoussevitch, OK (2017–2024) “Farey bryophylla”, arXiv:2409.01621

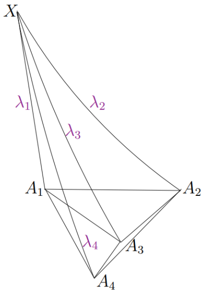
Proper “Farey type partition” with LR maps. Classification of maximal sets (on the absolute).

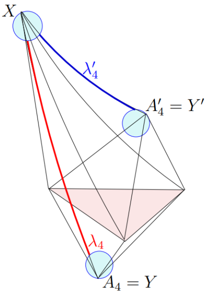
Too thin.

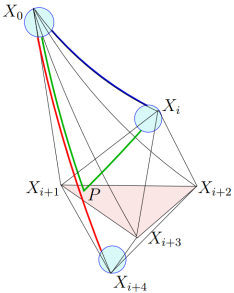
What in three dimensions?

$\mathbb{H}^3$ $\mathbb{Z}P^2$





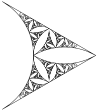


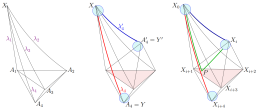


A. Felikson, P. Tumarkin, Kh. Serhiyenko, OK (2022–2023) "3D Farey graph, lambda lengths and SL_2-tilings", arXiv:2306.17118
Quite a solid theory for Eisenstein integers (on the absolute)
Too complex ($\mathbb{C}$), no combinatorial description.

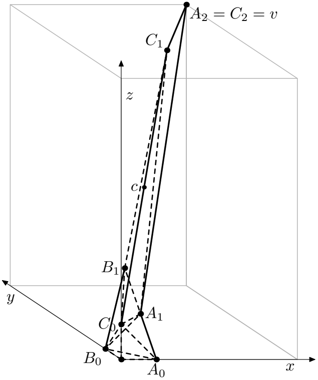
What in three dimensions?

$\mathbb{H}^3$





$\mathbb{Z}P^2$



M. van Son, OK (2024) “Geometry of multidimensional Farey summation algorithm and frieze patterns”

Farey summation

Hyperbolic and projective picture:

Farey addition:

$$\frac{2}{3} \oplus \frac{3}{4} = \frac{5}{7}.$$

Farey summation

Hyperbolic and projective picture:

Farey addition:

$$\frac{2}{3} \oplus \frac{3}{4} = \frac{5}{7}.$$

Integer geometric picture

Delone nose stretching (due to Arnold):

$$(2,3) + (3,4) = (5,7).$$

Farey summation in 3D

~~Hyperbolic and~~ projective picture:

Farey addition:

$$(2:3:4) \oplus (5:3:2) = (7:6:6).$$

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Delone nose straightening (due to Arnold):

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Idea of generalisation

In 2D. Start with a vector (a, b).
Origins of continued fractions is **Euclidean algorithm**:

$$(14, 3) \to (4 \cdot 3 + 2, 3) \to (2, 3), \qquad a_i = 4.$$

Here in the step we construct $[a_i : \ldots : a_n] \to [a_{i-1} : a_i : \ldots : a_n]$.

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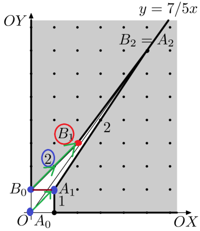
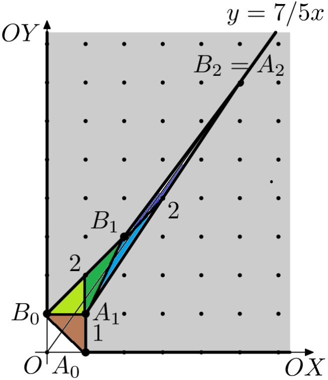
Stretching of noses is defined inversely
$[a_1 : \ldots : a_i] \to [a_1 : \ldots : a_i : a_{i+1}]$
(once you know a_i from Euclidean algorithm).



Idea of generalisation



Farrey summation algorithm: take all triangles in $\mathbb{Z}P^1$.
They are of type $v, w, v \oplus w$.

Idea of generalisation

In 3D. Start with a vector (a, b, c).
Origins of continued fractions is **Meester algorithm** (1989):

$$(17, 14, 3) \to (\underline{4} \cdot 3 + 5, \underline{4} \cdot 3 + 2, 3) \to (5, 2, 3), \qquad a_i = 4.$$

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Stretching of noses and the corresponding Farey summation algorithm (dual to Meester algorithm) is what we look for (O. R. Beaver and T. Garrity 2004).

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Stretching of noses and the corresponding Farey summation algorithm (dual to Meester algorithm) is what we look for (O. R. Beaver and T. Garrity 2004).
Remark: There are many subtractive algorithms in 3D.

Farey summation algorithm

Input: $(a, b, c) \in \mathbb{R}^3_+$.

Farey summation algorithm

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First setting: *First yard* is a triangle : $(1, 0, 0), (0, 1, 0), (0, 0, 1)$.

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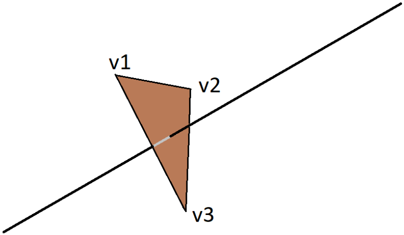
First setting: *First yard* is a triangle : $(1, 0, 0), (0, 1, 0), (0, 0, 1)$.
Step: Given previous yard v_1, v_2, v_3 we set $w = v_1 \oplus v_2 \oplus v_3$ and consider w, v_i, v_j as the next yard:



Farey summation algorithm

Input: $(a, b, c) \in \mathbb{R}^3_+$.
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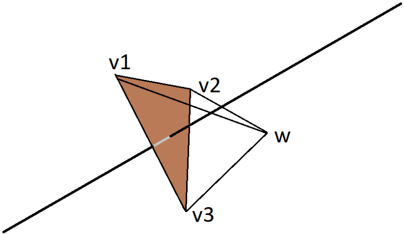

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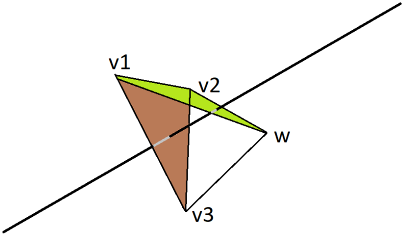

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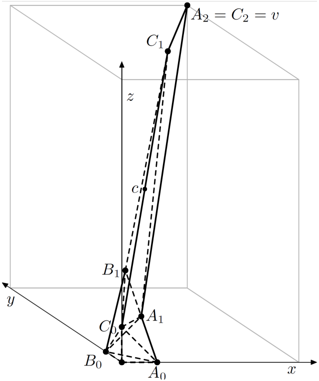
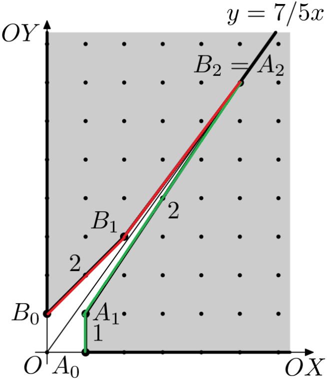
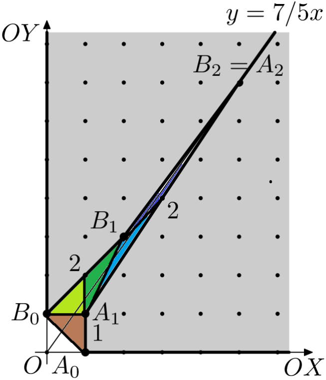
Output:



Terminology



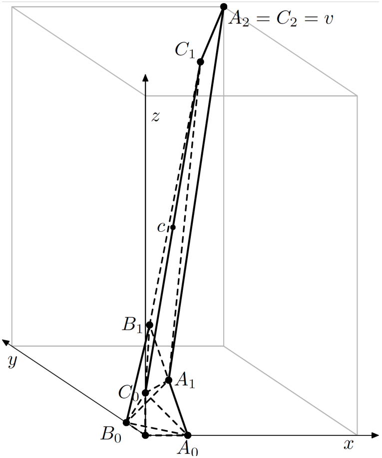


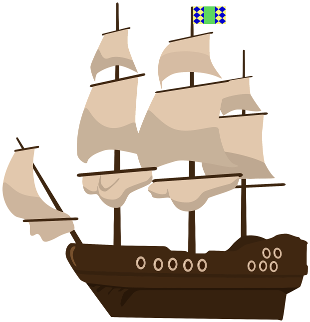
Existing terminology:

Klein sails and **Morier-Genoud—Ovsienko boat**

Sails form the boundary of boats.

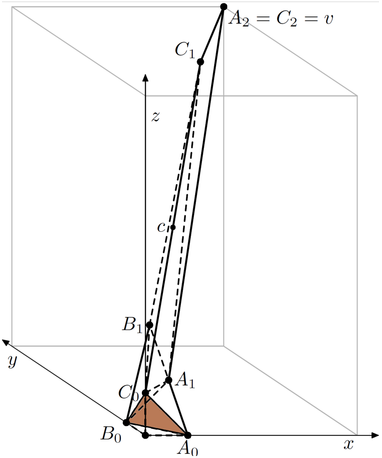
Terminology

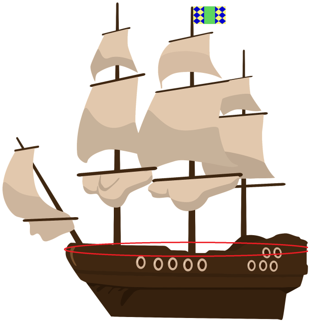




Farey polyhedron

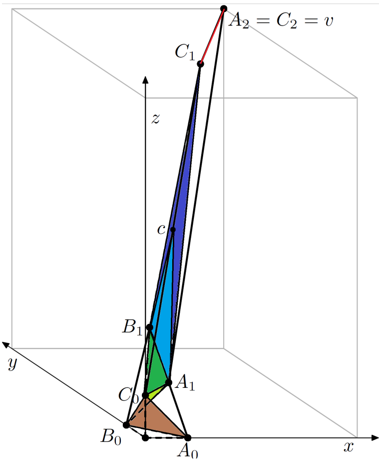
Terminology

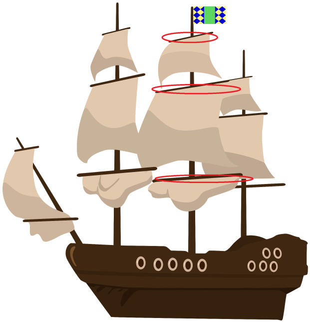




Deck

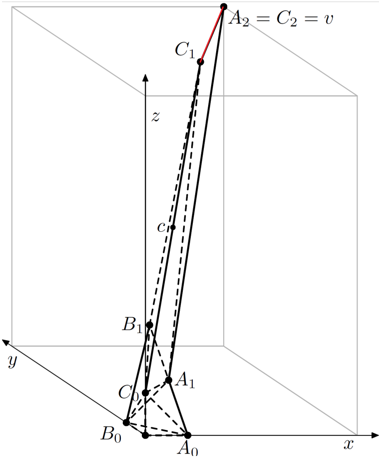
Terminology

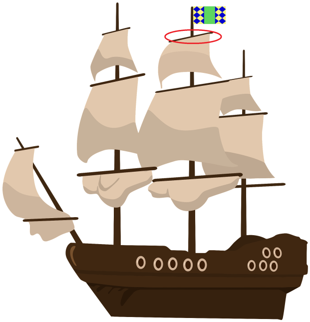




Yards

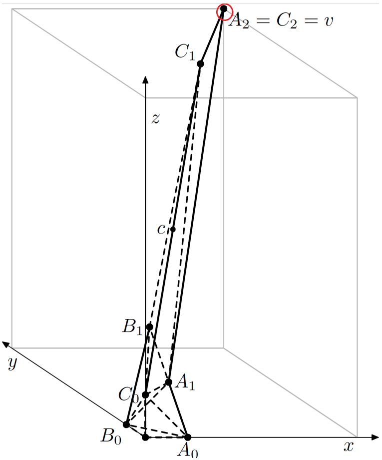
Terminology

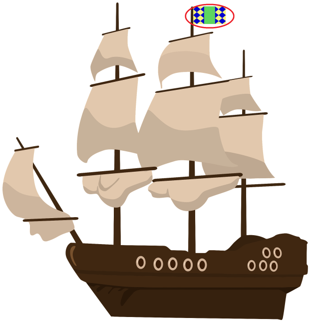




Crow's nest

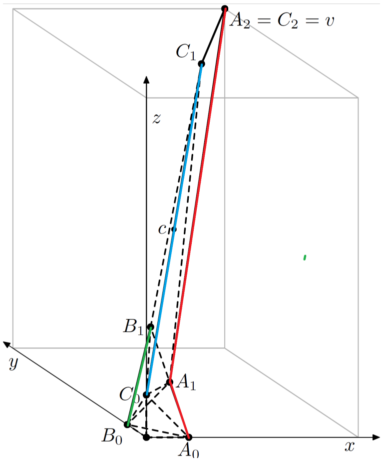
Terminology

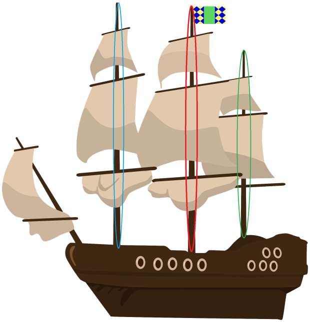




Pennant

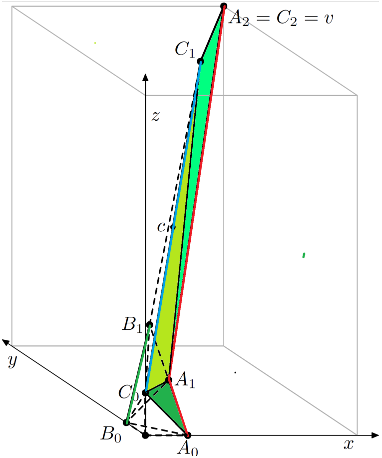
Terminology

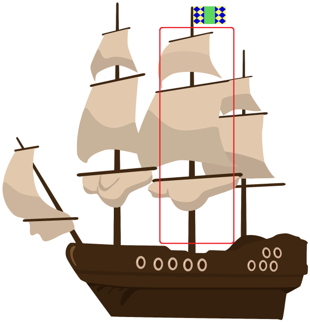




Masts

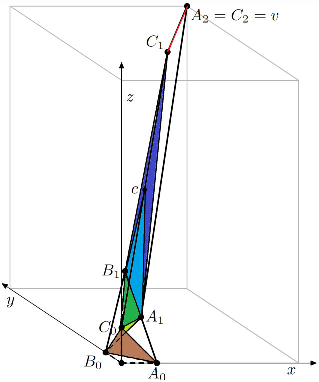
Terminology

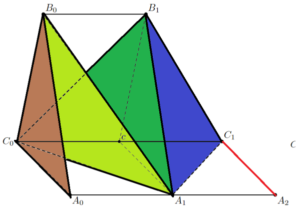




A sail (one of three)

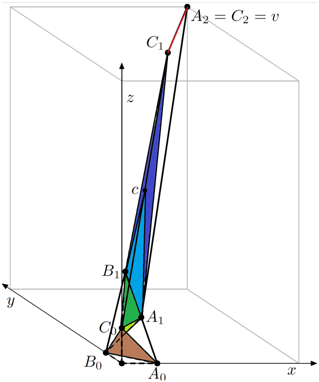
Prismatic diagram

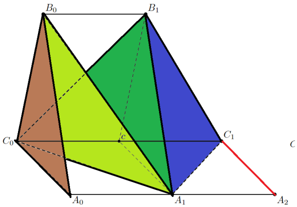




Prismatic diagram

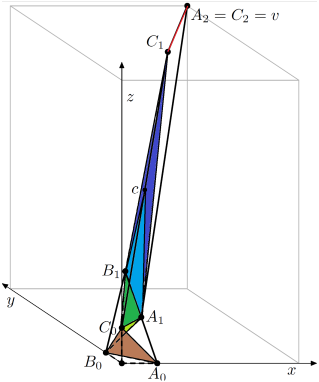
Prismatic diagram

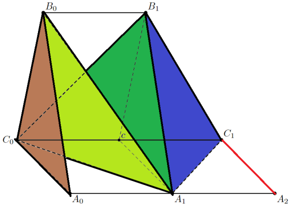




Theorem. There is a natural 1-1 correspondence between:
— Farey polyhedra (in general position);
— Prismatic diagrams;
— path-triangulations of polyhedra (dual graph is a broken line).

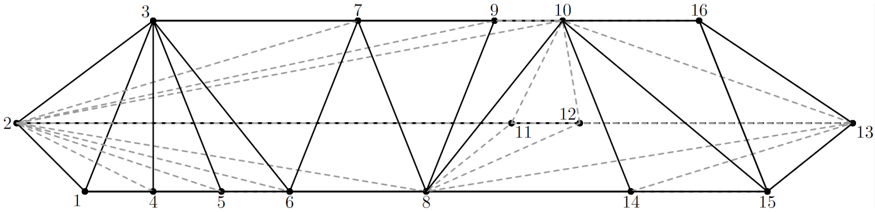
Prismatic diagram





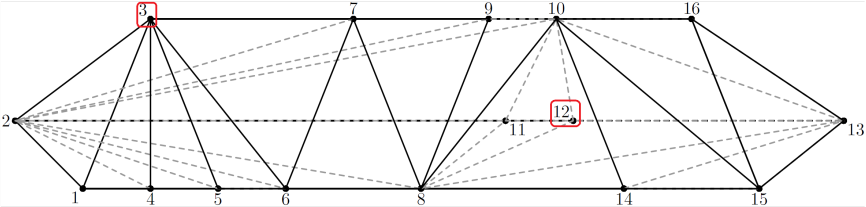
Remark. Not all rational triples are pennants of a Farey polyhedron with entirely 3D prismatic diagram.

λ-lengths (M. van Son, OK, 2024)



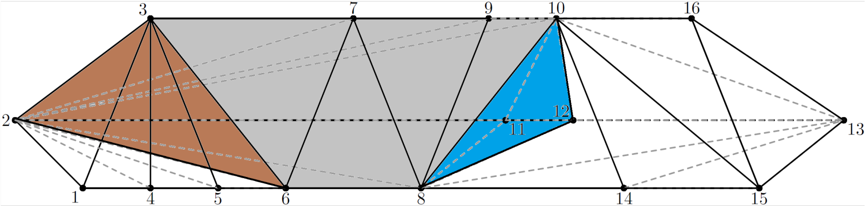
Prismatic diagram.

λ-lengths (M. van Son, OK, 2024)



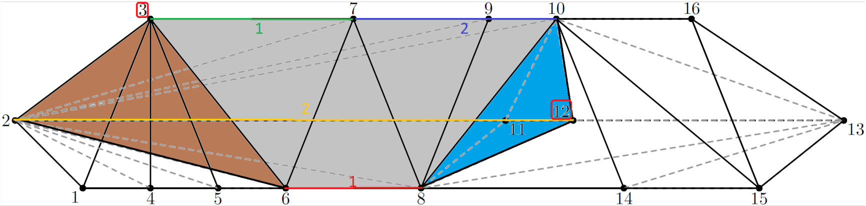
Consider two vertices.

λ-lengths (M. van Son, OK, 2024)



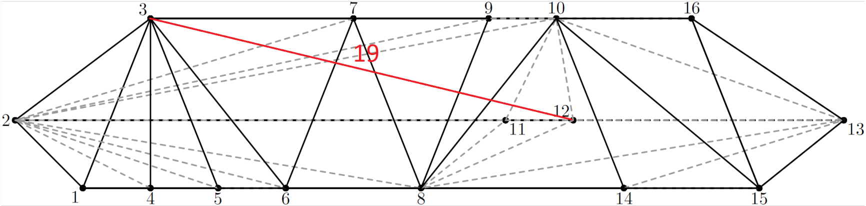
Take the minimal slice for two vertices.

λ-lengths (M. van Son, OK, 2024)



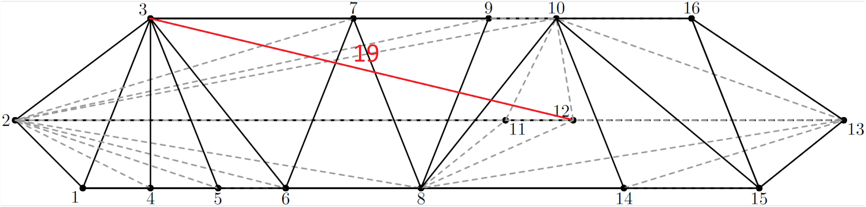
Consider it as a Farey polyhedron. On figure it is $[1; 0 : 1 : 2 : 2]$.

λ-lengths (M. van Son, OK, 2024)



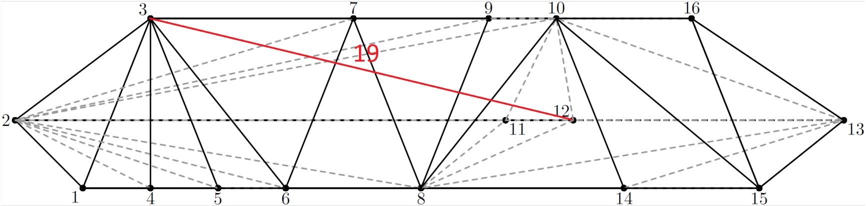
Consider it as a Farey polyhedron. On figure it is $[1; 0 : 1 : 2 : 2]$.
Definition. The λ-*length* is the maximal coordinate of the pennant.

λ-lengths (M. van Son, OK, 2024)



Consider it as a Farey polyhedron. On figure it is $[1; 0 : 1 : 2 : 2]$.
Definition. The λ-*length* is the maximal coordinate of the pennant.
On the picture: the pennant is $(8, 19, 14)$. Hence $\lambda(v, w) = 19$.

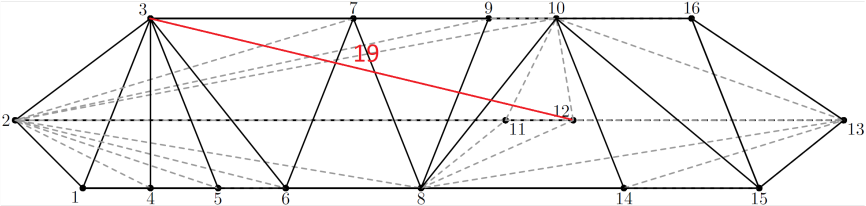
λ-lengths (M. van Son, OK, 2024)



Consider it as a Farey polyhedron. On figure it is $[1; 0 : 1 : 2 : 2]$.
Definition. The λ-*length* is the maximal coordinate of the pennant.
Best to compute it in the matrix form. Set

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

λ-lengths (M. van Son, OK, 2024)



Consider it as a Farey polyhedron. On figure it is $[1; 0 : 1 : 2 : 2]$.

Definition. The λ-*length* is the maximal coordinate of the pennant.

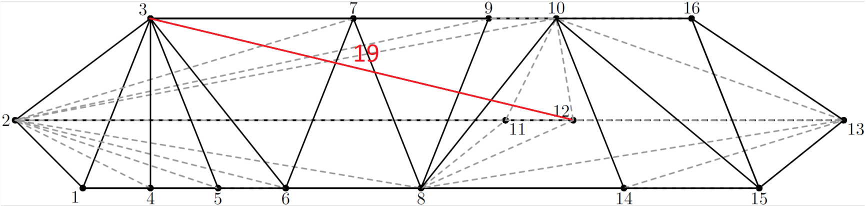
Then

$$A_1^1 A_2^0 A_3^1 A_1^2 A_2^2 = \begin{pmatrix} 3 & 8 & 1 \\ 7 & 19 & 2 \\ 5 & 14 & 2 \end{pmatrix}.$$

Take the maximal element: 19

(=generalised integer sine: Blackman, Dolan, OK 2023).

λ-lengths (M. van Son, OK, 2024)

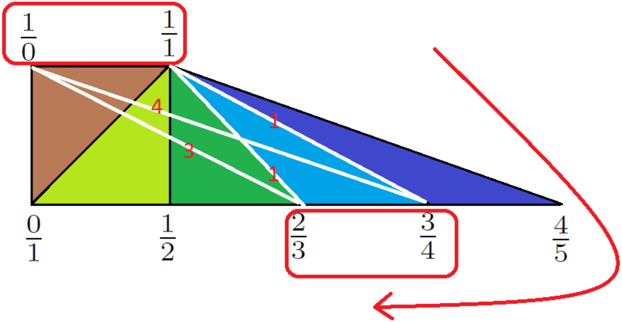


Consider it as a Farey polyhedron. On figure it is $[1; 0 : 1 : 2 : 2]$.

Definition. The λ-*length* is the maximal coordinate of the pennant.

Remark. All the λ-lengths are written via combinatorics of the triangulation (similarly to two-dimensional case).

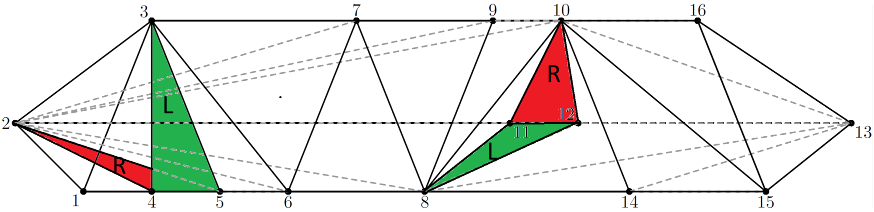
Ptolemy relation (recall)



Ptolemy relation for λ-lengths:

$$\det \begin{pmatrix} 4 & 1 \\ 3 & 1 \end{pmatrix} = 1$$

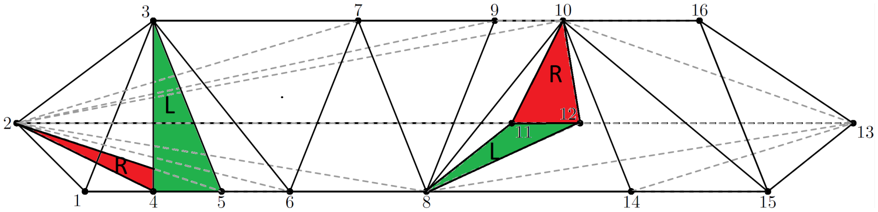
Ptolemy relation (M. van Son, OK, 2024)



Every triangle on the side of the prismatic diagram is either right (R) or left (L).

Here we assume that masts are naturally oriented.

Ptolemy relation (M. van Son, OK, 2024)

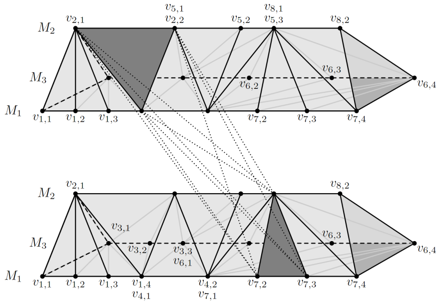


Theorem. Consider two triangles on sides V and W (separated by a nons-zero slice): the first oriented counterclockwise, the second oriented clockwise. Then the determinant of the matrix of lambda length $P(V, W)$ satisfies:

$$P(V,W) = \begin{cases} 1, & \text{if } V \text{ is a right triangle,} \\ 0, & \text{if } V \text{ is a left triangle.} \end{cases}$$

Here we assume that the Farey continued fractions does not have zeroes.

Ptolemy relation: example



The continued fraction is $[3; 1 : 2 : 1 : 2 : 3 : 3 : 1]$.

$$P(V, W) = \det \begin{pmatrix} 218 & 21 & 112 \\ 105 & 10 & 54 \\ 41 & 4 & 21 \end{pmatrix} = 1.$$

Frieze pattern in 3D.

Definition

Consider a Farey polyhedron P and with prismatic diagram D, let $V(D)$ be the set of vertices of D. Consider the function

$$\lambda : V(D) \times V(D) \to \mathbb{Z},$$

whose values on two vertices is the λ-length between the corresponding vertices in the Farey polyhedron. We call the collection $(\partial D \times \partial D, \lambda)$ the *frieze pattern* associated to the given Farey polyhedron.

Things to do.

I. Any link to q-multidimensional Euclidean algorithms. (What are 2D-eucldean algorithms)?

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I. Any link to q-multidimensional Euclidean algorithms. (What are 2D-eucldean algorithms)?

II. Any relation to cluster algebras?

III. Snake diagrams (and Markov numbers).

IV. Study/ compare different Euclidean algorithms.

Any relation to Voronoi polyhedra (interplay with cluster algebras, etc.)
(it is also based on matrix multiplication) ...

The end.

Thank you.