On operators generating higher brackets

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Objects: Higher Koszul brackets (a quick summary)

1. S_∞-structure of differential forms on a homotopy Poisson (P_∞) manifold.
2. An infinite series of odd brackets satisfying the Leibniz rule, along with a series of linked "higher Jacobi identities".
3. Can be defined by a Hamiltonian.
4. Cannot be defined by higher-order differential operators ("BV type"). A.: Can be defined by $\hbar$-differential operators.
5. The usual binary Koszul bracket is part of a classical diagram:

$$\begin{array}{ccc} \mathfrak{A}^k(M) & \xrightarrow{d_P} & \mathfrak{A}^{k+1}(M) \\ {\scriptstyle a^*}\Big\uparrow & & \Big\uparrow{\scriptstyle a^*:\ dx^a = P^{ab}x^*_b} \\ \Omega^k(M) & \xrightarrow{d} & \Omega^{k+1}(M)\,, \end{array} \tag{1}$$

   with vertical arrows also preserving the brackets.
   Q.: Do we have an analog for higher Koszul brackets?
6. Q.: What happens to the diagram under "quantization"?

Outline

Even and odd Poisson brackets

Even Poisson brackets. An even bracket on a commutative superalgebra s.t.

$$\{a, b\} = -(-1)^{\tilde{a}\tilde{b}}\{b, a\}\,, \tag{2}$$

$$\{a, \{b, c\}\} = \{\{a, b\}, c\} + (-1)^{\tilde{a}\tilde{b}}\{b, \{a, c\}\}\,, \tag{3}$$

$$\{a, bc\} = \{a, b\}c + (-1)^{\tilde{a}\tilde{b}}b\{a, c\}\,. \tag{4}$$

This bracket is linear without signs.

Odd Poisson brackets (antisymm. version). (Or Schouten, or Gerstenhaber, or antibracket.) An odd bracket s.t.

$$\{a, b\} = -(-1)^{(\tilde{a}+1)(\tilde{b}+1)}\{b, a\}\,, \tag{5}$$

$$\{a, \{b, c\}\} = \{\{a, b\}, c\} + (-1)^{(\tilde{a}+1)(\tilde{b}+1)}\{b, \{a, c\}\}\,, \tag{6}$$

$$\{a, bc\} = \{a, b\}c + (-1)^{(\tilde{a}+1)\tilde{b}}b\{a, c\}\,. \tag{7}$$

Linear with signs: $\{ka, b\} = k\{a, b\}$, $\{a, bk\} = \{a, b\}k$, and $\{ak, b\} = (-1)^{\tilde{k}}\{a, kb\}$. So the sign is on the comma.

Odd Poisson brackets (symm. version). A commutative superalgebra A with an odd bracket satisfying

$$\{a,b\} = (-1)^{\widetilde{a}\widetilde{b}}\{b,a\} \tag{8}$$

$$\{a,\{b,c\}\} = (-1)^{\widetilde{a}+1}\{\{a,b\},c\} + (-1)^{(\widetilde{a}+1)(\widetilde{b}+1)}\{b,\{a,c\}\}\,, \tag{9}$$

$$\{a,bc\} = \{a,b\}c + (-1)^{(\widetilde{a}+1)\widetilde{b}}b\{a,c\}\,. \tag{10}$$

Here the sign is on the opening bracket $\{$.
The two versions of odd bracket can be converted into one another by $\{a,b\}_{\mathrm{sym}} = (-1)^{\widetilde{a}}\{a,b\}_{\mathrm{anti}}$.

Ex. Even Poisson bracket on $C^\infty(M)$. Given a Poisson bivector $P = \frac{1}{2}P^{ab}x^*_b x^*_a$,

$$\{f, g\} = -(-1)^{\widetilde{a}(\widetilde{f}+1)} P^{ab} \frac{\partial f}{\partial x^b} \frac{\partial g}{\partial x^a} \,. \tag{11}$$

Ex. Canonical even Poisson bracket on Hamiltonians, $C^\infty(T^*M)$. $\{f, g\} = 0$, $\{H_X, f\} = X(f)$, $\{H_X, H_Y\} = H_{[X,Y]}$, where $H_X = X^a p_a$, for $X = X^a \frac{\partial}{\partial x^a}$. In particular, $\{p_a, x^b\} = \delta^b_a = -(-1)^{\widetilde{a}}(x^b, p_a\}$. In local coordinates,

$$(H\,,\,G) = (-1)^{\widetilde{a}(\widetilde{H}+1)} \frac{\partial H}{\partial p_a} \frac{\partial G}{\partial x^a} - (-1)^{\widetilde{a}\widetilde{H}} \frac{\partial H}{\partial x^a} \frac{\partial G}{\partial p_a} \,. \tag{12}$$

Ex. Schouten bracket: canonical odd Poisson bracket of multivector fields. Defined by the Leibniz rule with the initial conditions: for all $f, g \in C^\infty(M)$, and vector fields X, Y,

$$[\![f, g]\!] = 0\,,\ [\![P_X, f]\!] = X(f)\,,\ [\![P_X, P_Y]\!] = (-1)^{\widetilde{X}} P_{[X,Y]}\,. \tag{13}$$

On $C^\infty(\Pi T^*M)$, fiber coordinates in ΠT^*M are x^*_a, and $P_X = (-1)^{\widetilde{a}} X^a x^*_a$,

$$[\![F\,,\ G]\!] = (-1)^{\widetilde{a}(\widetilde{F}+1)} \left(\frac{\partial F}{\partial x^*_a} \frac{\partial G}{\partial x^a} + (-1)^{\widetilde{F}} \frac{\partial F}{\partial x^a} \frac{\partial G}{\partial x^*_a} \right)\,. \tag{14}$$

In particular, $[\![x^*_a, x^b]\!] = (-1)^{\widetilde{a}} \delta^b_a = [\![x^b, x^*_a]\!]$.

Ex. Koszul bracket: odd Poisson bracket of differential forms (symmetric version). Given a Poisson manifold M.

$$[f, g]_P = 0\,,\ [f, dg]_P = (-1)^{\widetilde{f}} \{f, g\}_P\,,\ [df, dg]_P = -(-1)^{\widetilde{f}} d\{f, g\}_P\,. \tag{15}$$

In particular,

$$[x^a, x^b]_P = 0\,,\ [x^a, dx^b]_P = -P^{ab}\,,\ [dx^a, dx^b]_P = dP^{ab}\,. \tag{16}$$

Outline

Poisson brackets as derived brackets

Notation. For an arbitrary smooth manifold (super or not) M,

$$\underbrace{\Omega(M) := C^\infty(\Pi TM)}_{\text{inhomogeneous diff forms}}, \quad \underbrace{\mathfrak{A}(M) := C^\infty(\Pi T^*M)}_{\text{multivector fields}}. \tag{17}$$

M: local coordinates x^a.
TM: $x^a, \delta x^a$, δ is even differential,
ΠTM: x^a, dx^a, d is odd differential,
T^*M: x^a, p_a,
ΠT^*M: x^a, x^*_a. E.g., a bivector has form $P = \frac{1}{2}P^{ab}(x)x^*_b x^*_a$.

δx^a and p_a have the same parities as the corresponding coordinates. dx^a and x^*_a have parities opposite to those of the corresponding coordinates. Under a change of coordinates, the variables p_a and x^*_a transform in the same way as the partial derivatives $\frac{\partial}{\partial x^a}$.

Even Poisson structure on M as a derived bracket.

Theorem.
An even bivector $P = \frac{1}{2}P^{ab}x_b^*x_a^*$, $[\![P, P]\!] = 0$, generates an even Poisson bracket on M:

$$\{f, g\}_P := \underbrace{[\![f, [\![P, g]\!]]\!]}_{\text{Schouten bracket}} \tag{18}$$

Odd Poisson structure on M as a derived bracket.

Theorem. An odd fiberwise quadratic Hamiltonian $H = \frac{1}{2}H^{ab}p_bp_a$, $\{H, H\} = 0$, generates an odd Poisson bracket on M:

$$\{f, g\}_H := \underbrace{-\{f, \{H, g\}\}}_{\text{canonical Poisson bracket of Hamiltonians}} \tag{19}$$

[Kosmann-Schwarzbach '94 and independently Voronov]

Outline

Poisson brackets from Lie algebroids

Lie algebroids were first introduced and studied by Jean Pradines in 1967. A Lie algebroid $E \to M$ is a vector bundle with a (super) Lie bracket on the space of its sections $\Gamma(E)$ and with a vector bundle morphism $a : E \to TM$, the "anchor", such that for all $u, v \in \Gamma(E)$ and $f \in C^\infty(M)$,

$$[u, fv] = a(u)(f)v + (-1)^{\tilde{u}\tilde{f}} f[u, v]\,, \tag{20}$$
$$a([u, v]) = [a(u), a(v)]\,. \tag{21}$$

Ex. Tangent algebroid $TM \to M$. The anchor is the identity map.
Ex. Poisson algebroid $T^*M \to M$. Given a Poisson manifold M.

$$[\delta f, \delta g]_P := \delta\{f, g\}_P\,,\ \ a : \delta x^a \mapsto P^{ab}\frac{\partial}{\partial x^b}\,. \tag{22}$$

(Using the anchor and linearity, extend the definition to non-exact 1-forms. This is a restriction of binary Koszul bracket.)

Q-manifold is a manifold endowed with a "homological vector field": odd Q such that

$$Q^2 = \frac{1}{2}[Q, Q] = 0\,. \tag{23}$$

Many objects in mathematical physics can be described in this way. The theory of Q-manifolds was initiated by A. Schwarz, A. Vaintrob, and M. Kontsevich.

Manifestations of a Lie algebroid $E \to M$.

$$\begin{array}{cccc} E & \Pi E & E^* & \Pi E^* \\ \downarrow & \downarrow & \downarrow & \downarrow \\ M & M & M & M \end{array}$$

Here $E^* \to M$ is the dual vector bundle and Π is the parity reversion functor.

$\Pi E \to M$ is obtained from $E \to M$ by reversing the parities of the fiber coordinates while the transition functions remaining the same. This gives non-trivial equivalent ways of describing a Lie algebroid.

$$\begin{array}{c} E \\ \downarrow \\ M \end{array}$$

$$[e_i, e_j] = (-1)^{\tilde{j}} Q_{ij}^k(x) e_k$$
$$a(e_i) = Q_i^a \frac{\partial}{\partial x^a}$$

$$\begin{array}{c} \Pi E \\ \downarrow \\ M \end{array}$$

ΠE is a Q manifold with

$$Q = \xi^i Q_i^a(x) \frac{\partial}{\partial x^a} + \frac{1}{2} \xi^i \xi^j Q_{ji}^k(x) \frac{\partial}{\partial \xi^k}$$

Homological v.f. of weight $+1$.

(Arkady Vaintrob)

$$\begin{array}{c} E^* \\ \downarrow \\ M \end{array}$$

$$\{x^a, x^b\} = 0\,, \quad \{u_i, x^a\} = Q_i^a(x)$$
$$\{u_i, u_j\} = (-1)^{\tilde{j}} Q_{ij}^k(x) u_k$$

Even Poisson bracket of weight -1.

$$\begin{array}{c} \Pi E^* \\ \downarrow \\ M \end{array}$$

$$\{x^a, x^b\} = 0\,, \quad \{\eta_i, x^a\} = Q_i^a(x)$$
$$\{\eta_i, \eta_j\} = (-1)^{\tilde{j}} Q_{ij}^k(x) \eta_k$$

Odd Poisson bracket of weight -1.

Ex. Manifestations of the tangent algebroid. $E = TM$.
On $\Pi E = \Pi TM$: de Rham differential $Q = d = dx^a \frac{\partial}{\partial x^a}$.
On $\Pi E^* = \Pi T^* M$: Schouten bracket.

Ex. Manifestations of the Poisson algebroid. $E = T^* M$.
On $\Pi E = \Pi T^* M$: Lichnerowicz differential $Q = d_P = [\![P, -]\!]$.
On $\Pi E^* = \Pi TM$: Koszul bracket.

Outline

Higher Poisson brackets

P_∞-structures. We replace Poisson bi-vector P with an even multivector

$$P = P_0 + P^a x_a^* + \frac{1}{2!} P^{ab} x_b^* x_a^* + \frac{1}{3!} P^{abc} x_c^* x_b^* x_a^* + \dots \tag{24}$$

squaring to zero: $[\![P, P]\!] = 0$. It generates on M a series of brackets by Voronov's "higher derived brackets" construction:

$$\{f_1, \dots, f_k\}_P := [\![\dots [\![P, f_1]\!], \dots, f_k]\!]\big|_M , \quad \big|_M = \big|_{x^*=0} . \tag{25}$$

Schouten brackets $[\![-,-]\!]$ are derivations of degree -1. Therefore, only the term of degree i contributes to the i-ary bracket. In particular,

$$\underbrace{\{x^a\} = -P^a}_{\text{odd bracket}},\ \underbrace{\{x^a, x^b\} = -(-1)^{\tilde{a}} P^{ab}}_{\text{even bracket}},\ \underbrace{\{x^a, x^b, x^c\} = \pm P^{abc}}_{\text{odd bracket}} \dots \tag{26}$$

Properties: antisymmetry, alternating parities, Leibniz and higher Jacobi.

S_∞ structures. Similarly, for an odd Hamiltonian $H = H(x^a, p_a) = H_0 + H^a p_a + \frac{1}{2!} H^{ab} p_b p_a + \frac{1}{3!} H^{abc} p_c p_b p_a + \ldots$ squaring to zero: $\{H, H\} = 0$, we have

$$\{f_1, \ldots, f_k\}_H := \{\ldots\{H, f_1\}, \ldots, f_k\}\Big|_M, \quad \Big|_M = \Big|_{p_a=0}. \qquad (27)$$

Properties: symmetry w.r.t. shifted parity, odd, Leibniz and higher Jacobi. For both P_∞ and S_∞ structures higher Jacobi are of the form ($\{-\} = d$):

$$n = 1 : d^2 = 0$$

$$n = 2 : d\{f, g\} = \{df, g\} \pm \{f, dg\}$$

$$n = 3 : \quad \{\{f, g\}, h\} \pm \{\{h, f\}, g\} \pm \{\{g, h\}, f\}$$

$$= \pm\, d\{f, g, h\} \pm \{df, g, h\} \pm \{f, dg, h\} \pm \{f, g, dh\}$$

$$n = 4 : \pm \sum_{shuffle} \{df_1, f_2, f_3, f_4\} \pm \sum_{shuffle} \{\{f_1, f_2\}, f_3, f_4\} \pm \sum_{shuffle} \{\{f_1, f_2, f_3\}, f_4\}$$

$$\pm\, d\{f_1, f_2, f_3, f_4\} = 0$$

$$n = 5 : \pm \sum_{shuffle} \{df_1, f_2, f_3, f_4, f_5\} \pm \sum_{shuffle} \{\{f_1, f_2\}, f_3, f_4, f_5\} \pm$$

$$\sum \{\{f_1, f_2, f_3\}, f_4, f_5\} \pm \sum \{\{f_1, f_2, f_3, f_4\}, f_5\} \pm d\{f_1, f_2, f_3, f_3, f_4, f_5\} =$$

Outline

Mackenzie-Xu transformation

MX map: for an arbitrary vector bundle $E \to M$,

$$\underbrace{T^* \underbrace{E}_{x^a, u^i}}_{p_a, p_i} \xrightarrow{MX} \underbrace{T^* \underbrace{E^*}_{x^a, u_i}}_{p_a, p^i} \tag{28}$$

$$MX^*(x^a) = x^a\,,\ MX^*(u_i) = p_i\,,\ MX^*(p_a) = -p_a\,,\ MX^*(p^i) = (-1)^{\widetilde{i}} u^i\,.$$

Property: anti-Poisson map of the canonical brackets:

$$MX^*(\{F, G\}) = -\{MX^*(F), MX^*(G)\}\,. \tag{29}$$

$$
\begin{tikzcd}
\underbrace{T^*E}_{p_a,p_i} \cong \underbrace{T^*E^*}_{p_a,p^i} \arrow[r] \arrow[d] \arrow[dr] & E^* \arrow[d, "u_i"] \\
E \arrow[r, "u^i"] & \underbrace{M}_{x^a}
\end{tikzcd}
$$

T^*E has two gradings:
$w_1 = \#u^i - \#p_i$ induced from the standard grading on $E \to M$, and
$w_2 = \#p_a + \#p_i$ as the standard grading of $T^*E \to E$.
Define $w_3 = w_1 + w_2 = \#u^i + \#p_a$.
Similarly, T^*E^* has gradings:
$w_1 = \#u_i - \#p^i$,
$w_2 = \#p_a + \#p^i$,
$w_3 = \#u_i + \#p_a$.
Then

$$MX(w_2) = w_3\,,\ MX(w_3) = w_2\,. \tag{30}$$

Ex. Tangent algebroid. Manifests through d and through the Schouten bracket. Using MX we can, for example, construct the master Hamiltonian of the Schouten bracket.

$$\begin{array}{lll}
\Pi TM: & d = dx^a \frac{\partial}{\partial x^a} \longrightarrow & \mathrm{H}_{\mathrm{deRham}} = dx^a p_a \\
& & \quad \Big\downarrow \mathrm{MX} \\
\Pi T^*M: & \text{(binary) Schouten br} & \mathrm{H}_{\mathrm{Sch}}
\end{array}$$

On ΠTM : x^a and dx^a.
On $T^*(\Pi TM)$: p_a and π_a.
On ΠT^*M : x^a and x^*_a.
On $T^*(\Pi T^*M$:) p_a and π^a. Then

$$\mathrm{MX}\colon\ T^*(\Pi T^*M) \to T^*(\Pi TM)$$
$$(x^a, x^*_a, p_a, \pi^a) \mapsto \left(x^a := x^a, dx^a = (-1)^{\widetilde{a}+1}\pi^a, p_a := -p_a, \pi_a := x^*_a\right). \tag{31}$$

and, therefore, $\mathrm{MX}^*(\mathrm{H}_{\mathrm{deRham}}) = \mathrm{MX}^*(dx^a p_a) = (-1)^{\widetilde{a}}\pi^a p_a$. A Hamiltonian that is linear in momenta maps into one that is quadratic in momenta.

Ex. Poisson algebroid.

Similarly, we can proceed with T^*M algebroid. Recall that it has the following manifestations:
on ΠT^*M: Lichnerowicz differential $Q = d_P$,
on ΠTM: Koszul bracket.

$$\begin{array}{lcc} \Pi T^*M : & d_P = [\![P, -]\!] \longrightarrow & \mathrm{H}_{\mathrm{d_P}} \\ & & \updownarrow \mathrm{MX} \\ \Pi TM : & \text{(binary) Koszul br} \longrightarrow & H_{\mathrm{Koszul}} \end{array}$$

Outline

Higher Koszul brackets

Khudaverdian, Th. Voronov'08.
Given a P_∞ manifold M with an even $P \in C^\infty(\Pi T^*M)$, $[\![P,P]\!]=0$,

$$P = P_0 + P^a x^*_a + \frac{1}{2!}P^{ab}x^*_b x^*_a + \frac{1}{3!}P^{abc}x^*_c x^*_b x^*_a + \dots \tag{32}$$

The idea for finding $H_{\text{Higher Koszul}}$:

$$\begin{array}{lll} \Pi T^*M : & \text{New } d_P = [\![P,-]\!] \longrightarrow & \text{Step 1: New } \mathrm{H}_{\mathrm{d_P}} \\ & & \Big\downarrow \mathrm{MX} \\ \Pi TM : & S_\infty\text{: Higher Koszul brs} & \text{Step 2: } H_{\text{Higher Koszul}} \end{array}$$

1) New $H_{d_P} \in C^\infty(T^*(\Pi T^*M))$. The Hamiltonian generates the same bracket as d_P: $d_P(Q) = \{H_{d_P}, Q\}$, where, on the right side, we have canonical Poisson bracket of Hamiltonians. The Schouten bracket can be also generated by a Hamiltonian: $[\![P,Q]\!] = \{\{H_{Sch},P\},Q\}$. This implies

$$H_{d_P} = \{H_{Sch},P\} = (-1)^{\tilde{a}}\frac{\partial P}{\partial x^*_a}p_a + (-1)^{\tilde{a}}\frac{\partial P}{\partial x^a}\pi^a\,. \tag{33}$$

Update our diagram accordingly:

$$
\begin{array}{llcl}
\Pi T^*M: & \text{New } d_P = [\![P, -]\!] & \longrightarrow & \mathrm{H}_{\mathrm{d_P}} = (-1)^{\widetilde{a}} \frac{\partial P}{\partial x^*_a} p_a + (-1)^{\widetilde{a}} \frac{\partial P}{\partial x^a} \pi^a \\
& & & \big\downarrow \mathrm{MX} \\
\Pi TM: & S_\infty\text{: Higher Koszul brs} & & \text{Step 2: } H_{\text{Higher Koszul}}
\end{array}
$$

2) Hamiltonian for Higher Koszul brackets. Applying MX to the odd linear in momenta Hamiltonian $H_{d_P} \in C^\infty(T^*(\Pi T^*M))$, we obtain a Hamiltonian from $C^\infty(T^*(\Pi TM))$:

$$
H_{HigherKoszul} = -(-1)^{\tilde{a}} \frac{\partial P}{\partial x^*_a}(x^b, \pi_b) p_a - \frac{\partial P}{\partial x^a}(x^b, \pi_b) dx^a \,. \qquad (34)
$$

Here we used

$$
\begin{aligned}
&\mathrm{MX}\colon\ T^*(\Pi TM) \to T^*(\Pi T^*M) \\
&(x^a, dx^a, p_a, \pi_a) \mapsto \left(x^a := x^a, x^*_a := \pi_a, p_a := -p_a, \pi^a := (-1)^{\widetilde{a}+1} dx^a\right) .
\end{aligned}
\qquad (35)
$$

$H_{HigherKoszul}$ can be used as the generating element in Voronov's higher derived brackets construction:

$$[\omega_1, \ldots, \omega_k]_P := \{\{H_{HigherKoszul}, \omega_1\}, \ldots, \omega_k\}\Big|_{\Pi TM} \tag{36}$$

In particular, we have

$$[f]_P = \{f\}_P \ , \tag{37}$$

$$[f_1, \ldots, f_k]_P = 0\,, \ k \geqslant 2 \ , \tag{38}$$

$$[f_1, df_2, \ldots, df_k]_P = (-1)^{\varepsilon}\{f_1, \ldots, f_k\}_P \ , \tag{39}$$

$$[df_1, df_2, \ldots, df_k]_P = (-1)^{\varepsilon+1} d\{f_1, \ldots, f_k\}_P \ , \tag{40}$$

where $\varepsilon = (k-1)\widetilde{f_1} + (k-2)\widetilde{f_2} + \cdots + \widetilde{f_{k-1}} + k$.
Higher Koszul brackets properties:
(1) all odd,
(2) symmetric,
(3) Leibniz,
(4) higher Jacobi
(an S_∞ structure)

Outline

Brackets generated by differential operators

By operators of order $\leqslant 2$.

Batalin-Vilkovisky (BV) algebra is a commutative (super) algebra A with an odd Poisson bracket and an odd differential operator $\Delta : A \to A$, $\operatorname{ord}\Delta \leqslant 2$, so that

$$\Delta(ab) = \Delta(a)b + (-1)^{\tilde{a}} a\Delta(b) + [a, b]\,. \tag{41}$$

Thm. $\Delta^2 = 0$ implies the Jacobi identity for the bracket (Lian-Zuckermann, E.Getzler, Penkava-Schwarz).

Classical example 1. $[\![-,-]\!]$ Schouten bracket of multivector fields. Here Δ is an odd second-order divergence operator, $\delta T = (-1)^{\tilde{a}} \dfrac{1}{\rho(x)} \dfrac{\partial}{\partial x^a} \left(\rho(x) \dfrac{\partial T}{\partial x^*_a}\right)$, defined using a choice of a volume element ρ on M.

(Since at least 1950s, see e.g. Kirillov's survey.)

Classical example 2. $[-,-]$ Koszul bracket of diff. forms on a Poisson manifold M. An odd second-order operator $\Delta = \partial_P := [d, i(P)]$. $i(P) = \frac{1}{2} P^{ab} \frac{\partial}{\partial dx^b} \frac{\partial}{\partial dx^a}$ (Koszul'85).

Generating by operators of order > 2.

Koszul'85: for arbitrary operator $\Delta : A \to A$, define a series of brackets, $\Phi^k_\Delta : A \times \cdots \times A \to A$,

$$\Phi^1_\Delta(a) = (\Delta - \Delta(1))(a) \tag{42}$$

$$\Phi^2_\Delta(a,b) = \Delta(ab) - \Delta(a)b - (-1)^{\tilde{a}\tilde{\Delta}} a\Delta(b) + \Delta(1)ab \tag{43}$$

$$\begin{aligned}\Phi^3_\Delta(a,b,c) &= \Delta(abc) - \Delta(ab)c \pm a\Delta(bc) \pm \Delta(ca)b \\ &+ \Delta(a)bc \pm \Delta(b)ca \pm \Delta(c)ab \\ &- \Delta(1)abc \\ &\vdots\end{aligned}$$

$$\Phi^k_\Delta(a,b,c) = [\dots[\Delta, a_1], \dots, a_k](1) = \dots \tag{44}$$

All $\Phi^k_\Delta = [-, \dots, -]$ are symmetric and of parity equal to the parity of Δ. Only for ord $\Delta \leqslant 2$, $\Delta(1) = 0$, Koszul proved that if $\tilde{\Delta} = 1$ and $\Delta^2 = 0$, then we have the Jacobi identity for $\Phi^2_\Delta = [-,-]$ and higher brackets vanish.

Olga Kravchenko: in general, for odd Δ of higher order, $\Delta^2 = 0$ implies the identities of an L_∞-algebra (i.e. higher Jacobi) for all $\Phi^n_\Delta = [-, \dots, -]$.

Fusun Akman: studied generalizations to non-commutative, non-associative case.

A problem: for $\operatorname{ord} \Delta > 2$, brackets Φ^n_Δ do not obey Leibniz rule. Bracket Φ^{n+1}_Δ is the obstruction for the Leibniz rule for Φ^n_Δ:

$$[f_1, \dots, f_{n-1}, fg]_\Delta = \\ [f_1, \dots, f_{n-1}, f]_\Delta g + (-1)^{(\tilde{\Delta} + \tilde{f}_1 + \dots + \tilde{f}_1)\tilde{f}} f [f_1, \dots, f_{n-1}, g]_\Delta \\ + [f_1, \dots, f_{n-1}, f, g]_\Delta \quad (45)$$

Both generating brackets by differential operators and by Hamiltonians are instances of Ted Voronov's higher derived brackets.

Outline

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Problem: can higher Koszul brackets be also described by differential operators? Using Koszul or Kravchenko directly, no.

Solution in a nutshell: use $\hbar$-differential operators.
Hamiltonian $H_{HigherKoszul}$ will be quantized into an "$\hbar$-differential operator",
canonical Poisson bracket will be quantized into $\frac{i}{\hbar}[-,-]$,
the higher Koszul brackets will be the limit of "quantum brackets".
Quantum brackets will be L_∞-structure plus $\hbar$-deformed Leibniz; higher Koszul brackets will be L_∞-structure plus strict Leibniz (i.e. give precisely S_∞-structure)

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A formal $\hbar$-differential operator is (see Voronov'18 and the theory in She'23) a (non-commutative) formal power series in $\hbar$ and $\hat{p}_i$,

$$L = \sum_{n=0}^{\infty} \left(L_0^{a_1 \ldots a_n}(x) \hat{p}_{a_1} \ldots \hat{p}_{a_n} + (-i\hbar) L_1^{a_1 \ldots a_{n-1}}(x) \hat{p}_{a_1} \ldots \hat{p}_{a_{n-1}} + \ldots + (-i\hbar)^n L_n^0(x) \right), \quad (46)$$

considered together with the "Heisenberg commutation relation"

$$[\hat{p}_a, f] = -i\hbar \frac{\partial f}{\partial x^a}, \quad (47)$$

which is homogeneous with respect to the total degree in $\hbar$ and $\hat{p}_a$s.
(1) Formal $\hbar$-differential operators have *grading* (not filtration) by $\#p_a + \#\hbar$, which is invariant under changes of variables.
(2) Each homogeneous component has a finite number of derivatives.

The principal symbol $\sigma(L) \in C^\infty(T^*M)$ is a (formal) Hamiltonian defined as follows: $L \mod \hbar$ with identification of $\hat{p}_a$ and p_a.

$$L = \sum_{n=0}^{\infty} \left(L_0^{a_1 \dots a_n}(x) \hat{p}_{a_1} \dots \hat{p}_{a_n} + \dots + (-i\hbar)^n L_n^0(x) \right) \mapsto$$

$$\sigma(L) = \sum_{n=0}^{\infty} L_0^{a_1 \dots a_n}(x) p_{a_1} \dots p_{a_n} \quad (48)$$

Theorem. For formal $\hbar$-differential operators,

$$\sigma(AB) = \sigma(A)\,\sigma(B)\,. \quad (49)$$

The commutator $[A, B]$ is always divisible by $\hbar$ and

$$\sigma(i\hbar^{-1}[A, B]) = \{\sigma(A), \sigma(B)\}\,, \quad (50)$$

where at the right-hand side there is the Poisson bracket on T^*M.

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[Voronov'05] For an operator L on an algebra, the quantum n-bracket and the classical n-bracket ($n = 0, 1, 2, 3, \ldots$) generated by L are respectively

$$\{f_1, \ldots, f_n\}_{L,\hbar} := (-i\hbar)^{-n}\,[\ldots[L, f_1], \ldots, f_n]\,(1)\,, \tag{51}$$

$$\{f_1, \ldots, f_n\}_{L} := (-i\hbar)^{-n}\,[\ldots[L, f_1], \ldots, f_n]\,(1) \pmod{\hbar}\,. \tag{52}$$

In order to avoid negative powers of $\hbar$, we assume that any n-fold commutator $[\ldots[L, f_1], \ldots, f_n]$ in the above formulas is divisible by $(-i\hbar)^n$. In particular, this is true for formal $\hbar$-differential operators.

The n-bracket generates the $(n+1)$-bracket as the obstruction to the Leibniz rule:

$$\begin{aligned}\{f_1, \ldots, f_{n-1}, fg\}_{L,\hbar} = \{f_1, \ldots, f_{n-1}, f\}_{L,\hbar}\, g + (-1)^{\varepsilon} f\, \{f_1, \ldots, f_{n-1}, g\}_{L,\hbar} \\ + (-i\hbar)\{f_1, \ldots, f_{n-1}, f, g\}_{L,\hbar}\,,\end{aligned}$$

where $(-1)^{\varepsilon} = (-1)^{(\tilde{L}+\tilde{f}_1+\ldots+\tilde{f}_{n-1})\tilde{f}}$.

Hence, the corresponding classical brackets satisfy the Leibniz rule, i.e are multiderivations, and can be generated by a Hamiltonian H.

<u>Thm.</u>[She'23] $H = \sigma(L)$.

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Operator generating Higher Koszul brackets.

[See She'23.] 1. Khudaverdian-Voronov obtained Hamiltonian for the Higher Koszul brackets as $H_{HigherKoszul} = MX^*\Big(\{H_{Sch}, P\}\Big)$. Alternatively, $H_{HigherKoszul} = -\{MX^*(H_{Sch}), MX^*(P)\}$.
Recall Mackenzie-Xu for our case:

$$\begin{gathered} \mathrm{MX}\colon\ T^*(\Pi T^*M) \to T^*(\Pi TM) \\ (x, p, x^*_a, \pi^a) \mapsto (x, dx^a := (-1)^{\widetilde{a}+1}\pi^a, -p, \pi_a := x^*_a)\,. \end{gathered} \tag{53}$$

Hence,

$$H_{HigherKoszul} = -\{H_{deRham}, P(x, \pi_a)\}\,.$$

2. Quantize:

$$\Delta_P = \frac{i}{\hbar}\Big[\underbrace{\hat{d}}_{-i\hbar d}, \underbrace{\hat{P}}_{P(x, -i\hbar\frac{\partial}{\partial dx})}\Big] = [d, \hat{P}]\,. \tag{54}$$

3. We proved that higher Koszul brackets can be generated by $\Delta_P = [d, \hat{P}]$ as the quantum brackets taken modulo $\hbar$.

Outline

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The de Rham and Poisson complexes. An even Poisson bivector $P = \frac{1}{2}P^{ab}x^*_b x^*_a$, $[\![P, P]\!] = 0$, induces:

1. On the differential forms $\Omega(M) = C^\infty(\Pi TM)$: Koszul bracket.
2. On the multivector fields $\mathfrak{A}(M) = C^\infty(\Pi T^*M)$: Lichnerowicz differential $d_P = [\![P, -]\!]$.

Besides this, we have the following canonical structures:

1. On the differential forms: de Rham differential d.
2. On the multivector fields: Schouten bracket $[\![-, -]\!]$.

The following commutative diagram arises:

$$\begin{tikzcd} \mathfrak{A}^k(M) \arrow[r, "d_P"] & \mathfrak{A}^{k+1}(M) \\ \Omega^k(M) \arrow[r, "d"] \arrow[u, "a^*"] & \Omega^{k+1}(M)\,, \arrow[u, "a^*:\ dx^a = P^{ab}x^*_b"'] \end{tikzcd} \tag{55}$$

with vertical arrows also preserving the brackets.

So, if P is a Poisson bivector, then:

$$\begin{array}{ccc} \mathfrak{A}^k(M), [\![-,-]\!] & \xrightarrow{d_P} & \mathfrak{A}^{k+1}(M), [\![-,-]\!] \\ {\scriptstyle a^*}\uparrow & & \uparrow {\scriptstyle a^*:\ dx^a = P^{ab}x^*_b} \\ \Omega^k(M), [-,-] & \xrightarrow{d} & \Omega^{k+1}(M), [-,-] \end{array} \tag{56}$$

What if P is replaced with an arbitrary multivector? (Even, and $[\![P,P]\!] = 0$.) Then:

1. On the differential forms $\Omega(M)$: higher Koszul brackets.
2. On the multivectors $\mathfrak{A}(M)$: an analog of Lichnerowicz's $d_P = [\![P,-]\!]$.

$$\begin{array}{ccc} \mathfrak{A}(M) & \xrightarrow{d_P} & \mathfrak{A}(M) \\ \uparrow & & \uparrow {\scriptstyle dx^a = (-1)^{\tilde{a}+1}\frac{\partial P}{\partial x^*_a}} \\ \Omega(M) & \xrightarrow{d} & \Omega(M)\,, \end{array} \qquad \begin{array}{c} (\mathfrak{A}(M), [\![-,-]\!]) \\ {\scriptstyle ???}\uparrow \\ (\Omega(M), S_\infty \text{ higher Koszul brs})\,, \end{array} \tag{57}$$

The left diagram shows the results of Khudaverdian-Voronov'08, and the right one – of Khudaverdian-Voronov'24. ??? should be an L_∞-morphism.

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The idea is easier to see in the context of Lie algebroids.
For a Lie algebroid $E \to M$, one can continue with the philosophy of manifestations, applying it to the anchor $a : E \to TM$:

On ΠE : $\quad \Pi E \xrightarrow{\;\mathrm{a}\;} \Pi TM \quad$ Q-morphism, i.e. $Q \circ a^* = a^* \circ d$

On ΠE^* : $\quad \Pi T^*M \xrightarrow{\;a^{\mathrm{dual}}\;} \Pi E^* \quad$ $(a^{\mathrm{dual}})^*$ maps br into Schouten br

Specifically, for the Poisson algebroid $T^*M \to M$:

On ΠE : $\quad \Pi T^*M \xrightarrow{\;\mathrm{a}\;} \Pi TM$

On ΠE^* : $\quad \Pi T^*M \xrightarrow{\;a^{\mathrm{dual}}\;} \Pi TM \quad$ $a^{\mathrm{dual}} = \pm a$

Equality $a^{\mathrm{dual}} = \pm a$ explains the famous diagram — why the same map is a chain map of complexes AND also respects the bracket structures.

Now we have an L_∞ algebroid $T^*M \to M$. We have the anchor $a : T^*M \rightsquigarrow TM$, which is an L_∞-morphism of L_∞-algebroids. The anchor has the following manifestations:

On ΠE : $\quad \Pi T^*M \xrightarrow{\text{a}} \Pi TM \quad$ a non-linear map!

On ΠE^* : $\quad \Pi T^*M \longrightarrow \Pi TM \quad$ dual for a non-linear map??

A problem: $a = (-1)^{\widetilde{a}+1} \frac{\partial P}{\partial x^*_a}$ is a non-linear map. How to get its dual? (On functions, we need an L_∞-morphism! How to get it?)

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Thick morphism (Voronov'14) $\Phi : M_1 \Longrightarrow M_2$ is a Lagrangian submanifold in $T^*M_2 \times -T^*M_1$ w.r.t. to symplectic form $\omega_2 - \omega_1$, specified by an even generating function of the form

$$S(x,q) = S^0(x) + \varphi^i(x)q_i + \frac{1}{2}S^{ij}(x)q_jq_i + \frac{1}{3!}S^{ijk}(x)q_kq_jq_i + \ldots \quad (58)$$

Here the local coordinates in M_1 and M_2 are x^a and y^i; and the momenta are p_a and q_i, respectively.

The pullback of a thick morphism $\Phi^* : C^\infty(M_2) \to C^\infty(M_1)$:

$$\Phi^*[g](x) = g(y) + S(x,q) - y^i q_i\,, \quad (59)$$

$$q_i = \frac{\partial g}{\partial y^i}(y)\,,\ y^i = (-1)^{\tilde{i}}\frac{\partial S}{\partial q_i}(x,q)\,. \quad (60)$$

Theorem. If odd Hamiltonians H_1 and H_2 are Φ-related, i.e. if

$$H_1\left(x, \frac{\partial S}{\partial x}(x,q)\right) = H_2\left((-1)^{\tilde{q}_i}\frac{\partial S}{\partial q_i}(x,q), q\right) \quad (61)$$

then $\Phi^* : C^\infty(M_2) \to C^\infty(M_1)$ is an L_∞-morphism of the S_∞ higher brackets structures defined by H_1 and H_2.

The L_∞-morphism mapping higher Koszul brackets into Schouten bracket.

1. Consider $a : \Pi T^*M \to \Pi TM$ as a thick morphism (a usual map $\varphi : M_1 \to M_2$ is a thick morphism with generating f. $S = \varphi^i(x)q_i$):

$$S = S(x^a, x^*_a; p_a, \pi_a) = x^a p_a + (-1)^{\tilde{a}+1} \frac{\partial P}{\partial x^*_a}(x, x^*)\pi_a\,. \tag{62}$$

2. Apply MX to the equations defining the Lagrangian submanifold, and then get

$$S^* = S^*(y, y^*, p_a, \pi_a) = y^a p_a + (-1)^{\tilde{a}} \frac{\partial P}{\partial x^*_a}(x, \pi_b) y^*_a\,. \tag{63}$$

3. Define $a^{\text{dual}}\colon\ \Pi T^*M \Longrightarrow \Pi TM$ as the thick morphism with S^*. As a is an S_∞ thick morphism, then a^{dual} is also an S_∞ thick morphism. This means that the pullback of this thick morphism,

$$(a^{\text{dual}})^*\colon\ C^\infty(\Pi TM) \to C^\infty(\Pi T^*M) \tag{64}$$

is an L_∞-morphism.

To sum up, we have the following:

When P is a multivector:

$$
\begin{tikzcd}
\mathfrak{A}(M) \arrow[r, "d_P"] & \mathfrak{A}(M) \\
\Omega(M) \arrow[u, "a^*"] \arrow[r, "d"] & \Omega(M)\,, \arrow[u, "a^*:\ dx^a = (-1)^{\tilde{a}+1}\frac{\partial P}{\partial x^a}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
(\mathfrak{A}(M), [\![-,-]\!]) \\
(\Omega(M), S_\infty:\ \text{Higher Koszul brs}) \arrow[u, "L_\infty:(a^{\text{thick dual}})^*"']
\end{tikzcd}
\tag{65}
$$

Here, $\mathrm{H}_{\mathrm{Sch}}$ and H_P are $a^{\text{thick dual}}$-related.

Quantizing this, we have:

$$
\begin{tikzcd}
\mathfrak{A}(M) \arrow[r, "-i\hbar d_P"] & \mathfrak{A}(M) \\
\Omega(M) \arrow[u, "a^*"] \arrow[r, "-i\hbar d"] & \Omega(M)\,, \arrow[u, "a^*"']
\end{tikzcd}
\qquad
\begin{tikzcd}
(\mathfrak{A}(M), -\hbar^2\delta_\rho) \\
(\Omega(M), S_{\infty,\hbar}:\Delta_P)\,, \arrow[u, "I\,???"']
\end{tikzcd}
\tag{66}
$$

Here, operators $-\hbar^2\delta_\rho$ and Δ_P are I-related.

Q. What is I? Construct I.

This will also imply an L_∞-morphism for quantum brackets.

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This is a quantum anchor problem.

Consider manifestations of the anchor in L_∞-algebroid T^*M and its quantization:

$$\begin{array}{lccc}
\Pi E: & a:\Pi T^*M \to \Pi TM & \overset{\hbar}{\rightsquigarrow} & \text{Same } a \text{ but } Q\text{s mult. by } -i\hbar \\
 & Q\text{-morphism} & & \updownarrow \hbar\text{-MX} \\
\Pi E^*: & a^{\mathrm{dual}}:\Pi T^*M \Rrightarrow \Pi TM & \overset{\hbar}{\rightsquigarrow} & (a^*)^\star:\Pi T^*M \to \Pi TM \\
 & \mathrm{H}_{\mathrm{Sch}}\,\&H_P \text{ are } a^{\mathrm{dual}}\text{-related} & & -\hbar^2\delta_\rho\&\Delta_P \text{ are intertwined by } I
\end{array}$$

To construct the desired $(a^{\mathrm{dual}})^\star$, we apply $\hbar$-MX to a in the right top quadrant. ($\hbar$-MX is a "quantum MX transformation" explained below.)

Anchor a is a usual map. With
$S = S(x^a, x^*_a; p_a, \pi_a) = x^a p_a + (-1)^{\tilde{a}+1}\frac{\partial P}{\partial x^*_a}(x, x^*)\pi_a$ we can write its pullback a^* as

(1) a pullback Φ^* of a thick morphism Φ, or

(2) a quantum pullback $\Phi^*_\hbar$ of a thick quantum morphism $\Phi_\hbar$.

We need the quantum pullback option. So,
$a^*\colon \ C^\infty(\Pi TM) \to C^\infty(\Pi T^*M)$ can be re-written as

$$f_1(x, x^*) = \int Ddy\, Đy^*\, e^{\frac{i}{\hbar}\left((-1)^{\widetilde{a}+1}\frac{\partial P}{\partial x^*_a} - dy^a\right)y^*_a} f_2(x, dy)\,, \tag{67}$$

where $Đy^* = (2\pi\hbar)^{-m}(i\hbar)^n(-1)^{\frac{n(n+1)}{2}}Dy^*$
By a theorem from She'23, operator $(a^*)^\star\colon \ C^\infty(\Pi TM) \to C^\infty(\Pi T^*M)$ can be written as follows:

$$g_2(x, y^*) = \int Ddx\, Đx^*\, e^{\frac{i}{\hbar}\left((-1)^{\widetilde{a}+1}\frac{\partial P}{\partial x^*_a}y^*_a - x^*_a\, dx^a\right)} g_1(x, dx)\,, \tag{68}$$

where $Đx^* = (2\pi\hbar)^{-m}(i\hbar)^n(-1)^{\frac{n(n+1)}{2}}Dx^*$.

Now, to sum up:

$$\begin{aligned} d_P a^* &= a^* d && (69)\\ (-i\hbar d_P) \circ a^* &= a^* \circ (-i\hbar d) && (70)\\ (a^*)^\star \circ d_P^\star &= d^\star \circ (a^*)^\star && (71)\\ (a^*)^\star \circ (-i\hbar d_P)^\star &= (-\hbar^2 \delta_{\boldsymbol{\rho}}) \circ (a^*)^\star\,. && (72) \end{aligned}$$

That would be the intertwining relation we are looking for if we had $(-i\hbar d_P)^\star = \Delta_P$. But we know that $(\Delta_P)^\star = -i\hbar d_P - i\hbar\delta_{\boldsymbol{\rho}}(P)$, so $(-i\hbar d_P)^\star \neq \Delta_P$ unless $\delta_{\boldsymbol{\rho}}(P) = 0$.
So, to get an intertwining between $\delta_{\boldsymbol{\rho}}$ and Δ_P, a correction is needed to the integral operator that we have obtained; and this has been done.

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She'23: Quantum Mackenzie-Xu ($\hbar$-MX or also $\star$ as the usual MX) is an anti-isomorphism (meaning the order of factors reversed) between the algebras of operators on dual vector bundles E and E^*, induced by the following pairing.

Given a volume element $\boldsymbol{\rho} = \rho(x)Dx$ on the base M, for functions $f = f(x,u) \in C^\infty(E)$, $g = g(x,u^*) \in C^\infty(E^*)$, define

$$\langle f, g\rangle_{\boldsymbol{\rho}} = \int\limits_{E\times_M E^*} \rho(x)Dx\, Du\, Du^*\; e^{-\frac{i}{\hbar}\langle u,u^*\rangle} f(x,u) g(x,u^*)\,, \tag{73}$$

The quantum Mackenzie-Xu ($\hbar$-MX) transformation of an operator $A\colon\; C^\infty(E_1) \to C^\infty(E_2)$ is the adjoint $A^\star\colon\; C^\infty(E_2^*) \to C^\infty(E_1^*)$:

$$\langle A(f), g\rangle = (-1)^{\tilde{A}\tilde{f}}\langle f, A^\star(g)\rangle \tag{74}$$

Let $E = \Pi TM$ and $E^* = \Pi T^*M$. Then $(f(x))^\star = f(x)$, and

$$\left(\frac{\partial}{\partial x^a}\right)^\star = -\rho^{-1}\circ\frac{\partial}{\partial x^a}\circ\rho\,; \quad (dx^a)^\star = -i\hbar\,(-1)^{\tilde{a}+1}\frac{\partial}{\partial x^*_a}\,;$$

$$\left(-i\hbar\frac{\partial}{\partial dx^a}\right)^\star = x^*_a\,; \quad d^\star = -i\hbar\,\delta_{\boldsymbol{\rho}} = -i\hbar\,(-1)^{\tilde{a}}\frac{1}{\rho(x)}\frac{\partial}{\partial x^a}\rho(x)\frac{\partial}{\partial x^*_a}\,.$$

Thank You!

Happy birthday, Vladimir and Valentin!