Representations of Posets, product formulas and Fractional Calabi-Yau Categories

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sketch

My aim is to explain some ideas relating

- combinatorics and partial orders
- isolated singularities of polynomial functions
- derived categories

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- derived categories

⚠️ This is mostly about ideas and conjectures. The *motto* :

Derived categories serve as a meeting point between combinatorics and singularity theory.

On the combinatorial side

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These are counted by the binomial coefficient $\binom{m+n}{m}$.

This formula can be written as follows

$$\binom{m+n}{m} = \prod_{i=1}^{m} \frac{m+n+1-i}{i},$$

which has the general shape

$$\prod_{e \in E} \frac{D-e}{e},$$

for some multi-set E and integer D (here $D = m+n+1$).

Fact: Many famous enumeration results in combinatorics involve formulas of this precise shape !

- Dyck paths, Dyck paths of slope m (Catalan and Fuss-Catalan numbers) A108
- Plane partitions (MacMahon), symmetric plane partitions A6366
- Alternating sign matrices (Mills-Robbins-Rumsey, Zeilberger, Kuperberg) A5130
- Various kinds of planar maps (Tutte)
- 2-stack sortable permutations (West, Zeilberger) A139
- intervals in the Tamari lattices A260
- clusters and tilting modules for Dynkin quivers (Fomin and Zelevinsky) A1700

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und so weiter...

No obvious general reason is known for these formulas to exist. Some partial reasons explain some cases.

For example, the number of clusters in a cluster algebra of finite type is given by

$$\prod_{i=1}^{m} \frac{h + d_i}{d_i},$$

where h and the $(d_1, \ldots, d_m)$ are the Coxeter number and the degrees of the associated finite Weyl group.

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Using the symmetry of the degrees $d_i \leftrightarrow h + 2 - d_{\varphi(i)}$ for some bijection φ, this is the same as

$$\prod_{i=1}^{m} \frac{2h + 2 - d_i}{d_i},$$

which has exactly the expected shape with $D = 2h + 2$.

On the singularity side

We will look at isolated quasi-homogeneous singularities of polynomial functions $f : \mathbb{C}^m \to \mathbb{C}$.

These were studied by Milnor, in famous and classical works.

Recall that $z = (z_1, \ldots, z_m) \in \mathbb{C}^m$ is a singular point of f if all partial derivatives of f vanish at z.

The word **"isolated"** means that f has an isolated singular point, that we assume to be $0 \in \mathbb{C}^m$.

We will also assume that $f(0) = 0$.

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The word **"quasi-homogeneous"** means that there exists integers $d_1, \ldots, d_m$ and D such that:

the total degree of f, when giving weight d_i to the variable z_i, is given by D.

Example: $f = x^3 + y^4$ has total degree 12 if x has weight 4 and y has weight 3.

So every quasi-homogeneous isolated singularity comes with the data of $(d_1, \ldots, d_m), D$.

Conversely, fix $(d_1, \ldots, d_m), D$ and pick f as a **generic quasi-homogeneous polynomial** w.r.t. this data.

Fact: There exists necessary and sufficient **conditions** on $(d_1, \ldots, d_m), D$ in order to ensure that f has an isolated singularity.

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Fact (A): The product $\prod_{i=1}^{m} \frac{D-d_i}{d_i}$ is an integer.

Fact (B): The product $\prod_{i=1}^{m} \frac{[D-d_i]_q}{[d_i]_q}$ is a polynomial in $\mathbb{N}[q]$.

Here $[d]_q$ denotes the q-integer $1 + q + q^2 + \cdots + q^{d-1}$.

Both facts are not clear *a priori*. They only belong clearly to $\mathbb{Q}$ and $\mathbb{Q}(q)$.

Milnor fiber and Milnor number

A quick little piece of *geometry* now, from Milnor.

Choose f with quasi-homogeneous isolated singularity. Let B be a small-enough ball around $0 \in \mathbb{C}^m$.

- The fibers $f^{-1}(z) \cap B$ for z in a small circle around $0 \in \mathbb{C}$ are all diffeomorphic.
- They form a locally-trivial fibration over the small circle.
- They have the homotopy type of a bouquet of μ spheres of dimension $m - 1$.

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Fact (A2): the integer $\mu = \prod_{i=1}^{m} \frac{D - d_i}{d_i}$ (**Milnor number** of the singularity)

Fact (B2): the polynomial $\prod_{i=1}^{m} \frac{[D - d_i]_q}{[d_i]_q}$ describes a filtration on the homology of the fibers

Also the dimension and graded dimension of the Jacobian ring of the singularity.

Homology and monodromy

Because fibers are bouquets 💐 of spheres 🌎 of the same dimension, only one interesting homology group H^{m-1} isomorphic to $\mathbb{Z}^\mu$.

Turning once around the small circle and following cycles by local triviality, one obtains a linear map

$H^{m-1} \to H^{m-1}$ which is called the *monodromy map* of the singularity.

You can think of the monodromy as a $\mu \times \mu$ matrix of integers (once a basis has been chosen).

Derived categories as a mediating object

Main idea : relate product formulas in combinatorics to Milnor formula for Milnor number μ of singularities

HOW ? Using **derived categories**

- on the combinatorial side, enrich combinatorial objets with **partial orders**

  and consider finite-dimensional modules over their incidence algebras over a field

  (basis are all pairs (x, y) such that $x \leq y$ and product is concatenation)

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- on the combinatorial side, enrich combinatorial objets with **partial orders**

  and consider finite-dimensional modules over their incidence algebras over a field

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- on the singularity side, consider a **categorification** $\mathscr{D}_{\mathrm{Mil}}$ of the Milnor fiber homology and monodromy

  This $\mathscr{D}_{\mathrm{Mil}}$ should be a triangulated category, recovering the geometric data when passing to K_0.

⚠️ WARNING: this kind of category is attributed to Seidel. I am not sure in which generality it is known to exist.

This $\mathscr{D}_{\text{Mil}}$ should be a *Directed Fukaya Category* 🦄 or a *Fukaya-Seidel category* 🦄, whatever it is.

Here some hand-waving about **A**-Model, **B**-model, mirror symmetry, Berglund-Hübsch invertible polynomials ✴️ ?

Motto / main idea

Suppose that you have a familly of combinatorial objects $(P_n)_n$ counted for each index n by a **combinatorial formula** of the shape

$$\prod_{e \in E} \frac{D - e}{e}$$

for some multi-sets E and integers D depending on the index n in some regular way.

(Example : Dyck paths and Catalan numbers)

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THEN 🎁

There should exist partial orders $\leq$ on the combinatorial objects such that the derived category $\mathscr{D}_P$ of modules over the incidence algebra of $(P_n, \leq)$ is **triangle-equivalent** to the derived category $\mathscr{D}_{\mathrm{Mil}}$ attached to the quasi-homogeneous singularity of a generic quasi-homogeneous polynomial with weights E and total weight D.

(Example: the natural partial order on Dyck paths by inclusion)

Claim: all the derived categories involved should be *fractional Calabi-Yau*.

Calabi-Yau (CY) and fractional Calabi-Yau (fCY) categories

Let T be triangulated category with finite-dimensional Hom spaces over a field.

A **Serre functor** on T is an auto-equivalence of T such that

$$\mathrm{Hom}(X, Y)^* \simeq \mathrm{Hom}(Y, SX)$$

functorially in both arguments.

This name comes from the Serre duality functor on coherent sheaves in algebraic geometry. Unique up to isomorphism.

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The category T is **Calabi-Yau** if S is isomorphic to a shift functor $[D]$.

This names comes from the properties of coherent sheaves on Calabi-Yau manifolds.

The category T is **fractional Calabi-Yau** if a power of S is isomorphic to a shift functor. (Kontsevich, around 2000)

meaning that $S^q \simeq [p]$ for some integers p and q. Abusingly, p/q is called the Calabi-Yau dimension.

Examples of fCY categories

- some examples come from algebraic geometry : pieces in semi-orthogonal decompositions of derived categories coming from **Fano manifolds**
- derived categories of representations of **Dynkin quivers** (types $\mathbb{ADE}$)
- some examples from singularity theory using categories of **matrix factorisations**

Combinatorics of **posets-with-product-formula** as a new source of examples !

Monodromy on both sides

On the geometry side, the expected categorification $\mathscr{D}_{\text{Mil}}$ of Milnor's fibration should be such that

- $K_0(\mathscr{D}_{\text{Mil}})$ is a free abelian group of rank μ,
- on which the Serre functor S induces the monodromy matrix.
- $\mathscr{D}_{\text{Mil}}$ depends only on the data $(d_1, \ldots, d_m), D$

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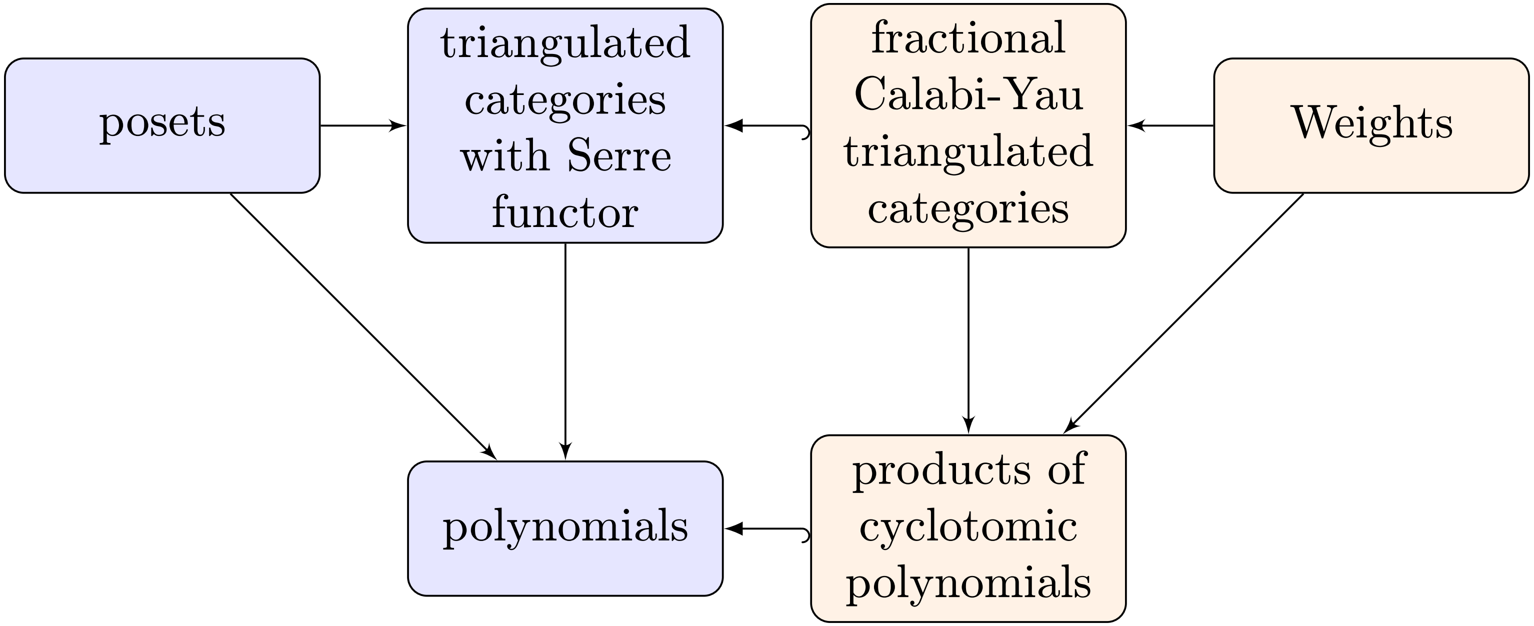
On the combinatorial side, the derived category $\mathscr{D}_P$ of modules over an incidence algebra of a partial order P has an Auslander-Reiten functor τ (equivalent to having a Serre functor S)

- $K_0(\mathscr{D}_P)$ is a free abelian group of rank the cardinality of P,
- τ induces a linear map on $K_0(T)$ with an easy matrix, computed directly from the partial order $\leq_P$


posets
triangulated categories with Serre functor
fractional Calabi-Yau triangulated categories
Weights
polynomials
products of cyclotomic polynomials

The Coxeter criterion

Suppose you have combinatorial objects counted by a product formula.

Suppose moreover that you have found partial orders on these objects.

How to convince yourself that they are "good" in the sense of our motto 🎁 ?

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💡 IDEA : compare the characteristic polynomials of monodromy !

Product formula $\implies$ weights E and degree D $\implies$ formula for characteristic polynomial (Milnor-Orlik)

So one has a guess for the characteristic polynomial, to compare with the one from the partial order (called the Coxeter polynomial of P)

If they match, one can hope to be on a good track !

Concrete example (almost for babies 👶🏼 👶🏾 👶🏿)

Consider the partial orders on Dyck paths by inclusion (being always below)

In size 3, there are 5 Dyck paths.

The general formula is the Catalan number $\frac{1}{n+1}\binom{2n}{n}$.

For $n = 3$, this gives the product formula

$$5 = \frac{6}{2}\frac{5}{3}$$

so that $E = \{2, 3\}$ and $D = 8$.

From this, one finds that the monodromy of a generic singularity has char. polynomial $t^5 - t^4 + t - 1$.

same concrete example, combinatorial side

On the other hand, one computes the matrix of the Auslander-Reiten translation τ and finds

$$\begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ -1 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

which has the same char. polynomial (up to technical details about shifts and signs).

In fact, this case is derived equivalent to representations of a quiver of Dynkin type $\mathbb{D}_5$, hence fCY.

Derived equivalences between partial orders

It can very well happen that two famillies of combinatorial objects share the same product formula.

It can also happen that one finds different partial orders both being good w.r.t. this product formula.

In this case, one should expect the derived categories of posets to be triangle-equivalent and fractional Calabi-Yau. This can be proved by purely algebraic means.

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Prototypical examples : Dyck words for inclusion and binary trees for rotation (Tamari order)

both counted by Catalan numbers, both apparently sharing the same char. polynomials.

It is known (Rognerud) that Tamari orders are indeed fractional Calabi-Yau.

Not known yet for Dyck paths under inclusion.

Formulas without good posets

Sometimes not easy to find good partial orders for which the motto 🎁 would work

- good candidate for the famous formula for alternating sign matrices, partial order found by J. Striker.
- no candidate known for the formula by Tutte counting intervals in the Tamari lattices
- no candidate known for the formula enumerating 2-stack sortable permutations

So there remains nice things to discover in the wild out there

🦕

Bonus track : factorising derived categories

Thom-Sebastiani sum of singularities : f and g with disjoint variables, consider $f+g$

in terms of weight data $(d_1,\ldots,d_m);D$ and $(e_1,\ldots,e_n);E$, this means:

- scale both data so that $E=D$, then take the disjoint union

For example $(1);3$ and $(1);4$ together give $(4,3);12$

The the monodromy of Milnor fibers is the tensor product of monodromies

Then the associated Fukaya-style categories should be tensor product of the smaller Fukaya-style categories.

On the combinatorial side, for the cartesian product of posets, one also gets that the derived category is the tensor product of categories for factor posets.

$\implies$ if you can factorise the weight data, you should be able to factorise the poset (up to derived equivalence)

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In [ ]:
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