Numeric-symbolic computation of the separatrix graph of a holomorphic differential equation

FELim

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(joint work with R. Schilling, Strasbourg)

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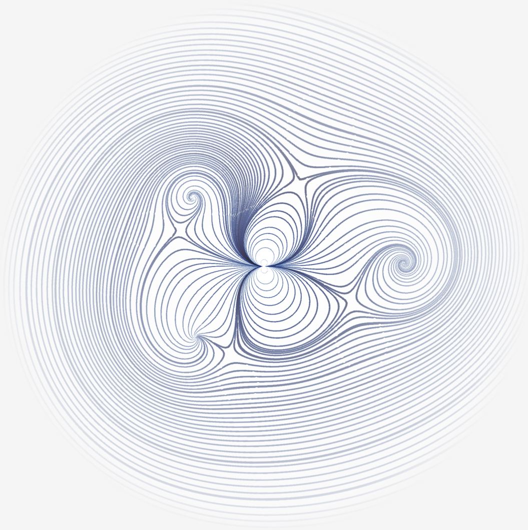
Rational flows on $\overline{\mathbb{C}}$



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$P, Q \in \mathbb{C}[z] \ , P \wedge Q = 1$

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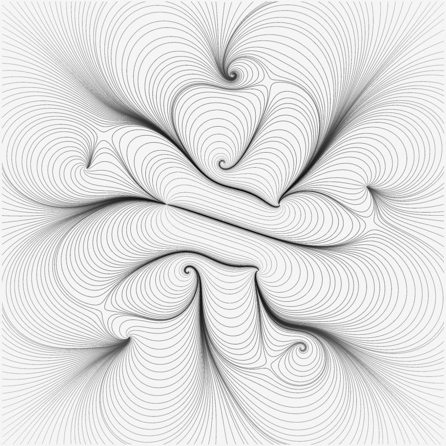
Singularities in affine chart $\mathbb{C} = \{z \neq \infty\}$

$PQ = 0$

- zero-like: $P^{-1}(0)$
- pole-like: $Q^{-1}(0)$

Example (some random degree-10 flow)



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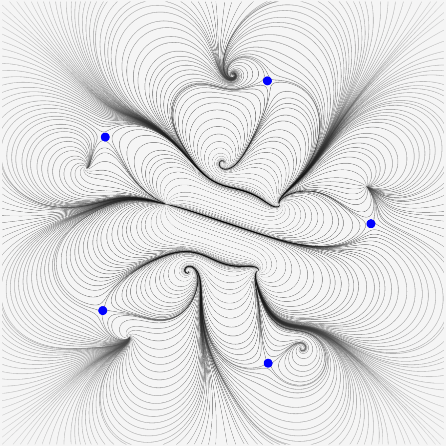

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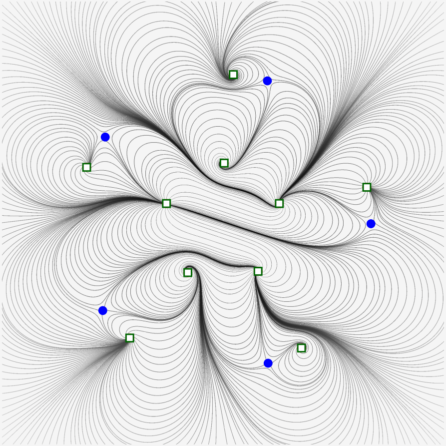

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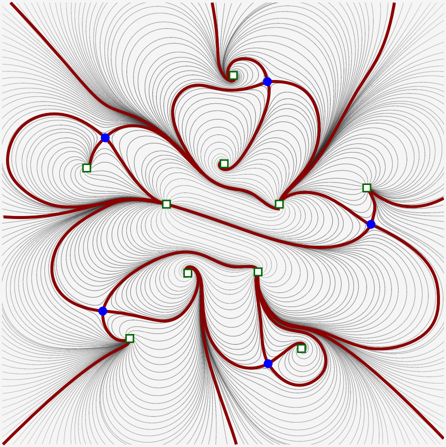

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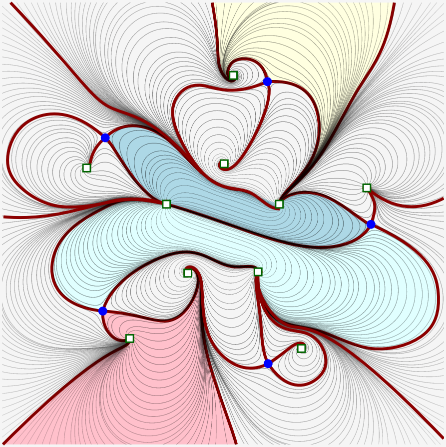

- Saddle points: poles $Q^{-1}(0)$
- Other stationary points: zeros $P^{-1}(0)$
- Separatrices
  (= trajectories passing through saddles)
- Zones

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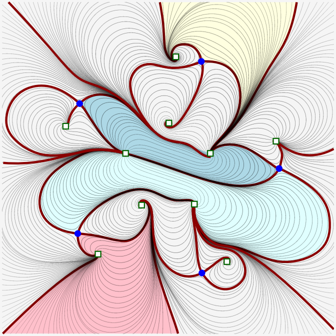

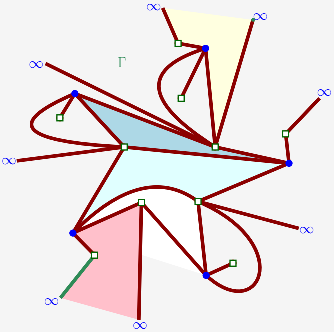

Fact

Topological class of the flow
= combinatorial class of the ***separatrix graph*** Γ

Douady's question

Questions

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   $\mathcal{C}$: decidable number field

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2. We produce *effective* algorithms for polynomials in $\mathcal{C}[z]$
   $\mathcal{C}$: decidable number field
3. Complexity = arithmetic complexity of the BSS-machine

Separatrix graph

Let $\gamma\ :\ t\in]\tau_-\,,\,\tau_+[\subset\mathbb{R}\to\gamma(t)$ be a maximal trajectory

- Either γ is periodic $\longrightarrow$ **cycle**

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- edges $\{\gamma : \tau_- \text{ or } \tau_+ \in \mathbb{R}\}$

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Theorem (A. Douady-F. Estrada-P. Sentenac / B. Branner-K. Dias)

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Conjecture

The mapping $\Gamma : \mathbb{C}(z) \longrightarrow \mathtt{PlanarGraphs}$ is surjective

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$$Q = 1 \quad , \quad P(z) = z^{k+1} + a_{k-1} z^{k-1} + \cdots + a_0 \quad , \quad k \in \mathbb{Z}_{>0}$$

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3. Simple roots: $k^4 \to (k\ln k)^2$

Bifurcation parameter

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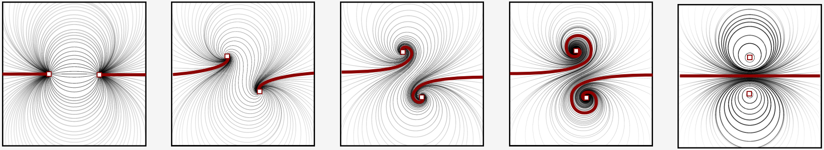

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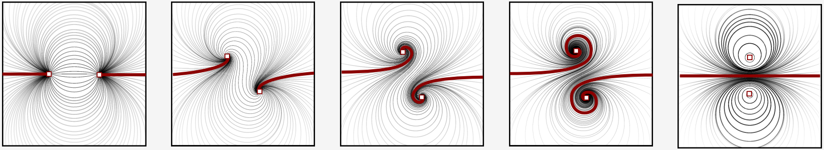

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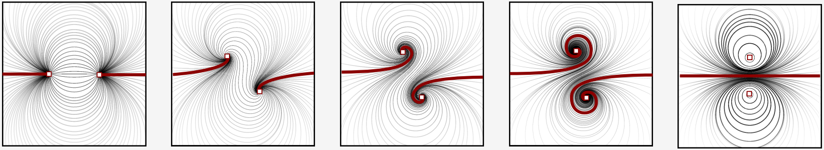

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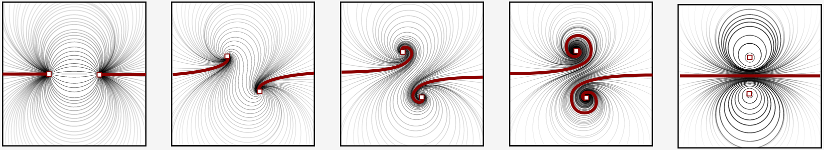
1. $\operatorname{Re}(\mathfrak{b}(P)) \to +\infty \iff$ roots collide **saddle-node bifurcation**



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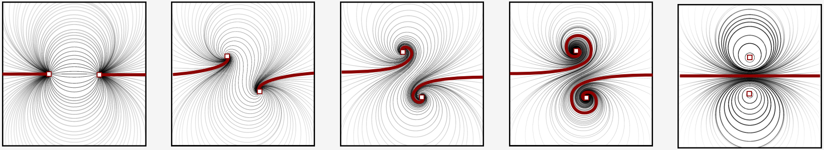
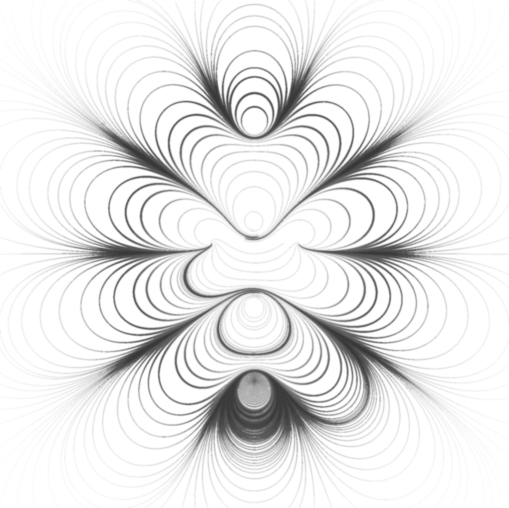
1. $\operatorname{Re}(\mathfrak{b}(P)) \to +\infty \iff$ roots collide **saddle-node bifurcation**
2. $\operatorname{Im}(\mathfrak{b}(P)) \to 0 \iff$ loop forms **center/focus bifurcation**



Example



Where do the separatrices land?

What is being computed

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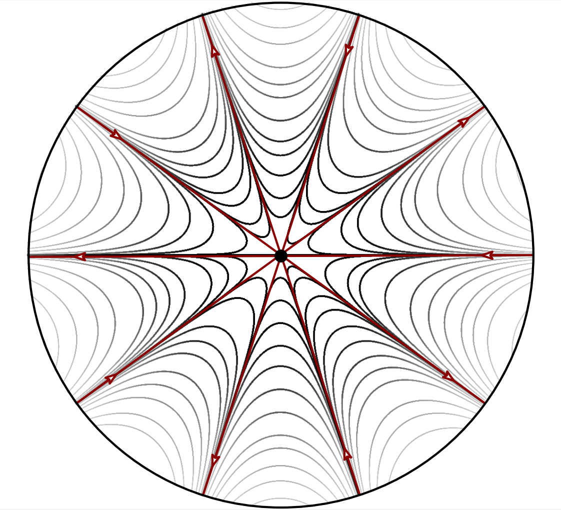
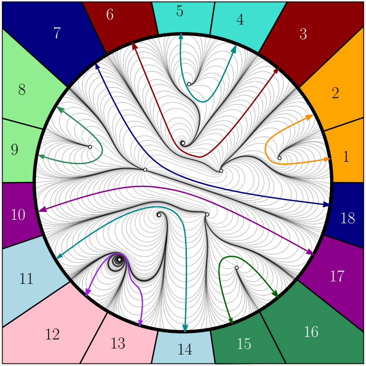


Near ∞ ($k = 5$)

What is being computed (generic case)

All roots are simple, no cycle (in particular $\mathcal{L} \cap \mathbb{R} = \{0\}$)



Douady-Estrada-Sentenac non-crossing involution:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
2	1	6	5	4	3	18	9	8	17	14	13	12	11	16	15	10	7

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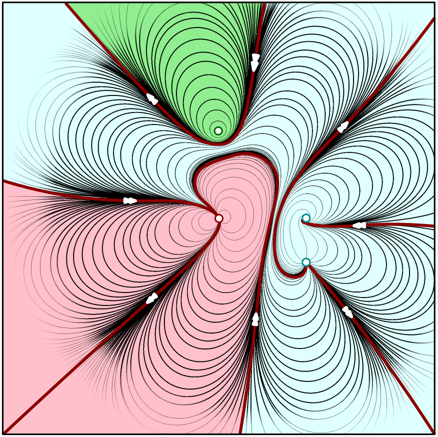
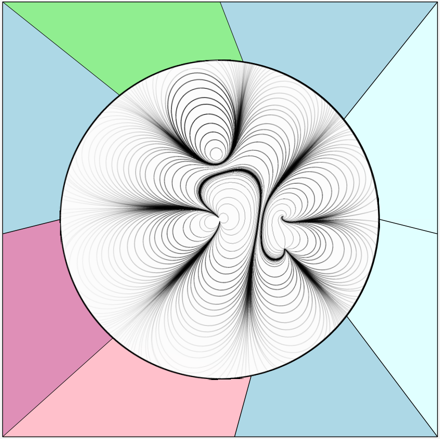
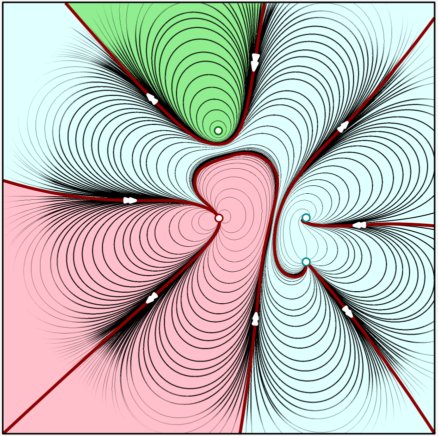
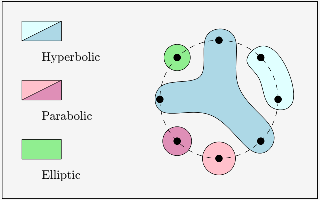


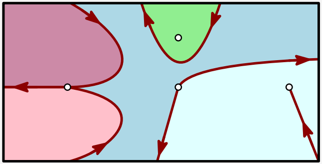


The zones are of different nature and define a sectors partition near ∞

What is being computed (general case)







Lemma

*The **polar partition** is a colored non-crossing partition of* $\mathbb{Z}/2k\mathbb{Z}$

What is being computed (general case)

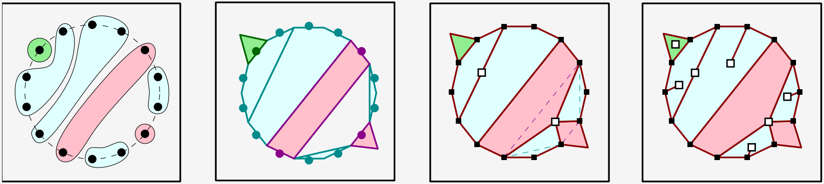
Definition

Here goes a boring, never ending list of properties defining a **colored non-crossing partition** of $\mathbb{Z}/2k\mathbb{Z}$, ascertaining that this is the right concept to state provable theorems. It is a combinatorial counterpart to the dynamical properties of holomorphic flows that have previously been discussed

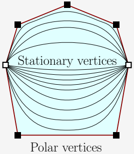
What is being computed (general case)

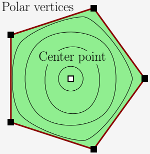
Main Proposition

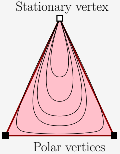
There is a 1-1 correspondence between classes of non-crossing colored partitions (up to cyclic action of $\mathbb{Z}/2k\mathbb{Z}$) and topological classes of flows



3 colors:







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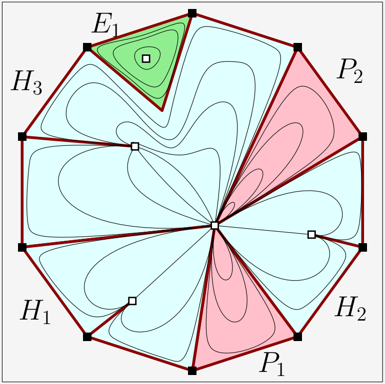
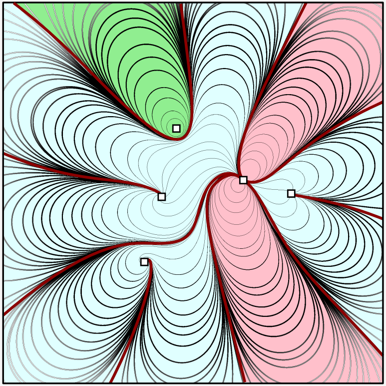
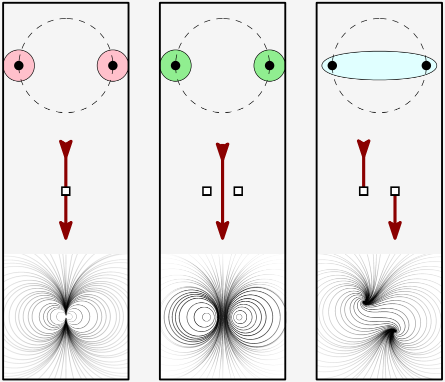


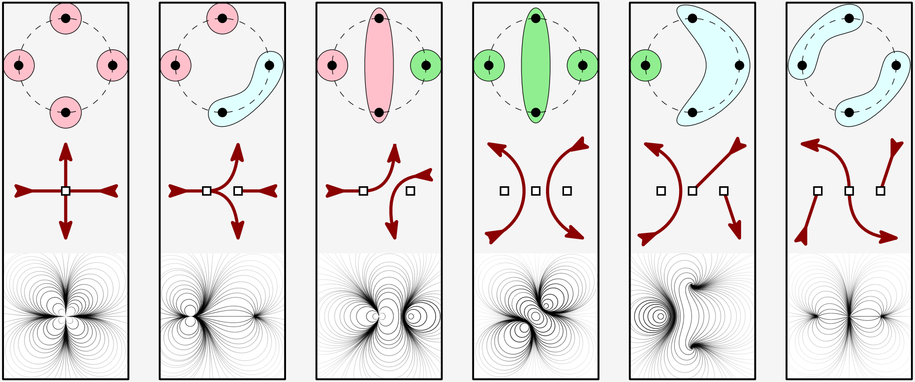


Examples



$k = 1$

Examples

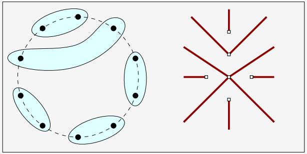


$k = 2$

Examples

Definition

Generic = structurally stable flows (open condition)
= hyperbolic blocks with 2 elements



Generic polar partition = non-crossing involution

Examples

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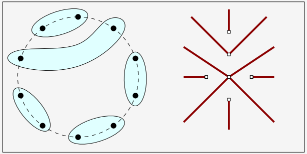

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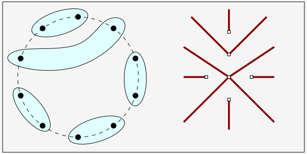

Generic polar partition = non-crossing involution

k	1	2	3	4	5	6	7
#NCInv(k)	1	1	2	3	6	14	34
#NonCrossing(k)	3	6	26	123	801	5686	43 846

(Computations by T. Tomasini)

How to compute a polar partition

Aim

To compute the polar connections near ∞ without going there

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1. Take $P \in \mathcal{C}[z]$, prepare it in the form $z^{k+1} + *z^{k-1} + \cdots$ and compute $\mathfrak{d}(P)$ and $|P|_k$

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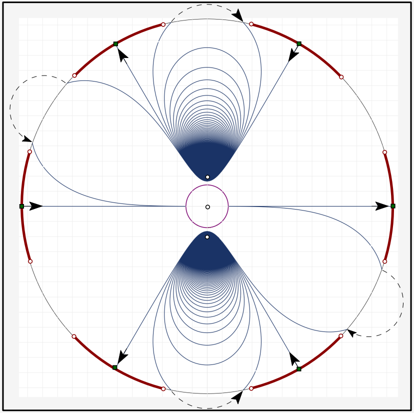
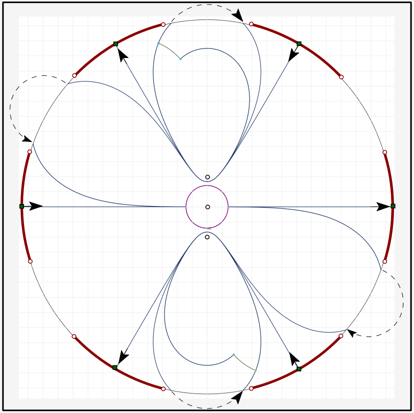
6. We end up with the incidence matrix of Γ

Simple example



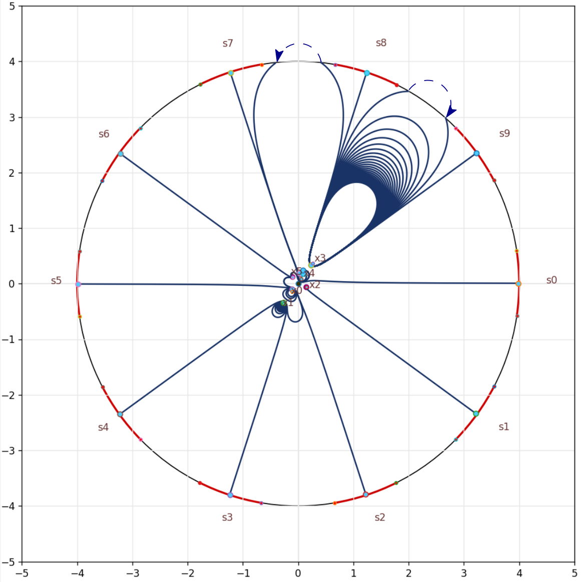
Naive approach

Computational shortcut: root-traps



Root-trapping is not a crime

A typical computation



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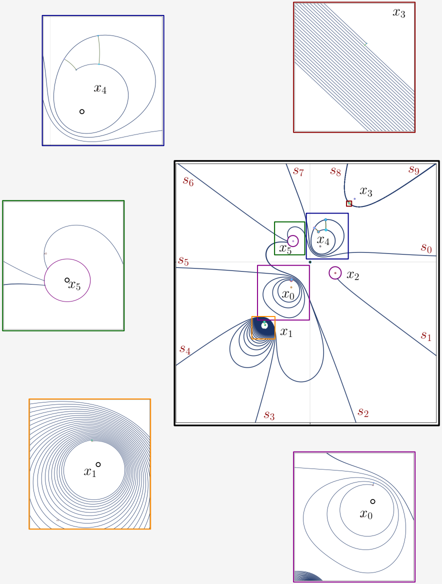

Determining the parameters: separatrix arcs

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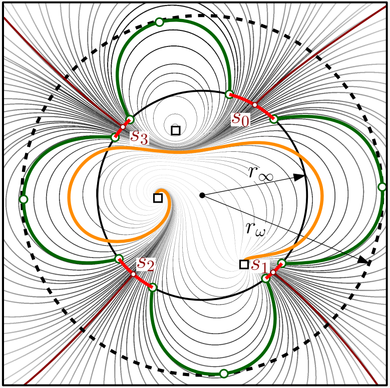
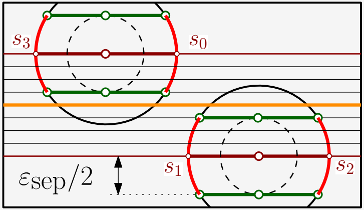
In z-space

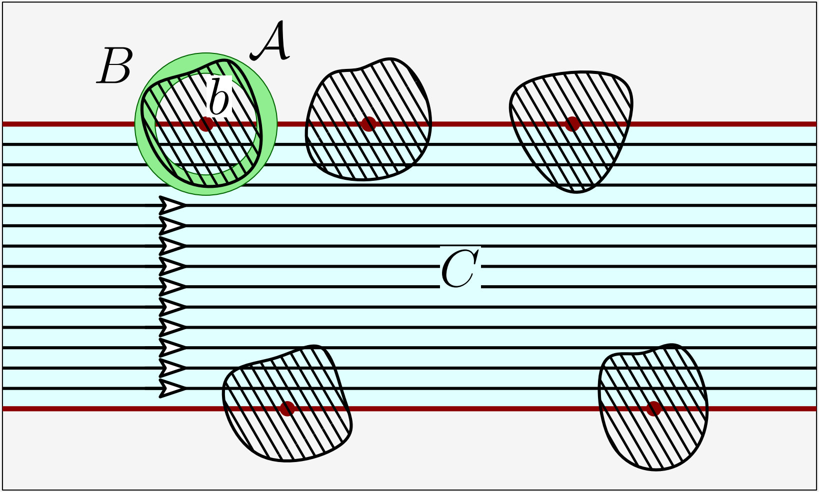


In t-space



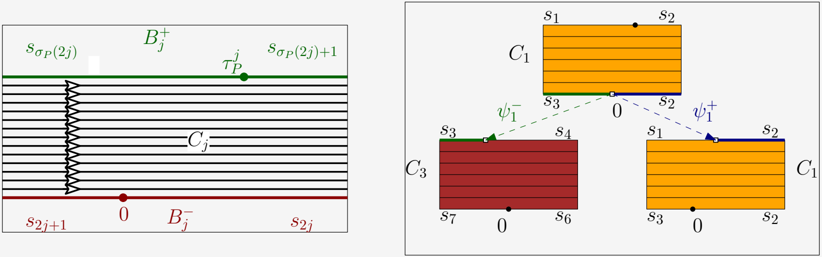
Trajectories in different coordinates, $\begin{cases} P(z) & \simeq z^{k+1} \\ t = T(z) & \simeq \frac{-1}{kz^k} \end{cases}$ if $r > r_\infty$

Determining the parameters: staying away



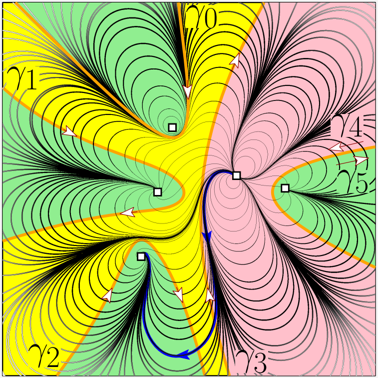
Safety annulus around the polar neighborhoods

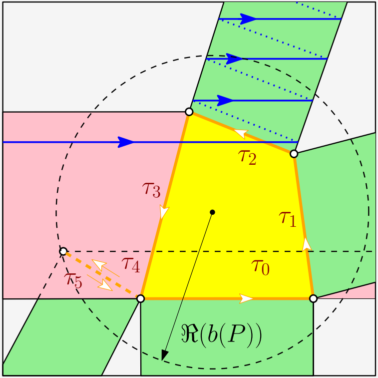
Determining the parameters: time-strips



Zone in t-space (left) and atlas of the time surface (right)

Determining the parameters: periodgon





Periodgon in t-space (A. Chéritat, M. Klimes, C. Rousseau)

Its edges belong to $\mathcal{L} = \mathbb{Z}/2\mathbb{Z}\left[\text{residues of } \frac{1}{P}\right]$

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Main Lemma

We can compute *a priori* bounds on:

for correct computation of Γ

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4. Taylor solver order and time-step

for correct computation of Γ

Applications

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1. Atlas of Riemann surfaces of certain logarithmic sums

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Further work

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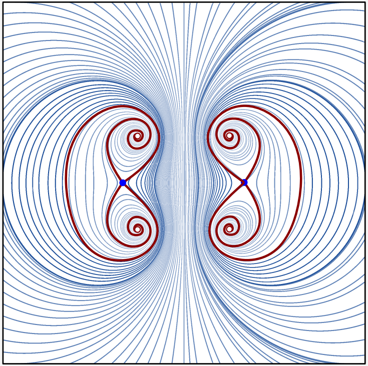

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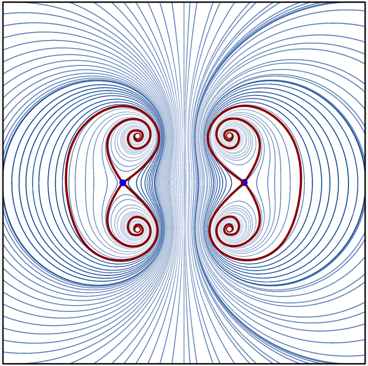

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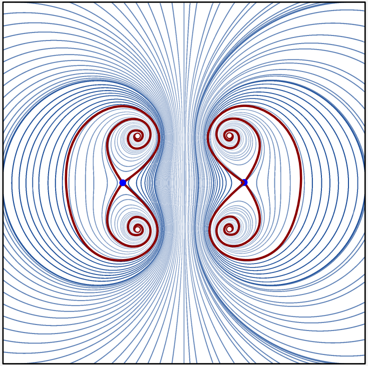

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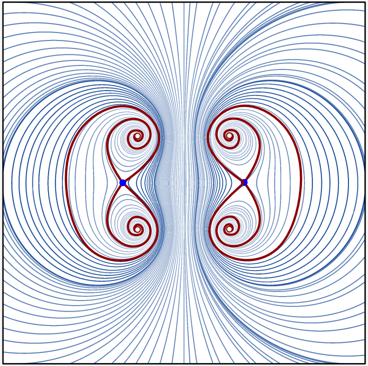

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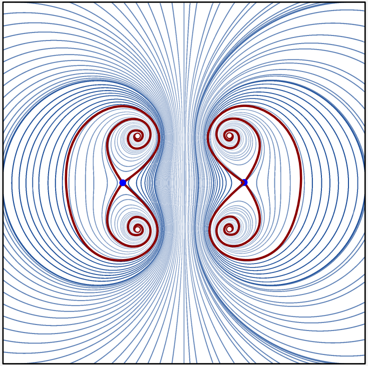

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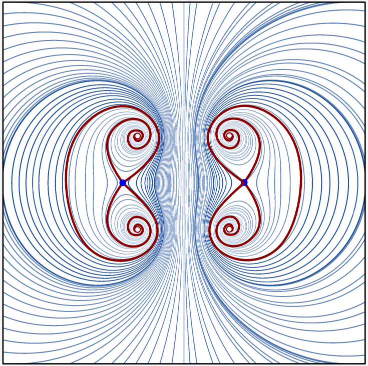
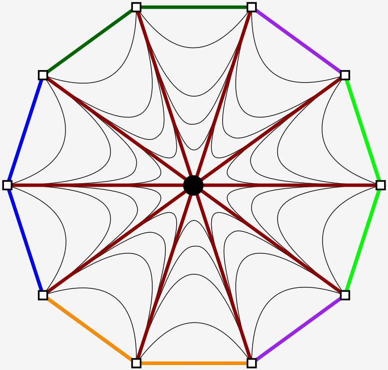
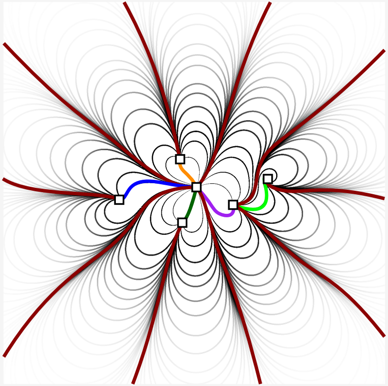


Conjecture

Any finite graph can be obtained as the separatrix graph of an algebraic flow

Thank you for your attention





(and enjoy a nice meal!)