A differential equation for decoding derivative codes a.k.a Hermite interpolation with outliers

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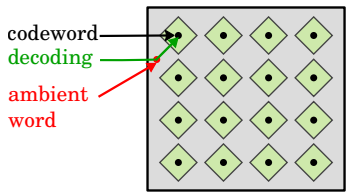
Coding Theory, Redundancy and Decoding

- A correcting code C encodes a message of length k into a codeword of length $n > k$

$$m \mapsto c \in C \subset \mathbb{F}_q^n$$

- Some noise modifies the codeword
- Decoding: given an "erroneous codeword", recover the message



- Message should be close (Hamming distance)
- d is the minimum distance
- less than $(d-1)/2$ errors $\implies$ unique decoding

combinatorics A good code has many codewords and a large minimum distance

algorithmics And an associated decoding algorithm

Difficult Tasks

- Difficult to build good codes
- Decoding is intractable

Reed-Solomon codes

Definition

Given $L = \{x_1, \ldots, x_n\} \subset \mathbb{F}_q$, and $k \leq n$
the Reed-Solomon codes RS$[L, k]$ is the code

$$\{(f(x_1), \ldots, f(x_n)) | \deg f(X) < k\} \subset \mathbb{F}_q^n$$

It has minimum distance $d = n - k + 1$ (*Singleton bound*: best distance achievable)

- $k = 1$ (constants) $\frac{n}{2}$ errors: majority vote
- $k = 2$ (lines) $\frac{n-1}{2}$ errors.
- $\frac{d-1}{2} = \frac{n-k}{2}$ errors can be corrected with a unique solution
- $f(X)$ coincides in $t = \frac{n+k}{2}$ positions

Lagrange interpolation with outliers

Problem

Given $x_1, \ldots, x_n$, $y = (y_1, \ldots, y_n) \in \mathbb{F}_q^n$, k and $t \leq n$, find

$$\mathcal{L}_y^t = \{f(X) \in \mathbb{F}[X], \deg f(X) < k, |C(f(X), y)| \geq t\}$$

- $t = n$: unique solution
- $t \geq \frac{n+k}{2}$: unique solution
- $t = k$: $\binom{n}{k}$ solutions
- $t < k$: problem is ill-founded
- $k < t < \frac{n+k}{2}$: "list-decoding"

Unique decoding: Berlekamp-Welch

Code: $\mathbb{F}$, $k \leq n$, $x_1, \ldots, x_n \in \mathbb{F}$, $x_i \neq x_j$

Decoding Radius: $t = \frac{n+k}{2}$

"Received word": $y = (y_1, \ldots, y_n) \in \mathbb{F}^n$

Bivariate Interpolation

Find $A(X, Y) = A_0(X) + A_1(X)Y$ such that

- $A(x_i, y_i) = 0$, $i = 1, \ldots, n$
- $\deg A_0(X) < t$
- $\deg A_1(X) < t - (k-1)$

Root-Finding

return $f(X) = A_1(X)/A_0(X)$ if

- $f(X)$ is a polynomial of degree $< k$
- $C(f(X), y) \geq t$

$A(X, Y) = A_0(X) + A_1(X)Y$ is such that

- $A(x_i, y_i) = 0,\ i = 1, \ldots, n$
- $\deg A_0(X) < t$
- $\deg A_1(X) < t - (k-1)$

Correctness

Let $f(X)$ with $C(f(X), y) \geq t$ and $\deg f(X) < k$

1. $\deg A_0(X) + A_1(X)f(X) < t$
2. $f(x_i) = y_i \implies A(x_i, f(x_i)) = 0$
3. $C(f(X), y) \geq t \implies A(X, f(X))$ has at least t roots $\implies A(X, f(X)) = 0$

Non triviality

1. linear system with n equations
2. number of unknowns $t + t - (k-1) = 2t + 1 - k \geq 2\frac{n+k}{2} + 1 - k = n + 1$
3. $\implies$ there exists a non zero $A(X, Y)$

Sudan

- Look for
$$A(X, Y) = A_0(X) + A_1(X)Y + \cdots + A_\ell(X)Y^\ell$$
with $\mathrm{wdeg}_{1,k-1} A(X, Y) < t$ such that
$$A(x_i, y_i) = 0, \quad i = 1, \ldots, n$$
- Finding $A(X, Y)$ is solving a homogoneous linear system
- Solve for $f(X)$ solution to $A(X, f(X)) = 0$

The existence of a non zero $A(X, Y)$ is ensured by

$$\text{number of unknowns} > \text{number of equations}$$

$$\implies t \geq \sqrt{2n(k-1)},\ |\mathcal{L}| \leq \sqrt{2/R}$$

Guruswami-Sudan

- Look for
$$A(X, Y) = A_0(X) + A_1(X)Y + \cdots + A_\ell(X)Y^\ell$$
with $\mathrm{wdeg}_{1,k-1} A(X, Y) < t$ such that
$$A(x_i, y_i) = 0, \quad i = 1, \ldots, n$$
with multiplicity s
- Solve for $f(X)$ solution to $A(X, f(X)) = 0$

The existence of a non zero $A(X, Y)$ is ensured by

$$\text{number of unknowns} > \text{number of equations}$$

$$\implies t \geq \sqrt{n(k-1)(1+\tfrac{1}{s})} \text{ and } |\mathcal{L}| \leq \sqrt{ns(s+1)/k}$$

"Capacity" of list decoding for general codes

With the notation $\tau = (n - t)/n$, and ℓ the maximum list size

Existence result (ELias 1957)

There exists a family of (τ, ℓ) list-decodable codes over $\mathbb{F}_q$ of rate

$$R \geq 1 - H_q(\tau) - 1/\ell$$

with

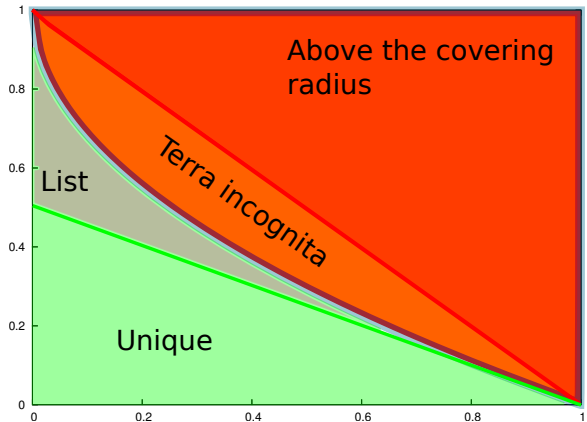
$$H_q(\tau) = -\tau \log_q(\tau) - (1 - \tau) \log_q(1 - \tau) + \tau \log_q(q - 1)$$

Using $\ell = O(\frac{1}{\varepsilon})$, and $q = \exp(O(1/\varepsilon))$ gives

$$R \geq 1 - \tau - \varepsilon$$

Same result for linear codes (Mosheiff, Resch, Ron-Zewi, Silas,Wootters 2019)

Case of Reed-Solomon codes



- Decoding of Reed-Solomon is NP-hard (Guruswami-Vardy 2004)
- Existence of Reed-Solomon codes with better list decoding radius (Goldberg-Shangguan-Tamo 2021, Ferber-Kwan-Sauermann 2020, Wooters 2014)

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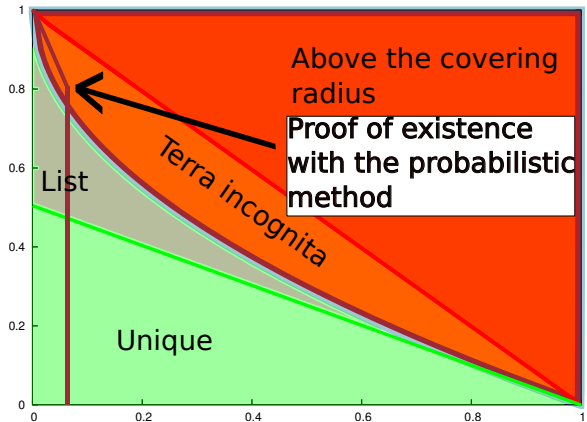

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Derivative codes (Guruswami-Wang 2011)

For $m \leq k \leq nm$, the derivative code $\mathrm{DRS}^m[n,k]$ has "codewords" given by

$$\begin{pmatrix} f(x_1) & \dots & f(x_n) \\ f^{(1)}(x_1) & \dots & f^{(1)}(x_n) \\ \vdots & & \vdots \\ f^{(m-1)}(x_1) & \dots & f^{(m-1)}(x_n) \end{pmatrix}$$

where $f(X)$ is a polynomial of degree less than k

Beware!

- "messages" have alphabet $\mathbb{F}_q$
- "codewords" have alphaet $\mathbb{F}_q^m$
- The rate is $R = (k-1)/(nm)$
- the code

$$C \subset \left(\mathbb{F}_q^m\right)^n$$

is $\mathbb{F}_q$-linear

Hermite interpolation with errors

Problem

Given $x_1, \ldots, x_n$, k and $t \leq n$,

$$y = \begin{pmatrix} x_1 & \ldots & x_n \\ y_{01} & \ldots & y_{0n} \\ y_{11} & \ldots & y_{1n} \\ \vdots & & \vdots \\ y_{m-1,1} & \ldots & y_{m-1,n} \end{pmatrix}$$

find

$$\mathcal{L}_y^t = \{f(X) \in \mathbb{F}[X], \deg f(X) < k, |C(f(X), y)| \geq t\}$$

where $C(f(X), y)$ is the set of agreeing columns

Linear-algebraic list decoding, first version

Given a received word y, find

$$A(X, Y_1, \ldots, Y_{m-1}) = B(X) + A_0(X)Y_0 + \cdots + A_{m-1}(X)Y_{m-1}$$

such that

1. $\deg B(X) < t$
2. $\deg A_i(X) < t - (k-1)$
3. for $i = 1, \ldots, n$

$$A(x_i, y_{0,i}, \ldots, y_{m-1,i}) = 0$$

(Correctness) For any $f \in \mathcal{L}$

$$A(X, f(X), \ldots, f^{(m-1)}(X)) = 0$$

Analysis

(Decoding radius) The "more unknowns than equations" analysis gives

$$\begin{aligned} t &> \frac{n}{m+1} + \frac{m}{m+1} \cdot (k-1) \\ \frac{t}{n} &> \frac{1}{m+1} + \frac{m(k-1)}{n(m+1)} \\ &\sim \frac{1}{m+1} + mR \end{aligned}$$

With $\tau = 1 - t/n$:

$$\tau \sim 1 - \varepsilon - \varepsilon^{-1}R$$

We aim at

$$\tau \sim 1 - \varepsilon - R$$

Linear-algebraic list decoding, improved version

Derivation

$$D(p(X)Y_i) = p'(X)Y_i + p(X)Y_{i+1},$$

Given a received word y, find **for an auxiliary** s

$$A(X, Y_1, \ldots, Y_s) = B(X) + A_0(X)Y_0 + \cdots + A_{s-1}(X)Y_{s-1}$$

such that

1. $\deg B(X) < (m - s + 1)t$
2. $\deg A_i(X) < (m - s + 1)t - (k - 1)$
3. for $i = 1, \ldots, n$

$$\left(D^{(j)}A\right)(x_i, y_{0,i}, \ldots, y_{m-1,i}) = 0$$

for $j = 0, \ldots, m - s$

Reaching capacity

(Decoding radius) The "more unknowns than equations" analysis gives

$$t > \frac{n}{s+1} + \frac{s}{s+1} \cdot \frac{k-1}{m-s+1}$$

Optimisation

- $s = \Omega(\varepsilon^{-1})$
- $m = \Omega(\varepsilon^{-2})$

$$\frac{t}{n} \geq R + \varepsilon$$

Finite dimensional space

We are left with solving **a linear differential equation with polynomial coefficients**

$$B(X) + A_0(X)f(X) + \cdots + A_{s-1}(X)f^{(s-1)}(X) = 0 \quad (*)$$

- **looking for polynomial solutions**
- **of given bounded degree $k-1$**
- **which have t initial values in the received word y**

Case of a finite field

Proposition

If $A_{s-1}(X) \neq 0$, *all solution to* $(*)$ *belong to an affine space of dimension* $\leq s-1$

Exhaustive search when $\mathbb{F} = \mathbb{F}_q$ is finite $\implies$ list-size at most $q^s = q^{1/\varepsilon}$

Problem (reformulated)

Given

$$y = \begin{pmatrix} x_1 & \dots & x_n \\ y_{01} & \cdots & y_{0n} \\ y_{11} & \cdots & y_{1n} \\ \vdots & & \vdots \\ y_{m-1,1} & \cdots & y_{m-1,n} \end{pmatrix}$$

and

$$B(X) + A_0(X)f(X) + \cdots + A_{s-1}(X)f^{(s-1)}(X) = 0 \quad (*)$$

Find all solutions $f(X)$ which coincide with y in a at least t positions

Issue

$f(X)$ may be such that $A_{s-1}(x_i) = 0$ for $i \in C(f(X), y)$

Case $s = m$

$$\deg A_{s-1}(X) < (m - s + 1)t - (k - 1) = t - (k - 1) < t$$

Proposition

If $A_{s-1}(X) \neq 0$, for any $f(X) \in \mathcal{L}$, there exists $i \in [n]$ such that

$$i \in C(f(X), y) \text{ and } A_{s-1}(x_i) \neq 0$$

Proof. Suppose there is an $f(X) \in \mathcal{L}$ such that this is not true.

$i \in C(f(X), y) \implies (X - x_i) | A_{s-1}(X)$

$$\prod_{i \in C(f(X), y)} (X - x_i) \mid A_{s-1}(X)$$

But $|C(f(X), y)| \geq t$

List size

Over any field, when $s = m$

- Any $f(X) \in \mathcal{L}$ has a correct index i such that $A_{s-1}(x_i) \neq 0$
- Any $f(X)$ can be found from its Taylor series at this position

$\implies |\boldsymbol{\mathcal{L}}| \leq n$

More generally for $s \neq m$

When $A_{s-1}(X)$ has less than t zeros in $\{x_1, \ldots, x_n\}$

- any $f(X) \in \mathcal{L}$ has a correct index i such that $A_{s-1}(x_i) \neq 0$

$\implies |\boldsymbol{\mathcal{L}}| \leq \boldsymbol{n}$

Case $s = 2$, m general

First order linear equation

$$B(X) + A_0(X)f(X) + A_1(X)f'(X) = 0$$

w.l.o.g $\gcd(A_1(X), A_0(X)) = 1$

Let

$$Z(X) = \prod_{i \in I}(X - x_i)$$

- If $\deg Z(X) < t$
  Any $f(X) \in \mathcal{L}$ is determined by Taylor expansion at some point $x_i \implies |\mathcal{L}| \leq n$
- If $\deg Z(X) \geq k$, then
  $$B(X) + A_0(X)f(X) = \text{mod } Z(X)$$
  which uniquely determines $f(X) \implies |\mathcal{L}| \leq 1$
- $t \leq \deg Z(X) < k$?

First order approximation of $f(X)$

For x_i such that $Z(x_i) = 0$

$$B(x_i) + A_0(x_i)f(x_i) = 0$$
$$B(x_i) + A_0(x_i)y_{0i} = 0$$

thus

$$A_0(x_i)(y_{0i} - f(x_i)) = 0$$

and $f(x_i) = y_{0i}$ since $A_0(X)$ is coprime to $Z(X)$.

$\implies$ $f(X)$ is known modulo $Z(X)$

Write

$$f(X) = f_0(X) + f_1(X)Z(X)$$

with $f_1(X)$ to be determined

If $Z(x_i) \neq 0$
from $f(X) = f_0(X) + f_1(X)Z(X)$

$$f_1(x_i) = \frac{f(x_i) - f_0(x_i)}{Z(x_i)},$$

If $Z(x_i) = 0$
from $f'(X) = f_0'(X) + f_1'(X)Z(X) + f_1(X)Z'(X)$

$$f_1(x_i) = \frac{f'(x_i) - f_0'(x_i)}{Z'(x_i)}$$

We do not know $f(x_i)$ neither $f'(x_i)$, only y_{i0} and y_{i1}

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Set the first row of $y^{(1)}$ as

$$y_{0i}^{(1)} = \frac{y_{0i} - f_0(x_i)}{Z(x_i)}$$

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Other rows of $y^{(1)}$: iterative procedure

Suppose $f^{(u)}(x_i)$ known for $u < j$

Leibniz' rule on $f(X) = f_0(X) + f_1(X)Z(X)$

$$f^{(j)}(x_i) = f_0^{(j)}(x_i) + f_1^{(j)}(x_i)Z(x_i) + \sum_{u=0}^{j-1} \binom{j}{u} f_1^{(u)}(x_i) Z^{(j-u)}(x_i)$$

For $i \notin I$, replacing with received symbols

$$\frac{y_{ji}^{(1)} = y_{ji} - f_0^{(j)}(x_i) - \sum_{u=0}^{j-1} \binom{j}{v} y_{u,i}^{(1)} Z^{(j-u)}(x_i)}{Z(x_i)}$$

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For $i \in I$, the derivative at order $j+1$

$$f^{(j+1)}(x_i) = f_0^{(j+1)}(x_i) + f_1^{(j+1)}(x_i)Z(x_i) + f_1^{(j)}(x_i)Z'(x_i) + \dots$$

$$y_{ji}^{(1)} = \frac{y_{j+1,i} - f_0^{(j+1)}(x_i) - \dots}{Z'(x_i)}$$

Other rows of $y^{(1)}$: iterative procedure

Suppose $f^{(u)}(x_i)$ known for $u < j$

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$$y_{ji}^{(1)} = \frac{y_{j+1,i} - f_0^{(j+1)}(x_i) - \dots}{Z'(x_i)}$$

$\rightarrow$ Iterative computation of the rows of $y^{(1)}$

New instance of the problem

A new received word $y^{(1)}$ and a new code $\mathrm{DRS}^{(m-1)}[n, k-t]$

A correct position of $f(X)$ w.r.t y is correct for $f_1(X)$ w.r.t $y^{(1)}$

The new t_1 for $m-1$ and $\deg f_1(X) < k-1-t$ is

$$t_1 = \frac{n}{3} + \frac{2}{3} \cdot \frac{k-1-t}{m-1-s+1} = \frac{n}{3} + \frac{2}{3}\frac{k-1-t}{m-2}$$

Then a miracle!

$$t_1 - t = \frac{2}{3} \cdot \frac{k-1-t(m-1)}{(m-2)(m-1)} \leq \mathbf{0}$$

$\rightarrow$ recursive algorithm with decreasing m
$\rightarrow$ until $m = 2$ $(= s = 2)$

Conclusion

- Hermite interpolation at order m with errors reduces to solving a differential equation
- The differential equation has order s, with auxiliary s
- Initial condition are "given"
- Linear list size for $s = 2$ and $s = m$
- No clue about the truth for other value of s