On linear recurrent equations having infinite sequences in the role of coefficients

S. Abramov, G. Pogudin

Throughout the talk, R stands for the ring of two-sided sequences having rational number terms with respect to termwise addition and multiplication. For a linear difference equation

$$L(y(n)) := a_r(n)y(n+r) + \cdots + a_1(n)y(n+1) + a_0(n)y(n) = 0 \quad (1)$$

with computable $a_r(n), \ldots, a_0(n) \in R$ as coefficients, we consider the $\mathbb{Q}$-vector space V_L of solutions in R.
In the sequel, "equation" is always understood as an equation of the form (1), and "solution" is a solution belonging to R.

When working with these sequences the way they are represented plays an important role. In this talk, an algorithmic approach is used: the sequence is defined by an algorithm (each sequence has its own) for calculating the value of an element by the index of this element.

More formally, we will call a two-sided sequence of rational numbers $\{v(n)\}_{n\in\mathbb{Z}}$ *computable* if it is given by an algorithm computing the value of $v(n)$ for any given $n \in \mathbb{Z}$.
(Other approaches are also possible.)

We will show that in the case when the coefficients are computable sequences, there is no a priori relation between the order of the equations and the dimension of the solution space (in the contrast to, say, the constant coefficient case).

The example below shows that if the coefficients of (1) are arbitrary sequences, the dimension of the solution space can be infinite.

Example 1

Let $L = \Sigma_{k=0}^{r} c_k(n)\sigma^k$, *where*

$$(r+1) \nmid (n+k) \Rightarrow c_k(n) = 0, \tag{2}$$

and a sequence $a(n)$ *be such that*

$$(r+1) \mid n \Rightarrow a(n) = 0, \tag{3}$$

then by (2) *we have* $c_k(n)a(n+k) = 0$ *for all such* n, *that* $(r+1) \nmid (n+k)$. *In turn,* (3) *gives* $c_k(n)a(n+k) = 0$ *for all* n *such that* $(r+1) \mid (n+k)$. *Thus*

$$L(a(n)) = \Sigma_{k=0}^{r} c_k(n)a(n+k) = 0$$

for all $n \in \mathbb{Z}$, *i.e.* V_L *contains all numeric sequences satisfying* (3). *As the values of* $a(n)$ *for* n *such that* $(r+1) \nmid n$ *can be chosen arbitrary, we conclude that* $\dim V_L = \infty$.

The general statement:

Proposition 1

For every integers $r, d \geqslant 0$ (the case $d = \infty$ included), there exists an equation of the form (1) of order r with both $a_0(n)$ and $a_r(n)$ not identical zero which has d-dimensional solution space.

Theorem 1

(i) There is no algorithm that tests the existence of a non-zero solution to a given equation.
(ii) There is no algorithm that computes the dimension of the solution space of a given equation.

Proof relies on undecidability (due to A.M. Turing) of:

Input computable sequence $n \to a_n$;

Output `True` if it contains zero, and `False` otherwise.

It is possible to put the question more narrowly with the hope for its algorithmic solvability. Let it be known in advance that the equation under consideration has some nonzero solutions. Moreover, a set to which the value of dimension belongs is known. Is it possible to determine the dimension of the solution space in this case?
It turns out that the hopes for algorithmic solvability are not justified in this case either.

Theorem 2

For any subset of $S \subseteq \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$ with $|S| > 1$, there is no algorithm that computes the dimension of the solution space d for a given difference equation, for which it is known in advance that $d \in S$.

But for particular types of coefficients the problem can be still decidable.

Proposition 2

There is an algorithm which takes as input periodic sequences $a_0(n), \ldots, a_r(n)$ and computes the dimension of the solutions space of the equation

$$a_r(n)y(n+r) + \ldots + a_1(n)y(n+1) + a_0(n)y(n) = 0$$

in the ring of two-sided sequences.

Idea: Reduce to a constant-coefficient *system* by splitting $y(n)$ into sequences

$$y(nH),\ y(nH+1),\ \ldots,\ y(nH+H-1),$$

where H is the LCM of the periods.

Open question: Can we extend these decidability further, for example, for the case when the coefficients are

- quasi-polynomials?
- solutions of constant-coefficient recurrences themselves?
  (similar to $\mathcal{C}^2$-finite sequences by Jimenez-Pastor, Nuspl, and Pillwein but without requiring the leading term be always nonzero)

Back to the case of the infinite dimension of the solution space.

A sequence $\{a(n)\}_{n\in\mathbb{Z}}$ will be called *lacunary* if the difference between the consequent elements of the support can be arbitrary large. In other words, if for every N there exists $i, j \in \operatorname{supp}(a_n)$ such that

- $j - i > N$;
- there is no $k \in \operatorname{supp}(a)$ with $i < k < j$.

Proposition 3

The following statements are equivalent:

1. *the dimension of the solution space of* (1) *is infinite;*
2. (1) *has a lacunary solution.*

Example 2

Go back to Example 1. Recall that we considered on operator $L = \Sigma_{k=0}^{r} c_k(n)\sigma^k$, *where*

$$(r+1) \nmid (n+k) \Rightarrow c_k(n) = 0,$$

and we have shown that $\dim V_L = \infty$.
Indeed, the equation $L(y) = 0$ *has lacunar solutions, e.g., the sequence*

$$l(n) = \begin{cases} 1 \text{ if } n = 2^m(r+1)+1 \text{ for some } m \in \mathbb{Z}, \\ 0 \text{ otherwise.} \end{cases}$$