Information geometry and the quantum effective action

(a talk on the Legendre transform with pictures)

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Based on work with

Elizabeth Alexander (formerly University of Nottingham), **Björn Garbrecht** (Technische Universität München), **Yannick Kluth** (University of Manchester), **Jordan Nursey** (formerly University of Nottingham), **Paul M. Saffin** (University of Nottingham), esp.:

- Garbrecht, PM, 1509.07847
- PM, Saffin, 1905.09674
- Alexander, PM, Nursey, Saffin 1907.06503
- PM, Saffin, 2107.12914
- PM, Saffin, 2206.08865
- Kluth, PM, Saffin, 2311.17199

Aim

A reasonably pedagogical exposition of how Pete thinks about the **quantum effective action**, covering:

- The definition of the quantum effective action(s) and the role(s) played by **external sources**.
- Demystifying the **convexity** of the quantum effective action (and a side remark on the Maxwell construction).
- An alternative way to think about the **functional renormalization group**.
- The connection between the quantum effective action and **information geometry** through **Hessian structures**.

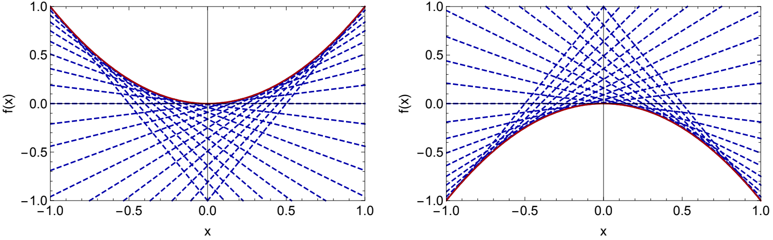
Note that I am not talking about effective actions in the sense of the operator product expansion.

What we want: A systematic way of dealing with non-perturbative phenomena in quantum field theory.

What we got: The quantum effective action (in various forms).
[R. Jackiw '74; J. M. Cornwall, R. Jackiw & E. Tomboulis '74; H. Verschelde & M. Coppens '92; M. E. Carrington '04; A. Pilaftsis & D. Teresi '13; J. Ellis, N. E. Mavromatos & D. P. Skliros '16]

What it's good for: Non-equilibrium phenomena
[J. S. Schwinger '61; G. Baym & L. P. Kadanoff '61; L. V. Keldysh '64; R. D. Jordan '86; E. Calzetta & B. L. Hu '88; J. P. Blaizot & E. Iancu '02; J. Berges '04; PM & A. Pilaftsis '13]
Symmetry breaking
[S. R. Coleman & E. J. Weinberg '73; J. Alexandre '12; J. Alexandre & A. Tsapalis '12]
Instantons/Solitons/Vacuum decay
[B. Garbrecht & PM '15; A. D. Plascencia & C. Tamarit '16]
Functional renormalisation group
[C. Wetterich '91 & '93; T. R. Morris '94; U. Ellwanger '94; M. Reuter '98; J. Berges, N. Tetradis & C. Wetterich '02; J. Pawlowski '07; H. Gies '12; O. J. Rosten '12]

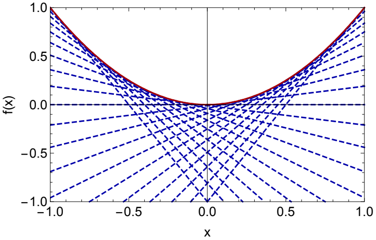
The Legendre transform

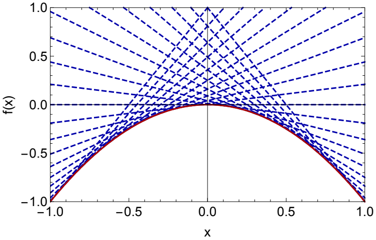


[M. Deserno '12 (unpublished); PM '16 (unpublished)]

- A function f that is strictly convex or concave on an interval $I \in \mathbb{R}$ has a second-derivative of definite sign.
- Its first derivative $f'(x)$ is monotonic, single-valued and invertible on I.
- We can express f as the set of ordered pairs $\{(x, f(x)) | x \in I, f(x) \in \mathbb{R}\}$ **or** the envelope of its tangents.
- The **Legendre transform** maps $\{(x, f(x))\}$ to $\{(x^*, f^*(x^*) = -^* f(x^*))\}$, specifying the gradients and intercepts of the tangents. ($* \equiv$ **convex conjugate**.)

The Legendre transform





[M. Deserno '12 (unpublished); PM '16 (unpublished)]

- Define $w(x) \equiv x^* x$.
- If $f(x)$ is convex (concave), $w(x) - f(x)$ will have a maximum (minimum):

$$f^*(x^*) \equiv \begin{cases} \min_{x \in I}\{f(x) - x^* x\}\ , & f(x) \text{ convex} \\ \max_{x \in I}\{f(x) - x^* x\}\ , & f(x) \text{ concave} \end{cases}$$

$${}^*f(x^*) \equiv \begin{cases} \max_{x \in I}\{x^* x - f(x)\}\ , & f(x) \text{ convex} \\ \min_{x \in I}\{x^* x - f(x)\}\ , & f(x) \text{ concave} \end{cases}$$

The 2PI effective action

For illustration, let's work with a **zero-dimensional** Euclidean quantum field theory:
[PM & P. M. Saffin '19]

$$S(\Phi) = \frac{m^2}{2}\Phi^2 + \frac{\lambda}{4!}\Phi^4$$

and write down the **partition function**

$$Z(J,K) = \mathcal{N}\int_{-\infty}^{+\infty} \mathrm{d}\Phi \, \exp\left[-\frac{1}{\hbar}\left(S(\Phi) - J\Phi - \frac{1}{2}K\Phi^2\right)\right]$$

in the presence of **external sources** J and K.

The **Schwinger function**

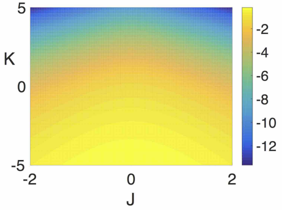
$$W(J,K) \equiv -\hbar \ln Z(J,K)$$

is **concave**.

Its gradients with respect to $-J$ and $-K/2$ are $\langle\Phi\rangle_{J,K}$ and $\langle\Phi^2\rangle_{J,K}$, respectively, i.e., the **one-** and **two-point correlation functions**.

The 2PI effective action

$W(J,K) \equiv -\hbar \ln Z(J,K)$ for $m^2 = -1$ and $\lambda = 6$, i.e. non-convex classical potential:



[PM & P. M. Saffin '19]

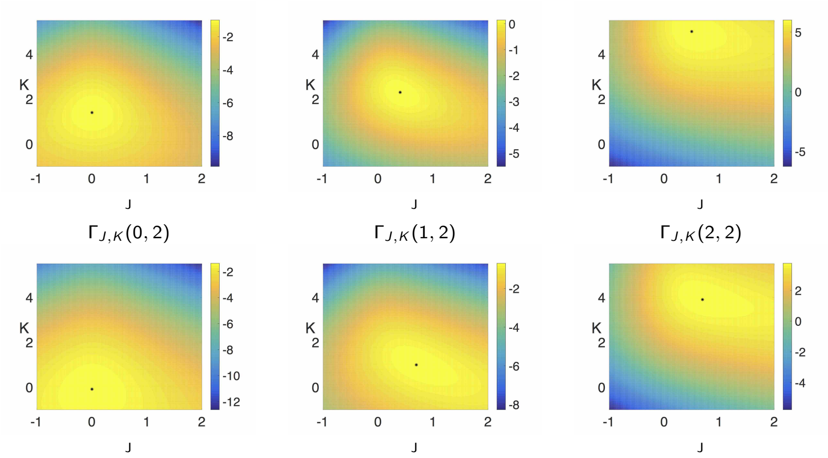
Introduce a function

$$\Gamma_{J,K}(\phi,\Delta) \equiv W(J,K) + J\phi + \frac{1}{2}K[\phi^2 + \hbar\Delta]$$

ϕ and Δ determine the value of the maximum of this function and its position in the (J,K) plane.

Δ is the connected two-point correlation function.

The 2PI effective action



$\Gamma_{J,K}(0,2)$ $\Gamma_{J,K}(1,2)$ $\Gamma_{J,K}(2,2)$

$\Gamma_{J,K}(0,1)$ $\Gamma_{J,K}(1,1)$ $\Gamma_{J,K}(2,1)$

[PM & P. M. Saffin '19]

The 2PI effective action

The **(double) Legendre transform**

$$\Gamma(\phi, \Delta) = \max_{J,K} \Gamma_{J,K}(\phi, \Delta)$$

corresponds to the value of these maxima as a function of ϕ and Δ.

The locations of the maxima correspond to **extremal sources** $\mathcal{J}$ and $\mathcal{K}$, defined by

$$\left.\frac{\partial \Gamma_{J,K}(\phi, \Delta)}{\partial J}\right|_{J=\mathcal{J}, K=\mathcal{K}} = 0 \qquad \left.\frac{\partial \Gamma_{J,K}(\phi, \Delta)}{\partial K}\right|_{J=\mathcal{J}, K=\mathcal{K}} = 0$$

The extremisation yields

$$\Gamma(\phi, \Delta) = W(\mathcal{J}, \mathcal{K}) + \mathcal{J}\phi + \frac{1}{2}\mathcal{K}[\phi^2 + \hbar\Delta]$$

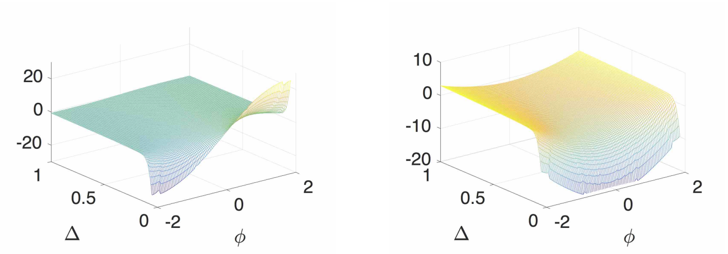
with

$$\phi = \hbar \frac{\partial}{\partial J} \ln Z(J, K)\Big|_{J=\mathcal{J}, K=\mathcal{K}} \qquad \hbar\Delta = 2\hbar \frac{\partial}{\partial K} \ln Z(J, K)\Big|_{J=\mathcal{J}, K=\mathcal{K}} - \phi^2$$

The 2PI effective action

Importantly, since the location of the maxima of $\Gamma_{J,K}(\phi, \Delta)$ depend on ϕ and Δ

$$\mathcal{J} \equiv \mathcal{J}(\phi, \Delta) \qquad\qquad \mathcal{K} \equiv \mathcal{K}(\phi, \Delta)$$



[PM & P. M. Saffin '19]

In corollary,

$$\phi \equiv \phi(\mathcal{J}, \mathcal{K}) \qquad\qquad \Delta \equiv \Delta(\mathcal{J}, \mathcal{K})$$

and they are related to the tangents to the Schwinger function.

The 2PI effective action

The extremal sources $\mathcal{J}$ and $\mathcal{K}$ are related to the tangents to $\Gamma(\phi, \Delta)$:

$$\frac{\partial \Gamma(\phi, \Delta)}{\partial \phi} = \mathcal{J}(\phi, \Delta) + \mathcal{K}(\phi, \Delta)\phi \qquad \frac{\partial \Gamma(\phi, \Delta)}{\partial \Delta} = \frac{\hbar}{2}\mathcal{K}(\phi, \Delta)$$

The right-hand sides are source terms, and the gradients of $\Gamma(\phi, \Delta)$ are the equations of motion for the one- and two-point functions ϕ and Δ.

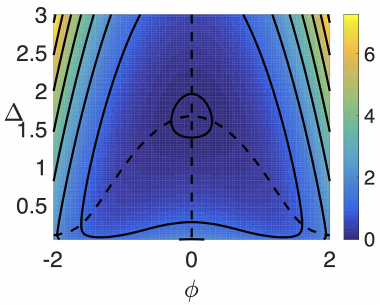
Since these are correct to all orders in $\hbar$, we are justified in calling $\Gamma(\phi, \Delta)$ a **quantum effective action**.

Why "2PI"?

The 2PI effective action: convexity

By definition of the Legendre transform, $\Gamma(\phi, \Delta)$ should be convex.

But for the non-convex classical potential with $m^2 = -2$ and $\lambda = 6$, we find



[PM & P. M. Saffin '19]

This doesn't look convex; **panic!?**

The 2PI effective action: convexity

Convenient to work with the variables $\phi' \equiv \phi$ and $\Delta' \equiv \phi^2 + \hbar\Delta$ and the rescaled sources $\mathcal{J}' \equiv \mathcal{J}$ and $\mathcal{K}' \equiv \mathcal{K}/2$:
[PM & P. M. Saffin '19]

$$\Gamma(\phi, \Delta) = W(\mathcal{J}, \mathcal{K}) + \mathcal{J}'\phi' + \mathcal{K}'\Delta'$$

$$\frac{\partial \Gamma(\phi, \Delta)}{\partial \phi'} = \mathcal{J}' \qquad \frac{\partial \Gamma(\phi, \Delta)}{\partial \Delta'} = \mathcal{K}'$$

$$\phi' = -\frac{\partial W(\mathcal{J}, \mathcal{K})}{\partial \mathcal{J}'} \qquad \Delta' = -\frac{\partial W(\mathcal{J}, \mathcal{K})}{\partial \mathcal{K}'}$$

We consider the product
[cf. the 1PI case in J. Alexandre & A. Tsapalis '12]

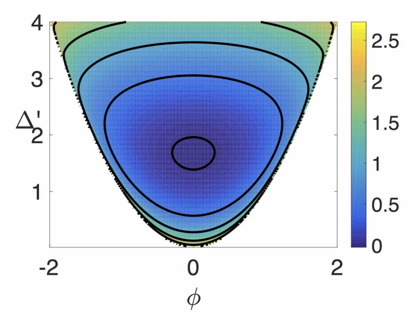
$$-\mathrm{Hess}(\Gamma)(\phi', \Delta') \cdot \mathrm{Hess}(W)(\mathcal{J}', \mathcal{K}') = \mathbb{I}$$

$-\mathrm{Hess}(W)(\mathcal{J}', \mathcal{K}')$ is a covariance matrix, i.e., positive definite.

Thus, $\mathrm{Hess}(\Gamma)(\phi', \Delta')$ is positive definite, and $\Gamma(\phi, \Delta)$ is therefore convex, but with respect to ϕ and Δ'.

The 2PI effective action: convexity

Plotting $\Gamma(\phi, \Delta)$ as a function of ϕ and $\Delta' = \phi^2 + \hbar\Delta$, we see that it is convex:



[PM & P. M. Saffin '19]

Note that this is for a non-convex classical potential, with $m^2 = -2$ and $\lambda = 6$.

The 2PI effective action: single saddle point

Stationarity/saddle-point condition:

$$\left.\frac{\partial S(\Phi)}{\partial \Phi}\right|_{\Phi=\varphi} - \mathcal{J}(\phi,\Delta) - \mathcal{K}(\phi,\Delta)\varphi = 0$$

Define the two-point function

$$\mathcal{G} = \left[G^{-1}(\varphi) - \mathcal{K}(\phi,\Delta)\right]^{-1} \qquad G^{-1}(\varphi) = \left.\frac{\partial^2 S(\Phi)}{\partial \Phi^2}\right|_{\Phi=\varphi} = m^2 + \frac{\lambda}{2}\varphi^2$$

and expand $\Phi = \varphi + \sqrt{\hbar}\hat{\Phi}$ to obtain

$$\Gamma(\phi,\Delta) = S(\varphi) + \hbar\Gamma_1(\varphi,\mathcal{G}) + \hbar^2\Gamma_2(\varphi,\mathcal{G}) + \hbar^2\Gamma_{1\mathrm{PR}}(\varphi,\mathcal{G}) + \mathcal{J}\,(\phi-\varphi) + \frac{1}{2}\mathcal{K}(\phi^2 - \varphi^2 + \hbar\Delta - \hbar\mathcal{G})$$

$$\Gamma_1(\varphi,\mathcal{G}) = \frac{1}{2}\left[\ln\left(\mathcal{G}^{-1}G(0)\right) + G^{-1}\mathcal{G} - 1\right] \qquad \Gamma_2(\varphi,\mathcal{G}) = \frac{1}{8}\lambda\mathcal{G}^2 - \frac{1}{12}\lambda^2\varphi^2\mathcal{G}^3 \qquad \Gamma_{1\mathrm{PR}}(\varphi,\mathcal{G}) = -\frac{1}{8}\lambda^2\varphi^2\mathcal{G}^3$$

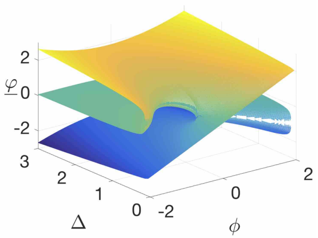
But $\varphi \equiv \varphi(\phi,\Delta)$, and we can expand the right-hand side around $\varphi - \phi = \mathcal{O}(\hbar)$:

$$\Gamma(\phi,\Delta) = S(\phi) + \hbar\Gamma_1(\phi,\Delta) + \hbar^2\Gamma_2(\phi,\Delta)$$

The 2PI effective action: multiple saddle points and the Maxwell construction

More generally, we have a set of saddle points $\{\varphi_i\} \equiv \{\varphi_i\}(\phi, \Delta)$, where type and number depend on (ϕ, Δ).

For $m^2 = -1$ and $\lambda = 6$, we have 1 to 3 saddles, depending on (ϕ, Δ):



[PM & P. M. Saffin '19]

Don't mix up your ϕ's and φ's!

If the saddle points are "reasonably well separated"

$$Z(\mathcal{J}, \mathcal{K}) \approx \sum_i Z_i(\mathcal{J}, \mathcal{K})$$

The 2PI effective action: method of external sources [B. Garbrecht & PM '16]

Folklore: The physical limit corresponds to vanishing external sources.

Reality: Setting $\mathcal{J}(\phi,\Delta)$ and $\mathcal{K}(\phi,\Delta)$ to zero constrains $\phi \equiv \phi(\mathcal{J},\mathcal{K})$ and $\Delta \equiv \Delta(\mathcal{J},\mathcal{K})$, yielding the CJT effective action with an important difference:
[J. M. Cornwall, R. Jackiw & E. Tomboulis '74]

We can choose the sources $\mathcal{J}(\phi,\Delta)$ and $\mathcal{K}(\phi,\Delta)$, such that the saddle point of the partition function coincides with the quantum trajectory by demanding

$$\left.\frac{\delta S[\Phi]}{\delta\Phi}\right|_{\Phi=\varphi} - \mathcal{J}(\phi,\Delta) - \mathcal{K}(\phi,\Delta)\varphi = \left.\frac{\delta\Gamma[\phi,\Delta]}{\delta\phi}\right|_{\phi=\varphi,\Delta=\mathcal{G}} = 0$$

This requires
[B. Garbrecht & PM '16; PM & P. M. Saffin '19]

$$\mathcal{J}(\varphi,\mathcal{G}) + \mathcal{K}(\varphi,\mathcal{G})\varphi = 0$$

This is important when the quantum trajectory is non-perturbatively far away from the classical trajectory, e.g., as in tunnelling problems in radiatively generated potentials.
[E. J. Weinberg '93; B. Garbrecht & PM '15 & '16]

The 2PI effective action: method of external sources

But we can do more:
[B. Garbrecht & PM '16]

- Setting $\mathcal{J}$ to zero and choosing $\mathcal{K}$ to be local yields the **2PPI effective action** of Verschelde and Coppens.
  [H. Verschelde & M. Coppens '92]

- Constraining the sources by, e.g., the Ward identities, yields results in the spirit of the **symmetry-improved effective action** of Pilaftsis and Teresi.
  [A. Pilaftsis & D. Teresi '13]

- Choosing $\mathcal{K}$ to be the regulator of the **renormalisation group evolution** yields ...
  [E. Alexander, PM, J. Nursey & P. M. Saffin '19]

The regulator-sourced 2PI effective action and exact flow equations

Starting from the 2PI effective action,
[E. Alexander, PM, J. Nursey & P. M. Saffin '19]

$$\partial_k \Gamma^{2\mathrm{PI}}[\phi,\Delta] = \frac{\delta \Gamma^{2\mathrm{PI}}[\phi,\Delta]}{\delta \phi_x}\,\partial_k \phi_x + \frac{\delta \Gamma^{2\mathrm{PI}}[\phi,\Delta]}{\delta \Delta_{xy}}\,\partial_k \Delta_{xy}$$

$$\partial_k \phi_x = -\partial_k \frac{\delta W[\mathcal{J},\mathcal{K}]}{\delta \mathcal{J}_x} = 0$$

$$\partial_k \Gamma^{2\mathrm{PI}}[\phi,\Delta] = \frac{\hbar}{2}\,\mathcal{K}_{xy}[\phi,\Delta]\partial_k \Delta_{xy}$$

Now choose $\mathcal{K}_{xy}[\phi,\Delta] = \mathcal{R}_{k,xy}$ to be the inverse Fourier transform of the **regulator**:

$$\partial_k \Gamma^{2\mathrm{PI}}[\phi,\Delta] = \frac{\hbar}{2}\,\mathrm{Tr}\,(\mathcal{R}_k * \partial_k \Delta)$$

$$\boxed{\partial_k \Gamma^{2\mathrm{PI}}[\phi,\Delta_k] = +\frac{\hbar}{2}\mathrm{STr}\,(\mathcal{R}_k \partial_k \Delta_k) \qquad \text{versus} \qquad \partial_k \Gamma^{1\mathrm{PI}}_{\mathrm{av}}[\phi,\mathcal{R}_k] = -\frac{\hbar}{2}\mathrm{STr}\,(\Delta_k \partial_k \mathcal{R}_k)}$$

Right-hand expression is the Wetterich-Morris-Ellwanger equation.
[C. Wetterich '91 & '93; T. R. Morris '94; U. Ellwanger '94; M. Reuter '98]

Closure and vertex functions [PM & P. M. Saffin '21 & '22; cf. A. N. Vasil'ev & A. K. Kazanskii '73]

It follows from the **convexity** of the 2PI effective action that
[J. M. Cornwall, R. Jackiw & E. Tomboulis '74 (footnote); cf. J. Berges, S. Borsányi, U. Reinosa & J. Serreau '05]

$$\Delta^{-1} = \frac{\delta^2\Gamma^{\mathrm{2PI}}}{\delta\phi^2} - \frac{2}{\hbar}\frac{\delta\Gamma^{\mathrm{2PI}}}{\delta\Delta} - \left(\frac{\delta^2\Gamma^{\mathrm{2PI}}}{\delta\phi\delta\Delta}\right)\left(\frac{\delta^2\Gamma^{\mathrm{2PI}}}{\delta\Delta^2}\right)^{-1}\left(\frac{\delta^2\Gamma^{\mathrm{2PI}}}{\delta\Delta\delta\phi}\right)$$

1PI effective action: n-point vertex functions are easily extracted by functional differentiation with respect to the one-point function:

$$\mathit{\Gamma}^{(n>2)}[\phi] = \frac{\delta^n\Gamma^{\mathrm{1PI}}}{\delta\phi^n} = -\Delta^{-n}\left\{\Delta\frac{\delta}{\delta\phi}\right\}^{n-2}\Delta$$

2PI effective action: the situation is more complicated. Starting from the connected function

$$\langle\phi^n\rangle_{\mathrm{c}} = \left(\hbar\frac{\delta}{\delta\mathcal{J}}\right)^{n-2}\left(-\hbar\frac{\delta^2 W}{\delta\mathcal{J}^2}\right) = \left(\hbar\frac{\delta}{\delta\mathcal{J}}\right)^{n-2}(\hbar\Delta)$$

the chain rule for derivatives w.r.t. $\mathcal{J} = \mathcal{J}[\phi, \Delta]$ gives

$$\mathit{\Gamma}^{(n>2)}[\phi,\Delta] = -\Delta^{-n}\left\{\Delta\left[\frac{\delta}{\delta\phi} - \frac{\delta^2\Gamma^{\mathrm{2PI}}}{\delta\phi\delta\Delta}\left(\frac{\delta^2\Gamma^{\mathrm{2PI}}}{\delta\Delta^2}\right)^{-1}\frac{\delta}{\delta\Delta}\right]\right\}^{n-2}\Delta$$

Hessian manifolds (stolen from information geometry) [Y. Kluth, PM & P. M. Saffin '23]

Given a functional $\overline{\Gamma}$ that depends on sources $\mathcal{Q}_A = (J_a, K'_\alpha)$, we can define a **metric**

$$g^{AB} = \bar{D}^A \bar{D}^B \overline{\Gamma} = \frac{\delta^2 \overline{\Gamma}}{\delta \mathcal{Q}_A \delta \mathcal{Q}_B}$$

where $\bar{D}$ is the **affine connection** on the **configuration space**.

Define a **dual connection** D whose affine coordinate frame is given by

$$\mathcal{P}^A = \frac{\delta \overline{\Gamma}}{\delta \mathcal{Q}_A}$$

where $\mathcal{P}^A = (\phi^a, \Delta'^\alpha)$ are the variables conjugate to the sources, i.e., the correlation functions.

The **dual potential** is related to the original functional $\overline{\Gamma}$ by the **Legendre transform**

$$\Gamma[\mathcal{P}] = -\overline{\Gamma} + \mathcal{Q}_A \mathcal{P}^A$$

and the **inverse metric** is

$$g_{AB} = \frac{\delta^2 \Gamma}{\delta \mathcal{P}^A \delta \mathcal{P}^B} \tag{1}$$

Hessian manifolds continued [Y. Kluth, PM & P. M. Saffin '23]

Associating $\overline{\Gamma} = -W$ and $\Gamma = \Gamma^{2\mathrm{PI}}$, the inverse relation for the Hessian metric

$$g_{AB}g^{BC} = \delta_A^C$$

is just the convexity condition of the 2PI effective action.

The n-point connected function is just

$$\langle \phi^{a_1} \dots \phi^{a_n} \rangle_{\mathrm{c}} = \bar{D}^{a_1} \dots \bar{D}^{a_n} W \qquad \text{with} \qquad \bar{D}^a = \frac{\delta}{\delta \mathcal{J}_a}$$

The chain rule to map between derivatives w.r.t. to sources and n-point correlation functions is

$$\bar{D}^a = g^{aB} D_B = \Delta^{ab} \left[\frac{\delta}{\delta \phi^b} - \frac{\delta^2 \Gamma}{\delta \phi^b \delta \Delta} \left(\frac{\delta^2 \Gamma}{\delta \Delta^\delta \delta \Delta^\epsilon} \right)^{-1} \frac{\delta}{\delta \Delta^\epsilon} \right]$$

This is precisely the operator that we found before for extracting the vertex functions!

Back to the functional renormalization group [Y. Kluth, PM & P. M. Saffin '23]

Suppose we take K_α to be a regulator of the functional renormalization group, and require the regulator to be constant along an RG trajectory specified by some RG time t then

$$\frac{\mathrm{d}K_\alpha}{\mathrm{d}t} = \text{const} \qquad \Rightarrow \qquad g_{\alpha B}\left[\frac{\mathrm{d}\mathcal{P}^B}{\mathrm{d}t^2} + 2\Gamma^B_{CD}\frac{\mathrm{d}\mathcal{P}^C}{\mathrm{d}t}\frac{\mathrm{d}\mathcal{P}^D}{\mathrm{d}t}\right] = 0$$

Assuming that the square bracket vanishes, this is just the **geodesic equation** for the affine connection $\bar{D}$

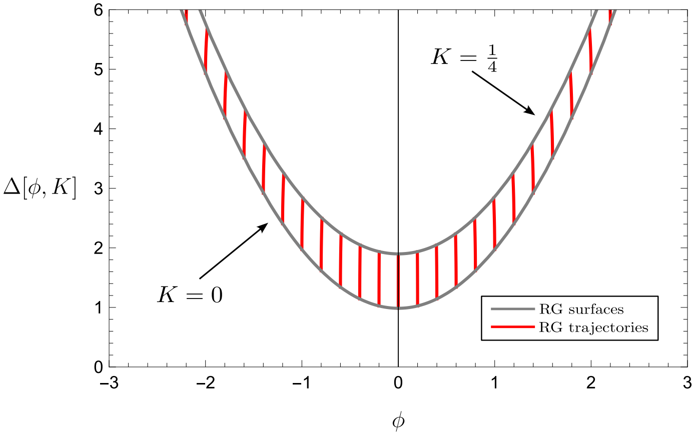
$$\frac{\mathrm{d}\mathcal{P}^B}{\mathrm{d}t^2} + 2\Gamma^B_{CD}\frac{\mathrm{d}\mathcal{P}^C}{\mathrm{d}t}\frac{\mathrm{d}\mathcal{P}^D}{\mathrm{d}t} = 0$$

where Γ^B_{CD} is the **Levi-Civita Christoffel symbol**.

We can now generate RG flows by solving the geodesic equation . . .

RG flows as geodesics [Y. Kluth, PM & P. M. Saffin '23]

For example, for the zero-dimensional ϕ^4 theory, we can evolve surfaces of constant RG time.



Concluding remarks

- *It pays to be pedantic when it comes to the quantum effective action.*

- We can exploit the sources to:
  - Organise the loop expansion and its (partial) resummation.
  - Improve symmetry properties.
  - Map between different realisations of the effective action.
  - Study the exact RG flow.

- We can obtain a geometric interpretation of the nPI quantum effective actions through Hessian structures:
  - Understand the role of the affine connection in the extraction of n-point correlation/vertex functions.
  - Recast RG flows in terms of geodesic flows on the configuration space.
  - And more to come . . .

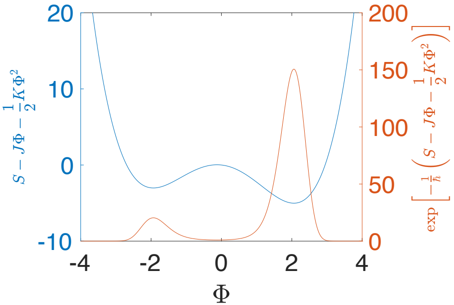
References I (in order of appearance)

R. Jackiw '74: https://inspirehep.net/literature/95439
J. M. Cornwall, R. Jackiw & E. Tomboulis '74: https://inspirehep.net/literature/1295
H. Verschelde & M. Coppens '92: https://inspirehep.net/literature/342538
M. E. Carrington '04: https://inspirehep.net/literature/642863
A. Pilaftsis & D. Teresi '13: https://inspirehep.net/literature/1233385
J. Ellis, N. E. Mavromatos & D. P. Skliros '16: https://inspirehep.net/literature/1408734
J. S. Schwinger '61: https://inspirehep.net/literature/9385
G. Baym & L. P. Kadanoff '61: https://inspirehep.net/literature/46492
L. V. Keldysh '64: https://inspirehep.net/literature/3567
R. D. Jordan '86: https://inspirehep.net/literature/228851
E. Calzetta & B. L. Hu '88: https://inspirehep.net/literature/235962
J. P. Blaizot & E. Iancu '02: https://inspirehep.net/literature/552014
J. Berges '04: https://inspirehep.net/literature/659724
PM & A. Pilaftsis '13: https://inspirehep.net/literature/1202693
S. R. Coleman & E. J. Weinberg '73: https://inspirehep.net/literature/81406
J. Alexandre '12: https://inspirehep.net/literature/1113766
J. Alexandre & A. Tsapalis '12: https://inspirehep.net/literature/1198187
B. Garbrecht & PM '16: https://inspirehep.net/literature/1394876
A. D. Plascencia & C. Tamarit '16: https://inspirehep.net/literature/1400843
C. Wetterich '91: https://inspirehep.net/literature/284270

References II (in order of appearance)

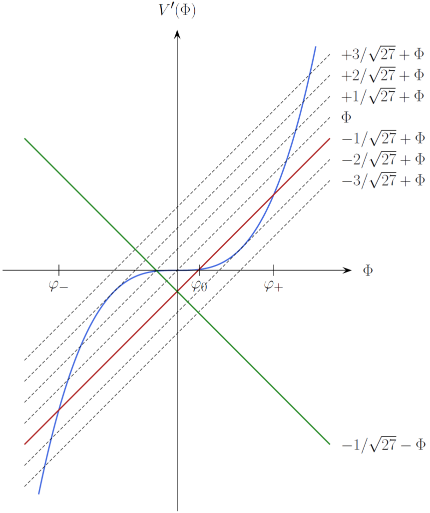
C. Wetterich '93: https://inspirehep.net/literature/33814
T. R. Morris '94: https://inspirehep.net/literature/358382
U. Ellwanger '94: https://inspirehep.net/literature/35872
M. Reuter '98: https://inspirehep.net/literature/418310
J. Berges, N. Tetradis & C. Wetterich '02: https://inspirehep.net/literature/527207
J. Pawlowski '07: https://inspirehep.net/literature/700965
H. Gies '12: https://inspirehep.net/literature/731416
O. J. Rosten '12: https://inspirehep.net/literature/848078
M. Deserno '12: https://www.andrew.cmu.edu/course/33-765/pdf/Legendre.pdf
PM & P. M. Saffin '19: https://inspirehep.net/literature/1736514
E. J. Weinberg '93: https://inspirehep.net/literature/33690
B. Garbrecht & PM '15: https://inspirehep.net/literature/1395262
E. Alexander, PM, J. Nursey & P. M. Saffin '19: https://inspirehep.net/literature/1744092
PM & P. M. Saffin '21: https://inspirehep.net/literature/1893652
PM & P. M. Saffin '22: https://inspirehep.net/literature/2097849
A. N. Vasil'ev & A. K. Kazanskii '73: https://inspirehep.net/literature/86611 & https://inspirehep.net/literature/86603
J. Berges, S. Borsányi, U. Reinosa & J. Serreau '05: https://inspirehep.net/literature/678968
Y. Kluth, PM & P. M. Saffin '23: https://inspirehep.net/literature/2727910
R. J. Rivers '84: https://inspirehep.net/literature/13759

Backup: middle saddle



[PM & P. M. Saffin '19]

Backup: saddles with $\mathcal{K} \neq 0$



[PM & P. M. Saffin '19]

The values on the right-hand side are $\mathcal{J} + \mathcal{K}\Phi$, with $|\mathcal{K}| = 1$.

The 2PI effective action: multiple saddle points and the Maxwell construction

Suppose there are two contributing saddle points, $\varphi_\pm(\phi,\Delta)=\tilde{\varphi}_\pm+\hbar\delta\varphi_\pm(\phi,\Delta)$:
[PM & P. M. Saffin '19]

$$\Gamma(\phi,\Delta)=\frac{(\tilde{\varphi}_+-\phi)\tilde{\Gamma}_-+(\phi-\tilde{\varphi}_-)\tilde{\Gamma}_+}{\tilde{\varphi}_+-\tilde{\varphi}_-}-\frac{1}{2}\mathcal{K}(\tilde{\varphi}_+-\phi)(\phi-\tilde{\varphi}_-)$$
$$-\hbar\ln\left[\left(\frac{\phi-\tilde{\varphi}_-}{\tilde{\varphi}_+-\phi}\right)^{\frac{\tilde{\varphi}_+-\phi}{\tilde{\varphi}_+-\tilde{\varphi}_-}}+\left(\frac{\tilde{\varphi}_+-\phi}{\phi-\tilde{\varphi}_-}\right)^{\frac{\phi-\tilde{\varphi}_-}{\tilde{\varphi}_+-\tilde{\varphi}_-}}\right]+\frac{\hbar}{2}\mathcal{K}\Delta$$

In the limit $\mathcal{K}\to 0$, we recover the 1PI result:
[J. Alexandre & A. Tsapalis '12]

$$\Gamma(\phi)=\frac{(\tilde{\varphi}_+-\phi)\tilde{\Gamma}_-+(\phi-\tilde{\varphi}_-)\tilde{\Gamma}_+}{\tilde{\varphi}_+-\tilde{\varphi}_-}-\hbar\ln\left[\left(\frac{\phi-\tilde{\varphi}_-}{\tilde{\varphi}_+-\phi}\right)^{\frac{\tilde{\varphi}_+-\phi}{\tilde{\varphi}_+-\tilde{\varphi}_-}}+\left(\frac{\tilde{\varphi}_+-\phi}{\phi-\tilde{\varphi}_-}\right)^{\frac{\phi-\tilde{\varphi}_-}{\tilde{\varphi}_+-\tilde{\varphi}_-}}\right]$$

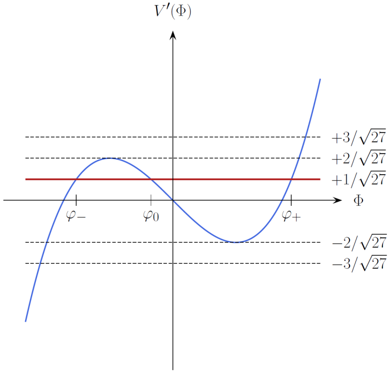
giving the **Maxwell construction** in the limit $\hbar\to 0$:

$$\Gamma(\phi)=\frac{(\tilde{\varphi}_+-\phi)\tilde{V}_-+(\phi-\tilde{\varphi}_-)\tilde{V}_+}{\tilde{\varphi}_+-\tilde{\varphi}_-}$$

For how this works in higher dimensions, see [R. J. Rivers '84; PM & P. M. Saffin '19].

The 2PI effective action: multiple saddle points and the Maxwell construction

- $\Gamma(\phi)$ is monotonic only for $\tilde{\varphi}_- < \phi < \tilde{\varphi}_+$.
- We hit branch points at $\phi = \tilde{\varphi}_\pm$ when we no longer have multiple saddles.
- For $\phi > \tilde{\varphi}_+$ or $\phi < \tilde{\varphi}_-$, $\Gamma(\phi) \to V(\phi)$.



[PM & P. M. Saffin '19]

The values on the right-hand side are $\mathcal{J} \equiv \mathcal{J}[\phi]$, with $\mathcal{K} = 0$.

Interlude: the 1PI average effective action

The average 1PI effective action is defined as
[C. Wetterich '91]

$$\Gamma^{\text{1PI}}_{\text{av}}[\phi,\mathcal{R}_k] = W[\mathcal{J},\mathcal{R}_k] + \mathcal{J}_x\phi_x + \frac{1}{2}\phi_x\mathcal{R}_{k,xy}\phi_y \qquad \phi_x = -\frac{\delta W[\mathcal{J},\mathcal{R}_k]}{\delta \mathcal{J}_x}$$

where $\mathcal{R}_{k,xy}$ is the inverse FT of the **regulator** (eliminates fluctuations with $q^2 < k^2$).

Requiring

$$\partial_k\phi_x = -\partial_k\frac{\delta W[\mathcal{J},\mathcal{R}_k]}{\delta \mathcal{J}_x} \overset{!}{=} 0$$

implies $\mathcal{J}[\phi] \equiv \mathcal{J}_k[\phi]$ and

$$\partial_k W[\mathcal{J}_k,\mathcal{R}_k] = -\phi_x\partial_k\mathcal{J}_{k,x} - \frac{1}{2}\left(\hbar\Delta_{k,xy} + \phi_x\phi_y\right)\partial_k\mathcal{R}_{k,xy}$$

$$\Delta_{k,xy} = -\frac{\delta^2 W[\mathcal{J}_k,\mathcal{R}_k]}{\delta\mathcal{J}_{k,x}\delta\mathcal{J}_{k,y}}$$

The **Wetterich-Morris-Ellwanger equation**:
[C. Wetterich '93; T. R. Morris '94; U. Ellwanger '94]

$$\partial_k\Gamma^{\text{1PI}}_{\text{av}}[\phi,\mathcal{R}_k] = -\frac{\hbar}{2}\,\text{Tr}\left(\Delta_k * \partial_k\mathcal{R}_k\right)$$